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THE HEIGHTS OF SYMMETRIC PEAKS AND THE DEPTHS OF SYMMETRIC VALLEYS OVER COMPOSITIONS OF AN INTEGER

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Abstract

A composition $\pi = \pi_1\pi_2\cdots\pi_k$ of a positive integer n is an ordered collection of one or more positive integers whose sum is n. The number of summands, namely k, is called the number of parts of π. In this paper, we introduce two statistics over compositions of an integer n with exactly k parts: heights of symmetric peaks and depths of symmetric valleys over all compositions of n. We derive an explicit formula for the generating functions of compositions of n with exactly k parts according to the number of symmetric peaks (resp., valleys) and the total heights (resp., depths) of peaks (resp., valleys). Additionally, we present a combinatorial proof for the total heights (resp., depths) of peaks (resp., valleys) over all compositions of n with exactly k parts. Furthermore, we investigate these statistics in the context of random words, where the letters are generated by geometric probabilities.

1. Introduction

This paper is divided into three sections. In the first and second sections, we focus on compositions of an integer n as follows: a *composition* π of a positive integer n with k parts is a sequence $\pi_1\pi_2\cdots\pi_k$ of positive integers over the set $[n] = \{1, 2 \ldots, n\}$ such that $\sum_{i=1}^{k}\pi_i = n$. For instance, the compositions of 5 are 5, 41, 14, 32, 23, 113, 131, 311, 221, 212, 122, 1112, 1211, 1121, 2111, and 11111. For more details on compositions see [6] for example. Let C_n (resp., $C_{n,k}$) be the set of all compositions of n (resp., compositions of n with exactly k parts), with $|C_0| = |C_{0,0}| = 1$, $|C_{0,k}| = 0$ for $k \geq 1$, and $|C_{n,k}| = 0$ for $n < 0$. The number

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of compositions of n with exactly k parts is given by $\binom{n-1}{k-1}$, and the total number of compositions of n is 2^{n-1} (see [7]). Any composition can be represented as a bargraph, where a *bargraph* is a first-quadrant lattice path with steps $(0, 1)$, $(1, 0)$, and $(0, -1)$ that starts at the origin and ends on the x-axis. The column heights of the bargraph are exactly the parts of the composition; see [9]. For example, the composition 312 corresponds to a bargraph with three adjacent columns of heights 3, 1, and 2 (see Figure 1). We say that the composition $\pi \in C_{n,k}$ contains a

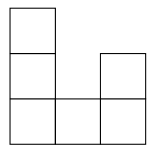

Figure 1: The bargraph corresponding to the composition 312.

symmetric peak (resp., valley) $\pi_i\pi_{i+1}\pi_{i+2}$ if there exists $1 \leq i \leq k-2$ such that $\pi_i < \pi_{i+1}$ and $\pi_i = \pi_{i+2}$ (resp., $\pi_i > \pi_{i+1}$ and $\pi_i = \pi_{i+2}$). Mansour, Moreno, and Ramírez [8] found the total number of symmetric peaks and symmetric valleys over all compositions of n. We define the value $\pi_{i+1} - \pi_i$ (resp., $\pi_i - \pi_{i+1}$) to be the *height of a symmetric peak (resp., the depth of a symmetric valley)*. The study of symmetric peaks and valleys interested researchers in various combinatorial structures. Asakly [1] found the number of symmetric and non-symmetric peaks over words. More recently, Flórez and Ramírez [5] proposed the idea of symmetric and non symmetric peaks in Dyck paths. Following this, Elizalde [4] generalized their results.

In Sections 1 and 2, we obtain the generating function for the number of compositions of an integer n with exactly k parts according to the statistics of the number of symmetric peaks and the sum of the heights of symmetric peaks. By using the theory of generating functions, we derive the formula of the sum of the heights of symmetric peaks. A similar discussion leads us to derive the formula of the sum of the depths of symmetric valleys. We provide combinatorial formulas to count the sum of heights and depths of symmetric peaks and symmetric valleys, respectively.

In Section 3, we study geometrically distributed words. To implement this, we need the following definition: if $0 < p \leq 1$, then a discrete random variable X is said to be *geometric* if $P\{X = k\} = pq^{k-1}$ for all integers $k \geq 1$, where $p+q = 1$. We say that a word $\omega = \omega_1\omega_2 \cdots \omega_m$ over the alphabet of positive integers is *geometrically distributed* if the letters of ω are independent and identically distributed geometric random variables. The study of geometrically distributed words has been a recent topic of interest in enumerative combinatorics; see for example, [2, 3], and the references therein. In the end of the section, we count the total number of symmetric peaks (resp., valleys) and the sum of heights of symmetric peaks (resp., depths of valleys) in a discrete geometrically distributed sample.

