



**TWO SETS OF INTEGERS SUCH THAT ALL ELEMENTS OF THE
SUMSET OF THE TWO SETS ARE PERFECT SQUARES**

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Abstract

This paper is concerned with the problem of finding two sets of integers, $\{a_1, a_2, \dots, a_m\}$ and $\{b_1, b_2, \dots, b_n\}$, such that all mn sums $a_i + b_j, i = 1, \dots, m, j = 1, \dots, n$, are perfect squares. A method is known for generating numerical examples of such sets when $m = 2$ or 3 and n is arbitrary. When both m and n exceed 2 , only one two-parameter solution with $(m, n) = (4, 4)$ has been published. In this paper we obtain several multi-parameter solutions of the problem in three cases when (m, n) is $(3, 3)$ or $(5, 3)$ or $(4, 4)$, and we indicate how more such solutions may be obtained.

1. Introduction

This paper is concerned with the problem of finding two sets of integers, $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$, such that all elements of the sumset $A + B$, that is, the mn sums $a_i + b_j, i = 1, \dots, m, j = 1, \dots, n$, are perfect squares. The problem has been mentioned in Guy's book, 'Unsolved Problems in Number Theory' [2, Problem D14, pp. 266-267] where a method is described for generating numerical solutions of the problem when $m = 2$ or 3 , and n is arbitrary.

When $(m, n) = (4, 4)$, Guy[2, p. 267] has presented a numerical solution, as well as a parametric solution found by Jean Lagrange by simply stating the related 4×4 matrices of squares whose $(i, j)^{\text{th}}$ entry is $a_i + b_j$. We thus get the numerical quadruples $A = \{324, 54756, 119716, 264196\}$ and $B = \{0, 79200, 227205, 1258560\}$, and also a solution in terms of arbitrary parameters u and v given by

$$\begin{aligned} a_1 &= 9(18u^2 + 50uv + 37v^2)^2, \\ a_2 &= 36(u^2 + 13uv + 16v^2)^2, \\ a_3 &= 36(9u^2 + 53uv + 64v^2)^2, \\ a_4 &= 4(3u^2 - 25uv - 48v^2)^2, \end{aligned}$$

and

$$\begin{aligned} b_1 &= 0, \\ b_2 &= 20(u - 4v)(11u + 19v)(u + 4v)(u + v), \\ b_3 &= 36(3u + 11v)(7u + 12v)(3u + 4v)(u + v), \\ b_4 &= 756(u - v)(3u + 2v)(u + 3v)(u + 2v), \end{aligned}$$

such that all 16 sums $a_i + b_j$ are squares. Apart from the above solution, no other parametric solutions seem to have been published for any other values of (m, n) .

In this paper we obtain several multi-parameter solutions of the problem when (m, n) is $(3, 3)$ or $(5, 3)$ or $(4, 4)$ and we show how more such parametric solutions may be obtained. We obtain all our parametric solutions by using semi-magic matrices that satisfy certain conditions. In Section 2 we prove a couple of preliminary lemmas, and in the next three sections we apply the lemmas to obtain parametric solutions for the three cases mentioned above. We conclude the paper with a couple of open problems.

2. Preliminary Lemmas and Remarks

We first give a preliminary lemma about two sets of rational numbers, namely, $A = \{a_i : i = 1, \dots, m\}$ and $B = \{b_j : j = 1, \dots, n\}$, such that all mn sums $a_i + b_j, i = 1, \dots, m,$ and $j = 1, \dots, n,$ are squares of rational numbers. It immediately follows from this lemma that to obtain two sets of integers $\{a_i : i = 1, \dots, m\}$ and $\{b_j : j = 1, \dots, n\}$ such that all mn sums $a_i + b_j$ are squares, it suffices to obtain two sets of rational numbers with the same property.