2. Heights of Symmetric Peaks

2.1. The Generating Function for the Number of Compositions of n According to the Sum of Symmetric Peak Heights

Let $\mathrm{Sp}(\pi)$ and $\mathrm{Hsp}(\pi)$ denote the number of symmetric peaks and the sum of the heights of symmetric peaks in a composition π, respectively. Consider the generating function for the number of compositions of n with exactly k parts according to the statistics Sp and Hsp to be $\mathrm{HSP}(x, y, q, h)$. That is,

$$\mathrm{HSP}(x, y, q, h) = \sum_{n,k\geq 0} \sum_{\pi \in C_{n,k}} x^n y^k q^{\mathrm{Sp}(\pi)} h^{\mathrm{Hsp}(\pi)}.$$

We derive an explicit formula for the generating function $\mathrm{HSP}(x, y, q, h)$. For that, we denote the generating function for the number of compositions $\pi = \pi_1\pi_2 \cdots \pi_k \in C_{n,k}$, such that $\pi_j = a_j$ for all $j = 1, 2, \ldots, s$ and $s < k$ according to the statistics Sp and Hsp by

$$\mathrm{HSP}(x, y, q, h|a_1a_2 \cdots a_s) = \sum_{n,k\geq 0} \sum_{\pi=a_1a_2\cdots a_s\pi_{s+1}\cdots\pi_k \in C_{n,k}} x^n y^k q^{\mathrm{Sp}(\pi)} h^{\mathrm{Hsp}(\pi)}.$$

Theorem 1. *The generating function for the number of compositions of n with exactly k parts, according to the number of symmetric peaks and the sum of heights of symmetric peaks, is given by*

$$\mathrm{HSP}(x, y, q, h) = \frac{1}{1 - \sum_{a\geq 1} \frac{x^a y}{1-y^2x^{2a+1}\left(\frac{qh}{1-hx} - \frac{1}{1-x}\right)}}. \tag{1}$$

Proof. By definition,

$$\mathrm{HSP}(x, y, q, h) = 1 + \sum_{a\geq 1} \mathrm{HSP}(x, y, q, h|a). \tag{2}$$

Any composition of n that starts with a can be decomposed as in Figure 2. The corresponding generating function is

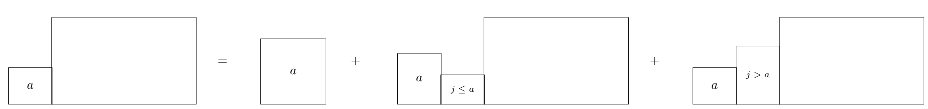

Figure 2: Decomposition of a composition starting with a according to the value of the next part j.

$$\begin{aligned}\mathrm{HSP}(x,y,q,h|a) &= x^a y + \sum_{j=1}^{a} \mathrm{HSP}(x,y,q,h|aj) + \sum_{j\geq a+1} \mathrm{HSP}(x,y,q,h|aj)\\ &= x^a y + x^a y \sum_{j=1}^{a} \mathrm{HSP}(x,y,q,h|j) + \sum_{j\geq a+1} \mathrm{HSP}(x,y,q,h|aj). \quad (3)\end{aligned}$$

For all $b \geq a+1$,

$$\begin{aligned}\mathrm{HSP}(x,y,q,h|ab) &= x^{a+b}y^2 + \sum_{j=1, j\neq a}^{b-1} \mathrm{HSP}(x,y,q,h|abj) + \mathrm{HSP}(x,y,q,h|aba)\\ &\quad + \mathrm{HSP}(x,y,q,h|abb) + \sum_{j\geq b+1} \mathrm{HSP}(x,y,q,h|abj)\\ &= x^{a+b}y^2 + x^{a+b}y^2 \sum_{j=1, j\neq a}^{b-1} \mathrm{HSP}(x,y,q,h|j)\\ &\quad + qx^{a+b}y^2h^{b-a}\,\mathrm{HSP}(x,y,q,h|a) + x^{a+b}y^2\,\mathrm{HSP}(x,y,q,h|b)\\ &\quad + x^a y \sum_{j\geq b+1} \mathrm{HSP}(x,y,q,h|bj)\\ &= x^{a+b}y^2 + x^{a+b}y^2 \sum_{j=1}^{b-1} \mathrm{HSP}(x,y,q,h|j)\\ &\quad + (h^{b-a}q-1)x^{a+b}y^2\,\mathrm{HSP}(x,y,q,h|a)\\ &\quad + x^{a+b}y^2\,\mathrm{HSP}(x,y,q,h|b) + x^a y \sum_{j\geq b+1} \mathrm{HSP}(x,y,q,h|bj).\end{aligned}$$