Lemma 1. *If there exist two sets of rational numbers, namely, $A = \{a_i : i = 1, \dots, m\}$ and $B = \{b_j : j = 1, \dots, n\}$, such that all mn sums $a_i + b_j, i = 1, \dots, m,$ and $j = 1, \dots, n,$ are squares, then if t is an arbitrary integer and k is either an arbitrary integer or a rational number so chosen that $k^2 a_i, i = 1, \dots, m,$ and $k^2 b_j, j = 1, \dots, n,$ are all integers, and we define $a'_i = k^2 a_i - t, i = 1, \dots, m,$ $b'_j = k^2 b_j + t, i = 1, \dots, n,$ then the two sets $A' = \{a'_i : i = 1, \dots, m\}$ and $B' = \{b'_j : j = 1, \dots, n\}$ consist of integers such that all mn sums $a'_i + b'_j, i = 1, \dots, m,$ and $j = 1, \dots, n,$ are perfect squares.*

Proof. The numbers $a'_i, i = 1, \dots, m$ and $b'_j, j = 1, \dots, n,$ are clearly integers. Further, the mn sums $a'_i + b'_j$ may be written as $k^2(a_i + b_j), i = 1, \dots, m,$ and $j = 1, \dots, n$ and are hence perfect squares. This proves the lemma. \square

In view of the above lemma, two such sets of integers, A and $B,$ can be presented in a standard form in which all the integers of the two sets are nonnegative, they do not have a common squared factor, the integers of both the sets are in ascending order of magnitude and the smallest integer of the set B is 0. In this standard

form, all the integers of the set A are necessarily perfect squares. The numerical examples of such sets obtained in this paper will be presented in this standard form with $b_1 = 0$. The sets A and B that we obtain in terms of certain arbitrary parameters will be presented generally, but not always, with $b_1 = 0$ and such that the elements of the two sets do not have a common squared factor.

A square matrix is said to be a *semi-magic matrix* if the sum of the entries in each row and each column is the same. This common sum is referred to as the *semi-magic sum*. The following lemma relates the existence of two sets of integers $A = \{a_i, i = 1, \dots, m\}$ and $B = \{b_i, j = 1, \dots, n\}$, where $(m, n) = (3, 3)$ or $(5, 3)$ or $(4, 4)$, such that all mn sums $a_i + b_j$ are perfect squares to certain 3×3 semi-magic matrices which satisfy certain conditions.

Lemma 2. *There exist two triples of integers $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ such that all nine sums $a_i + b_j, i = 1, 2, 3, j = 1, 2, 3$, are squares if and only if there exists a 3×3 semi-magic integer matrix E with squared entries. When such a matrix E exists and is defined by*

$$E = \begin{bmatrix} e_{11}^2 & e_{12}^2 & e_{13}^2 \\ e_{21}^2 & e_{22}^2 & e_{23}^2 \\ e_{31}^2 & e_{32}^2 & e_{33}^2 \end{bmatrix}, \tag{2.1}$$

then the two triples of integers $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ defined by

$$\begin{aligned} a_1 &= e_{13}^2, & a_2 &= e_{31}^2, & a_3 &= e_{22}^2, \\ b_1 &= 0, & b_2 &= e_{21}^2 - e_{13}^2, & b_3 &= e_{11}^2 - e_{22}^2, \end{aligned} \tag{2.2}$$

are such that all nine sums $a_i + b_j, i = 1, 2, 3, j = 1, 2, 3$, are squares.

Further, the two triples A and B defined by (2.2) may be extended to a quintuple $A_1 = \{a_1, a_2, a_3, a_4, a_5\}$ and a triple $B_1 = \{b_1, b_2, b_3\}$ such that all 15 sums $a_i + b_j, i = 1, \dots, 5, j = 1, 2, 3$, are squares if and only if, in addition to the matrix E defined by (2.1), there exists a 3×3 semi-magic integer matrix F with squared entries, namely,

$$F = \begin{bmatrix} f_{11}^2 & f_{12}^2 & f_{13}^2 \\ f_{21}^2 & f_{22}^2 & f_{23}^2 \\ f_{31}^2 & f_{32}^2 & f_{33}^2 \end{bmatrix},$$

such that the entries of the two matrices E and F satisfy the following conditions:

$$e_{11}^2 = f_{11}^2, \quad e_{22}^2 = f_{22}^2, \quad e_{33}^2 = f_{33}^2. \tag{2.3}$$

When two such matrices E and F exist, and $a_i, b_i, i = 1, 2, 3$, are defined, as before, by (2.2) and a_4, a_5 are defined by

$$a_4 = f_{13}^2, \quad a_5 = f_{31}^2, \tag{2.4}$$

then all 15 sums $a_i + b_j, i = 1, \dots, 5, j = 1, 2, 3$, are squares.