By Equation (3), we get

$$\begin{aligned}\mathrm{HSP}(x,y,q,h|ab) &= x^{a+b}y^2 + x^{a+b}y^2 \sum_{j=1}^{b-1} \mathrm{HSP}(x,y,q,h|j)\\ &\quad + (qh^{b-a}-1)x^{a+b}y^2\,\mathrm{HSP}(x,y,q,h|a)\\ &\quad + x^{a+b}y^2\,\mathrm{HSP}(x,y,q,h|b) + x^a y(\mathrm{HSP}(x,y,q,h|b)\\ &\quad - x^b y - x^b y \sum_{j=1}^{b} \mathrm{HSP}(x,y,q,h|j))\\ &= (qh^{b-a}-1)x^{a+b}y^2\,\mathrm{HSP}(x,y,q,h|a) + x^a y\,\mathrm{HSP}(x,y,q,h|b).\end{aligned}$$

Summing over all $b \geq a+1$, we have

$$\begin{aligned}\sum_{j\geq a+1} \mathrm{HSP}(x,y,q,h|aj) &= \sum_{j\geq a+1} (qh^{j-a}-1)x^{a+j}y^2\,\mathrm{HSP}(x,y,q,h|a)\\ &\quad + x^a y \sum_{j\geq a+1} \mathrm{HSP}(x,y,q,h|j)\\ &= x^{2a+1}y^2\,\mathrm{HSP}(x,y,q,h|a)\left(\frac{qh}{1-hx}-\frac{1}{1-x}\right)\\ &\quad + x^a y \sum_{j\geq a+1} \mathrm{HSP}(x,y,q,h|j).\end{aligned}$$

Thus, by Equation (3), we have

$$\begin{aligned}\mathrm{HSP}(x,y,q,h|a) &= x^a y + \sum_{j=1}^{a} \mathrm{HSP}(x,y,q,h|aj)\\ &\quad + x^{2a+1}y^2\,\mathrm{HSP}(x,y,q,h|a)\left(\frac{qh}{1-hx}-\frac{1}{1-x}\right)\\ &\quad + x^a y \sum_{j\geq a+1} \mathrm{HSP}(x,y,q,h|j)\\ &= x^a y + x^a y \sum_{j=1}^{a} \mathrm{HSP}(x,y,q,h|j)\\ &\quad + x^{2a+1}y^2\,\mathrm{HSP}(x,y,q,h|a)\left(\frac{qh}{1-hx}-\frac{1}{1-x}\right)\\ &\quad + x^a y \sum_{j\geq a+1} \mathrm{HSP}(x,y,q,h|j)\\ &= x^a y + x^a y(\mathrm{HSP}(x,y,q,h)-1)\\ &\quad + x^{2a+1}y^2\,\mathrm{HSP}(x,y,q,h|a)\left(\frac{qh}{1-hx}-\frac{1}{1-x}\right).\end{aligned}$$

Therefore, we derive

$$\mathrm{HSP}(x,y,q,h|a) = \frac{x^a y\,\mathrm{HSP}(x,y,q,h)}{1-x^{2a+1}y^2\left(\frac{qh}{1-hx}-\frac{1}{1-x}\right)}.$$

Hence, by Equation (2), we have

$$\mathrm{HSP}(x,y,q,h)-1 = \mathrm{HSP}(x,y,q,h)\sum_{a\geq 1}\frac{x^a y}{1-x^{2a+1}y^2\left(\frac{qh}{1-hx}-\frac{1}{1-x}\right)}.$$

This leads to the required result. □

Note that, by substituting $h = 1$ in Theorem 1, we obtain the generating function for the number of compositions of n with k parts according to the number of symmetric peaks

$$\mathrm{HSP}(x, y, q, 1) = \frac{1}{1 - \sum_{a\geq 1} \frac{x^a y}{1-y^2x^{2a+1}(\frac{q-1}{1-x})}}, \tag{4}$$

which is in accord with the result in [8].

Let $\mathrm{hsp}(n)$ denote the sum of heights of symmetric peaks in C_n, and let $\mathrm{hsp}(n,k)$ denote the sum of heights of symmetric peaks in $C_{n,k}$. Then, we have $\mathrm{hsp}(n) = \sum_{k=0}^{n} \mathrm{hsp}(n,k)$.

According to Theorem 1, we can conclude this lemma.

Lemma 1. *The generating functions for the sequences* $\mathrm{hsp}(n)$ *and* $\mathrm{hsp}(n,k)$ *are given by*

$$\sum_{n\geq 0} \mathrm{hsp}(n)x^n = \frac{\partial}{\partial h}\mathrm{HSP}(x,1,1,h)\mid_{h=1} = \frac{x^4}{(1-2x)^2(1-x^3)}. \tag{5}$$

$$\sum_{n,k\geq 0} \mathrm{hsp}(n,k)x^n y^k = \frac{\partial}{\partial h}\mathrm{HSP}(x,y,1,h)\mid_{h=1} = \frac{y^3x^4}{(1-x^3)(1-x-yx)^2}.$$

Using a computer algebra system (Maple), we computed the coefficients of x^n in Equation (5) and obtained

$$\sum_{n\geq 0} \mathrm{hsp}(n)x^n = x^4 + 4x^5 + 12x^6 + 33x^7 + 84x^8 + 204x^9 + 481x^{10} + O(x^{11}).$$

This leads to the following result.

Corollary 1. *The sum of the heights of symmetric peaks over all compositions of* n *is given by*

$$\mathrm{hsp}(n) = \left(\frac{7n-24}{49}\right)2^{n-1} + \frac{(-33-15i\sqrt{3})(-2)^n}{441(1+i\sqrt{3})^{n+1}} + \frac{(-33+15i\sqrt{3})(-2)^n}{441(1-i\sqrt{3})^{n+1}} + \frac{1}{3}. \tag{6}$$

Please see A393175 in [10]. For instance, when $n = 5$, we have three compositions with at least one symmetric peak: 131, 1211, and 1121. The sum of the heights of symmetric peaks is equal to 4 as obtained from Equation (6).

2.2. Combinatorial Derivation of the Number of Compositions of n with Exactly k Parts According to the Sum of Symmetric Peak Heights

The following theorem provides explicit formulas for the statistic $\mathrm{hsp}(n,k)$.

Theorem 2. *(a) For $n \geq 0$ and $k > 3$, we have*

$$\mathrm{hsp}(n,k) = (k-2)\sum_{m=2}^{n-k+1}\sum_{b=1}^{min\{m-1,t\}}\binom{n-2b-m-1}{k-4}(m-b)$$

where $t = \lfloor\frac{n-m-k+3}{2}\rfloor$.

(b) When $k = 3$ we have

$$\mathrm{hsp}(n,3) = \sum_{m=2}^{n-2} h_m$$

where h_m is defined as follows:

$$h_m = \begin{cases} \frac{3m-n}{2}, & \text{if } \frac{n-m}{2} \text{ is an integer} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. To prove (a), let $\pi_{i-1}\pi_i\pi_{i+1} = bmb$ be a symmetric peak in a composition of n with $k \geq 4$ parts, where $2 \leq i \leq k-1$. According to [8], the total number of symmetric peaks in C_n with k parts is given by

$$(k-2)\sum_{m=2}^{n-k+1}\sum_{b=1}^{\min\{m-1,t\}}\binom{n-2b-m-1}{k-4}.$$

Multiplying the above expression by $m-b$ for each pair of m and b such that $2 \leq m \leq n-k+1$ and $1 \leq b \leq \min\{m-1,t\}$ we get the desired result.

To prove (b), let $h_m = m-b$. If $\frac{n-m}{2}$ is an integer, then $b = \frac{n-m}{2}$. In this case, $h_m = m-b = m-\frac{n-m}{2}$. Otherwise, there is no symmetric peak bmb such that $\frac{n-m}{2}$ is an integer, and therefore, $h_m = 0$.

□

3. Depths of Symmetric Valleys

3.1. The Generating Function for the Number of Compositions of n According to the Sum of Symmetric Valleys Depth

Let $\mathrm{Sv}(\pi)$ and $\mathrm{Dsv}(\pi)$ denote the number of symmetric valleys and the sum of the depths of symmetric valleys in a composition in π, respectively. Denote the generating function for the number of compositions of n with exactly k parts according to the statistics Sv and Dsv by $\mathrm{DSV}(x,y,p,d)$; that is,

$$\mathrm{DSV}(x,y,p,d) = \sum_{n,k\geq 0}\sum_{\pi\in C_{n,k}} x^n y^k p^{\mathrm{Sv}(\pi)} d^{\mathrm{Dsv}(\pi)}.$$

We derive an explicit formula for the generating function $\mathrm{DSV}(x, y, p, d)$. To do this, we denote the generating function for the number of compositions $\pi = \pi_1\pi_2 \ldots \pi_k \in C_{n,k}$, such that $\pi_j = a_j$ for all $j = 1, 2, \ldots, s$ and $s < k$ according to the statistics Sv and Dsv, by

$$\mathrm{DSV}(x, y, p, d|a_1a_2 \cdots a_s) = \sum_{n,k\geq 0} \sum_{\pi=a_1a_2\cdots a_s\pi_{s+1}\cdots\pi_k\in C_{n,k}} x^n y^k p^{\mathrm{Sv}(\pi)} d^{\mathrm{Dsv}(\pi)}.$$

Theorem 3. *The generating function for the number of compositions of n with exactly k parts, according to the number of symmetric valleys and the sum of depths of symmetric valleys, is given by*

$$\mathrm{DSV}(x, y, p, d) = \frac{1}{1 - \sum_{a\geq 1} \frac{x^a y}{1-y^2x^{a+1}\left(\frac{px^{a-1}-d^{a-1}}{\frac{x}{d}-1} - \frac{x^{a-1}-1}{x-1}\right)}}. \tag{7}$$