Finally, the two triples of integers A and B defined by (2.2) may be extended to two quadruples of integers $A_2 = \{a_1, a_2, a_3, a_4\}$ and $B_2 = \{b_1, b_2, b_3, b_4\}$ such that all 16 sums $a_i + b_j, i = 1, \dots, 4, j = 1, \dots, 4$, are squares if and only if, in addition to the matrix E , there exist two 3×3 semi-magic integer matrices G and H with squared entries, namely,

$$G = \begin{bmatrix} g_{11}^2 & g_{12}^2 & g_{13}^2 \\ g_{21}^2 & g_{22}^2 & g_{23}^2 \\ g_{31}^2 & g_{32}^2 & g_{33}^2 \end{bmatrix}, \quad H = \begin{bmatrix} h_{11}^2 & h_{12}^2 & h_{13}^2 \\ h_{21}^2 & h_{22}^2 & h_{23}^2 \\ h_{31}^2 & h_{32}^2 & h_{33}^2 \end{bmatrix},$$

such that the entries of the three matrices E, G , and H satisfy the following conditions:

$$e_{12}^2 = g_{12}^2, \quad e_{13}^2 = g_{13}^2, \quad e_{21}^2 = g_{21}^2, \quad e_{31}^2 = g_{31}^2, \quad (2.5)$$

$$e_{11}^2 = h_{11}^2, \quad e_{22}^2 = h_{22}^2, \quad (2.6)$$

$$g_{11}^2 = h_{12}^2, \quad g_{22}^2 = h_{31}^2. \quad (2.7)$$

When such matrices E, G , and H exist, and $a_i, b_i, i = 1, 2, 3$, are defined, as before, by (2.2) and a_4, b_4 are defined by

$$a_4 = g_{22}^2, \quad b_4 = g_{11}^2 - g_{22}^2, \quad (2.8)$$

the two quadruples of integers $A_2 = \{a_1, a_2, a_3, a_4\}$ and $B_2 = \{b_1, b_2, b_3, b_4\}$ are such that all 16 sums $a_i + b_j, i = 1, \dots, 4, j = 1, \dots, 4$, are squares.

Proof. If there exist two triples of integers $\{a_i, a_2, a_3\}$ and $\{b_i, b_2, b_3\}$ such that all nine sums $a_i + b_j$ are perfect squares, then it is readily seen that the matrix

$$E_1 = \begin{bmatrix} a_3 + b_3 & a_2 + b_2 & a_1 + b_1 \\ a_1 + b_2 & a_3 + b_1 & a_2 + b_3 \\ a_2 + b_1 & a_1 + b_3 & a_3 + b_2 \end{bmatrix} \quad (2.9)$$

is a semi-magic matrix with common sum $a_1 + a_2 + a_3 + b_1 + b_2 + b_3$, and all the entries of the matrix E are perfect squares.

Conversely, if the matrix E defined by (2.1) is semi-magic, we will show that the integers a_i, b_i defined by (2.2) are such that all nine sums $a_i + b_j, i = 1, 2, 3, j = 1, 2, 3$, are squares.

Since $b_1 = 0$, the three sums $a_i + b_1, i = 1, \dots, 3$, are clearly squares. Further, since E is a semi-magic matrix, we have,

$$\begin{aligned} a_1 + b_2 &= e_{21}^2, & a_1 + b_3 &= e_{13}^2 + e_{11}^2 - e_{22}^2 = e_{32}^2, \\ a_2 + b_2 &= e_{31}^2 + e_{21}^2 - e_{13}^2 = e_{12}^2, & a_2 + b_3 &= e_{31}^2 + e_{11}^2 - e_{22}^2 = e_{23}^2, \\ a_3 + b_2 &= e_{22}^2 + e_{21}^2 - e_{13}^2 = e_{33}^2, & a_3 + b_3 &= e_{11}^2. \end{aligned}$$

Thus, all nine sums $a_i + b_j, i = 1, 2, 3, j = 1, 2, 3$, are squares. This proves the first part of the lemma concerning the two triples A and B .

Next, we note that if there exists a quintuple $A_1 = \{a_1, \dots, a_5\}$ and a triple $B_1 = \{b_1, b_2, b_3\}$ such that all 15 sums $a_i + b_j, i = 1, \dots, 5, j = 1, 2, 3$, are squares, then all the entries of the two matrices E_1 , defined by (2.9), and the matrix F_1 defined by

$$F_1 = \begin{bmatrix} a_3 + b_3 & a_5 + b_2 & a_4 + b_1 \\ a_4 + b_2 & a_3 + b_1 & a_5 + b_3 \\ a_5 + b_1 & a_4 + b_3 & a_3 + b_2 \end{bmatrix},$$

are squares. Further, both E_1 and F_1 are semi-magic matrices, their common sums being $a_1 + a_2 + a_3 + b_1 + b_2 + b_3$ and $a_3 + a_4 + a_5 + b_1 + b_2 + b_3$, respectively. Finally, it is readily seen that the entries of the matrices E_1 and F_1 satisfy the conditions (2.3).