Proof. By definition,

$$\mathrm{DSV}(x, y, p, d) = 1 + \sum_{a\geq 1} \mathrm{DSV}(x, y, p, d|a). \tag{8}$$

Using the same decomposition as in the proof of Theorem 1 (see Figure 2), the corresponding generating function is

$$\begin{aligned}\mathrm{DSV}(x, y, p, d|a) &= x^a y + \sum_{j=1}^{a-1} \mathrm{DSV}(x, y, p, d|aj) + \sum_{j\geq a} \mathrm{DSV}(x, y, p, d|aj)\\ &= x^a y + \sum_{j=1}^{a-1} \mathrm{DSV}(x, y, p, d|aj) + x^a y \sum_{j\geq a} \mathrm{DSV}(x, y, p, d|j). \end{aligned}\tag{9}$$

For all $b \geq a$,

$$\begin{aligned}\mathrm{DSV}(x, y, p, d|ab) &= x^{a+b}y^2 + \sum_{j=1}^{b-1} \mathrm{DSV}(x, y, p, d|abj) + \mathrm{DSV}(x, y, p, d|aba)\\ &\quad + \sum_{j\geq b, j\neq a} \mathrm{DSV}(x, y, q, h|abj)\\ &= x^{a+b}y^2 + x^a y \sum_{j=1}^{b-1} \mathrm{DSV}(x, y, p, d|bj)\\ &\quad + x^{a+b}y^2 \sum_{j\geq b, j\neq a} \mathrm{DSV}(x, y, p, d|j) + x^{a+b}y^2 pd^{a-b}\, \mathrm{DSV}(x, y, p, d|a)\\ &= x^{a+b}y^2 + x^a y \sum_{j=1}^{b-1} \mathrm{DSV}(x, y, p, d|bj)\\ &\quad + x^{a+b}y^2\, \mathrm{DSV}(x, y, p, d|a)(pd^{a-b} - 1) + x^{a+b}y^2 \sum_{j\geq b} \mathrm{DSV}(x, y, p, d|j).\end{aligned}$$

By Equation (9), we get

$$\begin{aligned}\mathrm{DSV}(x,y,p,d|ab) &= x^{a+b}y^2 + x^a y\Big(\mathrm{DSV}(x,y,p,d|b) - x^b y - x^b y \sum_{j\geq b}\mathrm{DSV}(x,y,p,d|j)\Big)\\ &\quad + x^{a+b}y^2\,\mathrm{DSV}(x,y,p,d|a)(pd^{a-b}-1) + x^{a+b}y^2\sum_{j\geq b}\mathrm{DSV}(x,y,p,d|j)\\ &= x^{a+b}y^2\,\mathrm{DSV}(x,y,p,d|a)(pd^{a-b}-1) + x^a y\,\mathrm{DSV}(x,y,p,d|b).\end{aligned}$$

Thus, by Equation (9) and by summing over all $b \geq a$, we get

$$\begin{aligned}\mathrm{DSV}(x,y,p,d|a) &= x^a y + \sum_{j=1}^{a-1} x^{a+j}y^2\,\mathrm{DSV}(x,y,p,d|a)(pd^{a-j}-1)\\ &\quad + x^a y\sum_{j=1}^{a-1}\mathrm{DSV}(x,y,p,d|j) + x^a y\sum_{j\geq a}\mathrm{DSV}(x,y,p,d|j)\\ &= x^a y\,\mathrm{DSV}(x,y,p,d) + x^a y^2 p d^a\,\mathrm{DSV}(x,y,p,d|a)\left(\frac{\left(\frac{x}{d}\right)^a - \frac{x}{d}}{\frac{x}{d}-1}\right)\\ &\quad - x^a y^2\,\mathrm{DSV}(x,y,p,d|a)\left(\frac{x^a - x}{x-1}\right).\end{aligned}$$

Therefore, we derive

$$\mathrm{DSV}(x,y,p,d|a) = \frac{x^a y}{1 - y^2 x^{a+1}\left(\frac{px^{a-1}-d^{a-1}}{\frac{x}{d}-1} - \frac{x^{a-1}-1}{x-1}\right)}\,\mathrm{DSV}(x,y,p,d).$$

Hence, by Equation (8), we get

$$\mathrm{DSV}(x,y,p,d) = \frac{1}{1 - \sum_{a\geq 1}\frac{x^a y}{1-y^2x^{a+1}\left(\frac{px^{a-1}-d^{a-1}}{\frac{x}{d}-1} - \frac{x^{a-1}-1}{x-1}\right)}}.$$

□

Note that by substituting $d = 1$ in Equation (7) we obtain the generating function for the number of compositions of n with k parts according to the number of symmetric valleys

$$\mathrm{DSV}(x,y,p,1) = \frac{1}{1 - \sum_{a\geq 1}\frac{x^a y}{1-\left(\frac{y^2x^{a+1}(x^{a-1}-1)(p-1)}{x-1}\right)}}, \tag{10}$$

which is in accord with the result in [8].