Conversely, let $E = [e_{ij}^2]$ and $F = [f_{ij}^2]$ be two semi-magic matrices such that the conditions (2.3) are satisfied. We will now show that the integers $\{a_1, \dots, a_5\}$ and $\{b_1, b_2, b_3\}$ defined by (2.2) and (2.4) are such that all 15 sums $a_i + b_j, i = 1, \dots, 5, j = 1, 2, 3$, are squares.

The nine sums $a_i + b_j, i = 1, 2, 3, j = 1, 2, 3$, are squares as proved earlier. To prove that the remaining six sums are squares, we first note that since E is a semi-magic matrix, the sum of the second row and the third column of E is the same, hence $e_{21}^2 + e_{22}^2 = e_{13}^2 + e_{33}^2$, and thus $b_2 = e_{21}^2 - e_{13}^2 = e_{33}^2 - e_{22}^2$. In the following proof, it will be convenient to use the latter value of b_2 . We now note that

$$\begin{aligned} a_4 + b_1 &= f_{13}^2, \\ a_4 + b_2 &= f_{13}^2 + e_{33}^2 - e_{22}^2 = f_{13}^2 + f_{33}^2 - f_{22}^2 = f_{21}^2, \\ a_4 + b_3 &= f_{13}^2 + e_{11}^2 - e_{22}^2 = f_{13}^2 + f_{11}^2 - f_{22}^2 = f_{32}^2, \\ a_5 + b_1 &= f_{31}^2, \\ a_5 + b_2 &= f_{31}^2 + e_{33}^2 - e_{22}^2 = f_{31}^2 + f_{33}^2 - f_{22}^2 = f_{12}^2, \\ a_5 + b_3 &= f_{31}^2 + e_{11}^2 - e_{22}^2 = f_{31}^2 + f_{11}^2 - f_{22}^2 = f_{23}^2. \end{aligned}$$

Thus, all 15 sums $a_i + b_j, i = 1, \dots, 5, j = 1, 2, 3$, are squares. This proves the second part of the lemma concerning a quintuple and a triple.

Finally we note that if there exist two quadruples of integers $\{a_i : i = 1, \dots, 4\}$ and $\{b_j : j = 1, \dots, 4\}$ such that all 16 sums $a_i + b_j, i = 1, \dots, 4, j = 1, \dots, 4$, are perfect squares, then it is readily seen that the three matrices E_1 , defined by (2.9), and the matrices G_1 and H_1 defined by

$$G_1 = \begin{bmatrix} a_4 + b_4 & a_2 + b_2 & a_1 + b_1 \\ a_1 + b_2 & a_4 + b_1 & a_2 + b_4 \\ a_2 + b_1 & a_1 + b_4 & a_4 + b_2 \end{bmatrix}$$

and

$$H_1 = \begin{bmatrix} a_3 + b_3 & a_4 + b_4 & a_1 + b_1 \\ a_1 + b_4 & a_3 + b_1 & a_4 + b_3 \\ a_4 + b_1 & a_1 + b_3 & a_3 + b_4 \end{bmatrix}$$

are such that all the entries of these three matrices are perfect squares, all of them are semi-magic matrices (their common sums being $a_1 + a_2 + a_3 + b_1 + b_2 + b_3$, $a_1 + a_2 + a_4 + b_1 + b_2 + b_4$, and $a_1 + a_3 + a_4 + b_1 + b_3 + b_4$, respectively) and the relations (2.5), (2.6), and (2.7) are all satisfied.

Conversely, if there exist three semi-magic integer matrices $E = [e_{ij}^2]$, $G = [g_{ij}^2]$, and $H = [h_{ij}^2]$, with squared entries satisfying the conditions (2.5), (2.6), and (2.7), we will show that the integers a_i and b_j defined by (2.2) and (2.8) are such that all 16 sums $a_i + b_j, i = 1, \dots, 4, j = 1, \dots, 4$, are perfect squares.

We note that the nine sums $a_i + b_j, i = 1, 2, 3, j = 1, 2, 3$, are squares as proved earlier. Next, using the relations (2.2), (2.8) and the fact that G is a semi-magic matrix, we have,

$$\begin{aligned} a_1 + b_4 &= e_{13}^2 + g_{11}^2 - g_{22}^2 = g_{13}^2 + g_{11}^2 - g_{22}^2 = g_{32}^2, \\ a_2 + b_4 &= e_{31}^2 + g_{11}^2 - g_{22}^2 = g_{31}^2 + g_{11}^2 - g_{22}^2 = g_{23}^2, \\ a_4 + b_1 &= g_{22}^2, \\ a_4 + b_2 &= g_{22}^2 + e_{21}^2 - e_{13}^2 = g_{22}^2 + g_{21}^2 - g_{13}^2 = g_{33}^2, \\ a_4 + b_4 &= g_{11}^2. \end{aligned}$$

Finally, using the relations (2.6) and (2.7), and the fact that H is a semi-magic matrix, we have,

$$\begin{aligned} a_3 + b_4 &= e_{22}^2 + g_{11}^2 - g_{22}^2 = h_{22}^2 + h_{12}^2 - h_{31}^2 = h_{33}^2, \\ a_4 + b_3 &= g_{22}^2 + e_{11}^2 - e_{22}^2 = h_{31}^2 + h_{11}^2 - h_{22}^2 = h_{23}^2. \end{aligned}$$

Thus, all 16 sums $a_i + b_j, i = 1, \dots, 4, j = 1, \dots, 4$, are squares. This proves the final part of the lemma concerning the two quadruples and the proof is complete. \square

In the next three sections, we will obtain two sets of integers $\{a_1, a_2, \dots, a_m\}$ and $\{b_1, b_2, \dots, a_n\}$ in the three cases when (m, n) is either $(3, 3)$ or $(5, 3)$ or $(4, 4)$, respectively, such that, in each case, all mn sums $a_i + b_j, i = 1, \dots, m$, and $j = 1, \dots, n$, are squares by constructing semi-magic integer matrices satisfying the conditions of Lemma 2. For this purpose, we will repeatedly use the following semi-magic matrix of squares given by Euler (as quoted by Dickson [1, p. 530]):

$$M = \begin{bmatrix} (p^2 + q^2 - r^2 - s^2)^2 & (2qr + 2ps)^2 & (2qs - 2pr)^2 \\ (2qr - 2ps)^2 & (p^2 - q^2 + r^2 - s^2)^2 & (2pq + 2rs)^2 \\ (2qs + 2pr)^2 & (2rs - 2pq)^2 & (p^2 - q^2 - r^2 + s^2)^2 \end{bmatrix}. \quad (2.10)$$

3. Two Triples $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ Such That All Nine Sums $a_i + b_j$ Are Squares

The following theorem gives three pairs of triples $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, in terms of four parameters, such that all nine sums $a_i + b_j$ are perfect squares.

Theorem 1. *If $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are two triples of integers, defined in terms of arbitrary integer parameters p, q, r , and s , by*

$$\begin{aligned} a_1 &= 0, \\ a_2 &= 16pqrs, \\ a_3 &= (p + q + r + s)(p + q - r - s)(p - q + r - s)(p - q - r + s), \\ b_1 &= 4(pr - qs)^2, \\ b_2 &= 4(ps - qr)^2, \\ b_3 &= 4(pq - rs)^2, \end{aligned} \tag{3.1}$$

$$\begin{aligned} a_1 &= -(p + q + r + s)(p + q - r - s)(p - q + r - s)(p - q - r + s), \\ a_2 &= -(p - q + r + s)(p - q - r - s)(p + q + r - s)(p + q - r + s), \\ a_3 &= 0, \\ b_1 &= (p^2 - q^2 + r^2 - s^2)^2, \\ b_2 &= (p^2 - q^2 - r^2 + s^2)^2, \\ b_3 &= (p^2 + q^2 - r^2 - s^2)^2, \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} a_1 &= 4(ps - qr)^2, & a_2 &= 4(ps + qr)^2, \\ a_3 &= (p^2 - q^2 - r^2 + s^2)^2, & b_1 &= 4(p^2 - q^2)(r^2 - s^2), \\ b_2 &= 0, & b_3 &= 4(p^2 - r^2)(q^2 - s^2), \end{aligned} \tag{3.3}$$

then, for each of these three pairs of triples $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, all nine sums $a_i + b_j, i = 1, 2, 3, j = 1, 2, 3$, are perfect squares.