Let $\mathrm{dsv}(n)$ denote the sum of depths of symmetric valleys in C_n, and let $\mathrm{dsv}(n,k)$ denote the sum of depths of symmetric valleys in $C_{n,k}$. Then, we have $\mathrm{dsv}(n) = \sum_{k=0}^{n} dsv(n,k)$.

According to Theorem 3, we can conclude this lemma.

Lemma 2. *The generating functions for the sequences* $\operatorname{dsv}(n)$ *and* $\operatorname{dsv}(n,k)$ *are given by*

$$\sum_{n\geq 0} \operatorname{dsv}(n)x^n = \frac{\partial}{\partial d}\operatorname{DSV}(x,1,1,d)\mid_{d=1} = \frac{x^7+x^5-2x^6}{(1-2x)^2(1-x^3)(1-x^2)^2}. \tag{11}$$

$$\sum_{n,k\geq 0} \operatorname{dsv}(n,k)x^n y^k = \frac{\partial}{\partial d}\operatorname{DSV}(x,y,1,d)\mid_{d=1} = \frac{2x^7y^3-x^6y^3-x^4y^3}{(1-x^3)(1-x^2)(1-x-yx)^2}.$$

Using a computer algebra system (Maple), we determined the coefficients of x^n in Equation (11):

$$\sum_{n\geq 0} \operatorname{dsv}(n)x^n = x^5+2x^6+7x^7+17x^8+43x^9+101x^{10}+O(x^{11}).$$

We obtained the following result.

Corollary 2. *The sum of depths of symmetric valleys over all compositions of* n *is given by*

$$\begin{aligned}\operatorname{dsv}(n) &= \left(\frac{-6n+7}{108}\right)(-1)^n + \left(\frac{21n-79}{1323}\right)2^n \\ &\quad + \frac{(-33-15\sqrt{3}i)\,(-2)^n}{441(1+i\sqrt{3})^{n+1}} + \frac{(-33+15\sqrt{3}i)\,(-2)^n}{441(1-i\sqrt{3})^{n+1}} + \frac{1}{12}.\end{aligned} \tag{12}$$

Please see A393177 in [10]. For instance, when $n=8$, there are 14 compositions with at least one symmetric valley: 323, 3131, 1313, 2123, 3212, 21221, 21212, 22121, 12122, 12212, 212111, 111212, 112121, and 121211. The sum of depths of symmetric valleys is equal to 17 as obtained in Equation (12).

3.2. Combinatorial Derivation of the Number of Compositions of n with Exactly k Parts According to the Sum of Symmetric Valley Heights

The following theorem provides explicit formulas for the statistic $\operatorname{dsv}(n,k)$.

Theorem 4. *(a) For* $n\geq 0$, *and* $k>3$ *we have,*

$$\operatorname{dsv}(n,k) = (k-2)\sum_{m=1}^{\lfloor\frac{n-k+1}{3}\rfloor}\ \sum_{b=m+1}^{\lfloor\frac{n-m-(k-3)}{2}\rfloor}\binom{n-2b-m-1}{k-4}(b-m).$$

(b) When $k=3$ *we have*

$$\operatorname{dsv}(n,3) = \sum_{m=1}^{\lfloor\frac{n-2}{3}\rfloor} d_m$$

where d_m is defined as follows:

$$d_m = \begin{cases} \frac{n-3m}{2}, & \text{if } \frac{n-m}{2} \text{ is an integer} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We first consider the case $k > 3$ and $n \geq 0$. Let $\pi_{i-1}\pi_i\pi_{i+1} = bmb$ represent a symmetric valley within a composition π of n with $k \geq 4$ parts, where $2 \leq i \leq k-1$ and $1 \leq m \leq \lfloor \frac{n-k+1}{3} \rfloor$. Notice that the minimum value of b is $m+1$, and b reaches its maximum value when $\pi_j = 1$ for all $1 \leq j \leq k$ with $j \neq i-1, i, i+1$. Therefore, $m+1 \leq b \leq \lfloor \frac{n-m-(k-3)}{2} \rfloor$.