Proof. We first apply Lemma 2 taking E to be the matrix M defined by (2.10) and, from the relations (2.2) we obtain two triples $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. Next, in accordance with Lemma 1, we subtract t from each of the integers a_i and we add t to each of the integers b_i to get the following two triples, in terms of arbitrary integer parameters p, q, r, s , and t such that all nine sums $a_i + b_j, i = 1, 2, 3, j = 1, 2, 3$, are squares:

$$\begin{aligned} a_1 &= 4(pr - qs)^2 - t, \\ a_2 &= 4(pr + qs)^2 - t, \\ a_3 &= (p^2 - q^2 + r^2 - s^2)^2 - t, \\ b_1 &= t, \\ b_2 &= t - 4(p^2 - q^2)(r^2 - s^2), \\ b_3 &= t + 4(p^2 - s^2)(q^2 - r^2). \end{aligned}$$

Assigning to t the three values $4(pr - qs)^2$, $(p^2 - q^2 + r^2 - s^2)^2$, and $4(p^2 - q^2)(r^2 - s^2)$, in succession, we get the three pairs of triples defined by (3.1), (3.2), and (3.3), respectively, such that in each case, all nine sums $a_i + b_j, i = 1, 2, 3, j = 1, 2, 3$, are squares. This proves the lemma. \square

As a numerical example, taking $(p, q, r, s) = (5, 3, 2, 1)$ in the triples defined by (3.3), we get the two triples $A = \{4, 169, 484\}$ and $\{0, 192, 672\}$ such that all the nine elements of the sumset $A + B$ are perfect squares.

4. A Quintuple and a Triple

We will now apply Lemma 2 to obtain a quintuple a_1, \dots, a_5 and a triple b_1, b_2, b_3 such that all 15 sums $a_i + b_j$ are squares. The result thus obtained is stated in the following theorem.

Theorem 2. *If a_1, \dots, a_5 and b_1, b_2, b_3 are rational numbers, defined in terms of arbitrary rational parameters f, g, m, u_1, u_2, v_1 , and v_2 by*

$$\begin{aligned}
 a_1 &= 4(mu_1u_2(u_1v_1 - u_2v_2)f^2 - (u_1v_2 - u_2v_1) \\
 &\quad \times (m^2u_1u_2v_1v_2 - 1)fg - mv_1v_2(u_1v_1 - u_2v_2)g^2)^2, \\
 a_2 &= 4(mu_1u_2(u_1v_1 + u_2v_2)f^2 + (u_1v_2 + u_2v_1) \\
 &\quad \times (m^2u_1u_2v_1v_2 + 1)fg + mv_1v_2(u_1v_1 + u_2v_2)g^2)^2, \\
 a_3 &= ((u_1^2u_2^2(v_1^2 - v_2^2)m^2 + u_1^2 - u_2^2)f^2 \\
 &\quad - (v_1^2v_2^2(u_1^2 - u_2^2)m^2 + v_1^2 - v_2^2)g^2)^2, \\
 a_4 &= 4(mu_1u_2(u_1v_1 - u_2v_2)f^2 + (u_1v_2 - u_2v_1) \\
 &\quad \times (m^2u_1u_2v_1v_2 - 1)fg - mv_1v_2(u_1v_1 - u_2v_2)g^2)^2, \\
 a_5 &= 4(mu_1u_2(u_1v_1 + u_2v_2)f^2 - (u_1v_2 + u_2v_1) \\
 &\quad \times (m^2u_1u_2v_1v_2 + 1)fg + mv_1v_2(u_1v_1 + u_2v_2)g^2)^2, \\
 b_1 &= 0, \\
 b_2 &= -4(u_1^2 - u_2^2)(v_1^2 - v_2^2)(m^2u_1^2u_2^2f^2 - g^2)(f^2 - m^2v_1^2v_2^2g^2), \\
 b_3 &= 4(m^2u_1^2v_1^2 - 1)(m^2u_2^2v_2^2 - 1)(f^2u_1^2 - g^2v_1^2)(f^2u_2^2 - g^2v_2^2),
 \end{aligned}
 \tag{4.1}$$

then all 15 sums $a_i + b_j, i=1, \dots, 5, j = 1, 2, 3$, are squares of rational numbers.

Proof. We will obtain the rational numbers a_1, \dots, a_5 and b_1, b_2, b_3 by first finding two semi-magic matrices E and F of squares that satisfy the conditions (2.3) and then using the relations (2.2) and (2.4) given in Lemma 2. We begin with two matrices E and F obtained by replacing the parameters p, q, r, s in Euler’s matrix

M by p_1, q_1, r_1, s_1 and p_2, q_2, r_2, s_2 , respectively, that is, with the matrices

$$E = \begin{bmatrix} (p_1^2 + q_1^2 - r_1^2 - s_1^2)^2 & (2q_1r_1 + 2p_1s_1)^2 & (2q_1s_1 - 2p_1r_1)^2 \\ (2q_1r_1 - 2p_1s_1)^2 & (p_1^2 - q_1^2 + r_1^2 - s_1^2)^2 & (2p_1q_1 + 2r_1s_1)^2 \\ (2q_1s_1 + 2p_1r_1)^2 & (2r_1s_1 - 2p_1q_1)^2 & (p_1^2 - q_1^2 - r_1^2 + s_1^2)^2 \end{bmatrix} \quad (4.2)$$

and

$$F = \begin{bmatrix} (p_2^2 + q_2^2 - r_2^2 - s_2^2)^2 & (2q_2r_2 + 2p_2s_2)^2 & (2q_2s_2 - 2p_2r_2)^2 \\ (2q_2r_2 - 2p_2s_2)^2 & (p_2^2 - q_2^2 + r_2^2 - s_2^2)^2 & (2p_2q_2 + 2r_2s_2)^2 \\ (2q_2s_2 + 2p_2r_2)^2 & (2r_2s_2 - 2p_2q_2)^2 & (p_2^2 - q_2^2 - r_2^2 + s_2^2)^2 \end{bmatrix}. \quad (4.3)$$

The matrices E and F will satisfy the conditions (2.3) if the following conditions are satisfied:

$$p_1^2 + q_1^2 - r_1^2 - s_1^2 = p_2^2 + q_2^2 - r_2^2 - s_2^2, \quad (4.4)$$

$$p_1^2 - q_1^2 + r_1^2 - s_1^2 = p_2^2 - q_2^2 + r_2^2 - s_2^2, \quad (4.5)$$

$$p_1^2 - q_1^2 - r_1^2 + s_1^2 = p_2^2 - q_2^2 - r_2^2 + s_2^2. \quad (4.6)$$

Both Equations (4.5) and (4.6) will be satisfied if $p_1^2 - q_1^2 = p_2^2 - q_2^2$ and $r_1^2 - s_1^2 = r_2^2 - s_2^2$. Accordingly, a solution of the simultaneous Equations (4.5) and (4.6) may readily be written as follows:

$$\begin{aligned} p_1 &= eu_1 + fu_2, & p_2 &= eu_1 - fu_2, \\ q_1 &= eu_2 + fu_1, & q_2 &= eu_2 - fu_1, \\ r_1 &= gv_1 + hv_2, & r_2 &= gv_1 - hv_2, \\ s_1 &= gv_2 + hv_1, & s_2 &= gv_2 - hv_1, \end{aligned} \quad (4.7)$$

where $e, f, g, h, u_1, u_2, v_1$, and v_2 are arbitrary parameters.

Substituting the values of $p_i, q_i, r_i, s_i, i = 1, 2$, in Equation (4.4), and transposing all terms to the left-hand side, we get the condition,

$$(efu_1u_2 - ghv_1v_2)((e^2 + f^2)(u_1^2 + u_2^2) - (g^2 + h^2)(v_1^2 + v_2^2)) = 0. \quad (4.8)$$

For simplicity, we solve Equation (4.8) by equating to 0 the first factor on the left-hand side of Equation (4.8), and accordingly, we take

$$e = gmv_1v_2, \quad h = fmu_1u_2, \quad (4.9)$$

where m is an arbitrary rational parameter.

Thus, when the values of $p_i, q_i, r_i, s_i, i = 1, 2$, are given by (4.7) and the values of e and h are given by (4.9), then the semi-magic matrices E and F defined by (4.2) and (4.3), respectively, satisfy all the conditions (2.3). Now, on applying Lemma 2, we get the rational numbers a_1, \dots, a_5 and b_1, b_2, b_3 defined by (4.1) such that all 15 sums $a_i + b_j, i = 1, \dots, 5, j = 1, 2, 3$, are squares of rational numbers. This proves the theorem. \square

As a numerical example, on taking $(f, g, m, u_1, u_2, v_1, v_2) = (4, 1, 3, 2, 1, 3, 1)$ in the solution (4.1), we get the quintuple

$$A = \{198916, 532900, 1674436, 13468900, 19404025\}$$

and the triple

$$B = \{0, 3588000, 8527200\}$$

such that all 15 elements of the sumset $A + B$ are perfect squares.