The number of compositions of the form π is $\binom{n-2b-m-1}{k-4}$, which is equivalent to determining the number of solutions to $\ell_1 + \ell_2 \cdots + \ell_{k-3} = n-2b-m-(k-3)$, where $\ell_1, \ell_2, \ldots \ell_{k-3}$ are nonnegative integers. The depth of the symmetric valley bmb is $b-m$. Therefore, the total sum of depths of compositions of n with a symmetric valley in the i-th position can be expressed as follows:

$$\sum_{m=1}^{\lfloor \frac{n-k+1}{3} \rfloor} \sum_{b=m+1}^{\lfloor \frac{n-m-(k-3)}{2} \rfloor} \binom{n-2b-m-1}{k-4}(b-m).$$

There are $k-2$ options for choosing i. Multiplying the above expression by $k-2$ yields the desired result.

Next, we consider the case $k = 3$. Let $d_m = b - m$. If $\frac{n-m}{2}$ is an integer then $b = \frac{n-m}{2}$. In this case, $d_m = b - m = \frac{n-m}{2} - m$. Otherwise, there is no symmetric valley bmb such that $\frac{n-m}{2}$ is an integer, and therefore, $d_m = 0$.

□

4. A Discrete Geometrically Distributed Sample

Recall from Section 1 the definition of a geometric random variable and a geometrically distributed word. The generating function of a single letter is therefore,

$$G(z) = \mathbb{E}[z^X] = \sum_{k \geq 1} pq^{k-1}z^k = \frac{pz}{1-qz}.$$

The generating function of the sum of the letters in a word of length m is

$$G_m(z) = \left(\frac{pz}{1-qz}\right)^m.$$

Assume that $C(x, y, w)$ is the generating function for the number of compositions of n with exactly k parts, according to a fixed statistic. Applying the map $x \mapsto q$, $y \mapsto \frac{pz}{q}$, and $w \mapsto u$, the function $C(x, y, w)$ transforms into the generating function

$G(z,u)$ for discrete geometrically distributed samples of length n with respect to the same statistic. We say that a geometrically distributed word $\omega = \omega_1\omega_2\cdots\omega_m$ contains a *symmetric peak* (resp., *valley*) $\omega_i\omega_{i+1}\omega_{i+2}$, if there exists $1 \le i \le k-2$ such that $\omega_i < \omega_{i+1}$ and $\omega_i = \omega_{i+2}$ (resp., $\omega_i > \omega_{i+1}$ and $\omega_i = \omega_{i+2}$). We define the value $\omega_{i+1} - \omega_i$ (resp., $\omega_i - \omega_{i+1}$) to be the *height of a symmetric peak* (resp., *the depth of a symmetric valley).*

4.1. The Expectation

In this section we explore discrete geometrically distributed samples under the statistics of $\mathrm{Sp}, \mathrm{Sv}, \mathrm{Hsp}$, and Dsv. The following theorem provides explicit formulas for the expected values of these statistics.

Theorem 5. *Let $n \ge 2$ and let $p \in (0,1)$, with $q = 1-p$.*

(a) The expected value of the statistic Sp *in a sample of n geometric random variables is given by*

$$\frac{p^2 q}{1-q^3}(n-2).$$

(b) The expected value of the statistic Sv *in a sample of n geometric random variables is given by*

$$p^2\left(\frac{1}{1-q^2} - \frac{1}{1-q^3}\right)(n-2).$$

(c) The expected value of the statistic Hsp *in a sample of n geometric random variables is given by*

$$\frac{pq}{1-q^3}(n-2).$$

(d) The expected value of the statistic Dsv *in a sample of n geometric random variables is given by*

$$q\left(\frac{p}{1-q^3} - \frac{1}{(1+q)^2}\right)(n-2).$$

Proof. We provide the proof only for part (a), as the proofs of the other parts are similar. To prove (a), we apply the mapping $x \mapsto q$, $y \mapsto \frac{pz}{q}$, and $q \mapsto u$ in Equation (4) we obtain the generating function for the number of discrete geometrically distributed samples of length n according to the statistic Sp, which is given by

$$\mathrm{GSP}(z,u) = \frac{1}{1 - \sum_{a\ge 1} \frac{q^{a-1}pz}{1-pz^2q^{2a-1}(u-1)}}. \tag{13}$$

To find the expected value of the statistic Sp in a sample of n geometric random variables, we use the expression $\frac{[z^n]\frac{d}{du}\operatorname{GSP}(z,u)|_{u=1}}{[z^n]\operatorname{GSP}(u,1)}$. It is obvious that $\operatorname{GSP}(u,1) = \frac{1}{1-z}$, hence $[z^n]\operatorname{GSP}(u,1) = 1$. From Equation (13) we get

$$\frac{d}{du}\operatorname{GSP}(z,u)|_{u=1} = \frac{p^2z^3q}{(1-q^3)}\sum_{k\geq 0}\binom{k+1}{k}z^k.$$

Therefore,

$$[z^n]\frac{d}{du}\operatorname{GSP}(z,u)|_{u=1} = [z^{n-3}]\frac{p^2z^3q}{(1-q^3)}\sum_{k\geq 0}\binom{k+1}{k}z^k = \frac{p^2q}{1-q^3}(n-2).$$

To prove (b), we apply the mapping $x \mapsto q$, $y \mapsto \frac{pz}{q}$, and $p \mapsto u$, and by substituting $d = 1$ in Equation (7), we define $\operatorname{GSV}(z,u)$ as the generating function for the number of discrete geometrically distributed samples of length n according to the statistic Sv. By finding the coefficients of z^n in $\frac{\partial}{\partial u}\operatorname{GSV}(z,u)\,|_{u=1}$, similar to above, we obtain the required result.

To prove (c), we apply the mapping $x \mapsto q$, $y \mapsto \frac{pz}{q}$, and $h \mapsto u$, and by substituting $q = 1$ in Equation (1), we define $\operatorname{GHSP}(z,u)$ as the generating function for the number of discrete geometrically distributed samples of length n according to the statistic Hsp. By finding the coefficients of z^n in $\frac{\partial}{\partial u}\operatorname{GHSP}(z,u)\,|_{u=1}$, similar to above, we obtain the required result.

To prove (d), we apply the mapping $x \mapsto q$, $y \mapsto \frac{pz}{q}$, and $d \mapsto u$, and by substituting $p = 1$ in Equation (7), we define $\operatorname{GDSV}(z,u)$ as the generating function for the number of discrete geometrically distributed samples of length n according to the statistic Dsv. By finding the coefficients of z^n in $\frac{\partial}{\partial u}\operatorname{GDSV}(z,u)\,|_{u=1}$, similar to above, we obtain the required result.

□

4.2. The Variance

The following theorem provides formulas for the variances of several statistics on samples of geometric random variables.

Theorem 6. *Let $n \geq 2$ and let $p \in (0,1)$, with $q = 1-p$.*

(a) The variance of the number of symmetric peaks of a sample of n geometric random variables is given by

$$2(n-4)p^3q^2\left(\frac{p(n-5)}{24(1-q)^3} + \frac{1}{1-q^5}\right) + (n-2)\frac{p^2q}{1-q^3} - (n-2)^2\frac{p^4q^2}{(1-q^3)^2}.$$

(b) The variance of the sum of symmetric peak heights of a sample of n geometric random variables is given by

$$(n-5)(n-6)\frac{p^2q^2}{(1-q^3)^2} + 2(n-4)\frac{pq^2}{(1-q^5)} + \frac{(n-2)}{(1-q^3)}(2q^2+pq) - \frac{p^2q^2}{(1-q^3)^2}(n-2)^2.$$

(c) The variance of the number of symmetric valleys of a sample of n geometric random variables is given by

$$\frac{p^4q^2}{1-q^3}(n-5)(n-6)+\frac{2p^3}{1-q^5}(n-5)+p^2\left(\frac{1}{1-q^2}-\frac{1}{1-q^3}\right)(n-2)$$
$$-p^4\left(\frac{1}{1-q^2}-\frac{1}{1-q^3}\right)^2(n-2)^2.$$

(d) The variance of the sum of symmetric valley depths of a sample of n geometric random variables is given by

$$\frac{p^4}{(1-q^3)^2}(n-4)(n-5)+\frac{2p^3}{1-q^5}(n-4)+q\left(\frac{p}{1-q^3}-\frac{1}{(1+q)^2}\right)(n-2)$$
$$-q^2\left(\frac{p}{1-q^3}-\frac{1}{(1+q)^2}\right)^2(n-2)^2.$$

Proof. We provide the proof for only part (a), as the proofs for the other three statements are quite similar. To find the variance of the statistic Sp in a sample of n geometric random variables, we use the expression

$$\frac{[z^n]\frac{d^2}{du^2}\operatorname{GSP}(z,u)|_{u=1}}{[z^n]\operatorname{GSP}(u,1)}+\frac{[z^n]\frac{d}{du}\operatorname{GSP}(z,u)|_{u=1}}{[z^n]\operatorname{GSP}(u,1)}-\left(\frac{[z^n]\frac{d}{du}\operatorname{GSP}(z,u)|_{u=1}}{[z^n]\operatorname{GSP}(u,1)}\right)^2.$$

According to previous results, we have

$$\frac{[z^n]\frac{d}{du}\operatorname{GSP}(z,u)|_{u=1}}{[z^n]\operatorname{GSP}(u,1)}=\frac{p^2q}{1-q^3}(n-2).$$

In addition,

$$\frac{[z^n]\frac{d^2}{du^2}\operatorname{GSP}(z,u)|_{u=1}}{[z^n]\operatorname{GSP}(u,1)}=2(n-4)p^3q^2\left(\frac{p(n-5)}{4!(1-q)^3}+\frac{1}{1-q^5}\right).$$

Using a similar approach, we achieve $(b),(c)$, and (d). □

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