We note that more examples of a quintuple and a triple may be obtained by equating to 0 the second factor on the left-hand side of Equation (4.8) and proceeding as before. As these solutions are more cumbersome to write, we omit them.

5. Two Quadruples of Integers $\{a_i : i = 1, \dots, 4\}$ and $\{b_j : j = 1, \dots, 4\}$ Such That All Sixteen Sums $a_i + b_j$ Are Squares.

We will now obtain three examples of two quadruples $A = \{a_i : i = 1, \dots, 4\}$ and $B = \{b_j : j = 1, \dots, 4\}$, in parametric terms, such that all 16 sums $a_i + b_j, i = 1, \dots, 4, j = 1, \dots, 4$, are perfect squares. We also show how more such examples of quadruples may be constructed.

We can, and of course we will, apply Lemma 2 to find such quadruples. We, however, first note that if, instead of constructing the three semi-magic matrices E, G , and H as stipulated in Lemma 2, we just construct two semi-magic matrices E and G such that the conditions (2.5) are satisfied, and we use the relations (2.2) and (2.8) to construct two quadruples $\{a_i : i = 1, \dots, 4\}$ and $\{b_j : j = 1, \dots, 4\}$, then except for the two sums $a_3 + b_4$ and $a_4 + b_3$, all the remaining 14 sums $a_i + b_j$ are perfect squares. This follows from a perusal of the proof of Lemma 2 where the matrix H and the conditions (2.6) and (2.7) are needed only to ensure that the two sums $a_3 + b_4$ and $a_4 + b_3$ are squares — the matrices E and G and the relations (2.5) being sufficient to prove that 14 of the sums $a_i + b_j$ are squares.

In Section 5.1 we obtain, in parametric terms, two semi-magic matrices E and G with squared entries such that the conditions (2.5) are satisfied. We then use the relations (2.2) and (2.8) to obtain two quadruples $\{a_i : i = 1, \dots, 4\}$ and $\{b_j : j = 1, \dots, 4\}$ such that 14 of the sums $a_i + b_j$ are squares. Finally, we choose the parameters such that the two remaining sums $a_3 + b_4$ and $a_4 + b_3$ also become squares.

In Section 5.2 we use the matrices E and G of Section 5.1 and construct the matrix H such that all the conditions (2.5), (2.6) and (2.7) are satisfied, and we apply Lemma 2 to obtain two pairs of quadruples A and B such that all elements of the sumset $A + B$ are squares.

5.1. A Pair of Quadruples

Theorem 3. *If a_1, \dots, a_4 and b_1, \dots, b_4 are integers, defined in terms of arbitrary integer parameters $u_i, v_i, i = 1, 2$, by*

$$\begin{aligned}
 a_1 &= (v_1^4 - 6v_1^2v_2^2 + v_2^4)^2(v_1^2 - v_2^2)^2(u_1^2 + u_2^2)^2, \\
 a_2 &= (v_1^4 - 6v_1^2v_2^2 + v_2^4)^2(v_1^2 + v_2^2)^2(u_1^2 - u_2^2)^2, \\
 a_3 &= (v_1^2 - v_2^2)^2((v_1^2 + v_2^2)^2u_1^2 - 16u_1u_2v_1^2v_2^2 - (v_1^2 + v_2^2)^2u_2^2)^2, \\
 a_4 &= (v_1^2 - v_2^2)^2((v_1^2 + v_2^2)^2u_1^2 + 16u_1u_2v_1^2v_2^2 - (v_1^2 + v_2^2)^2u_2^2)^2, \\
 b_1 &= 0, \\
 b_2 &= (v_1^4 - 6v_1^2v_2^2 + v_2^4)^2((v_2 - v_1)u_1 + (v_1 + v_2)u_2) \\
 &\quad \times ((v_1 + v_2)u_1 + (v_1 - v_2)u_2)((v_1 - v_2)u_1 + (v_1 + v_2)u_2) \\
 &\quad \times ((v_1 + v_2)u_1 - (v_1 - v_2)u_2), \\
 b_3 &= -(v_1^2 - v_2^2)^2((v_1^2 + v_2^2)u_1 + 4u_1v_1v_2 - (v_1^2 + v_2^2)u_2) \\
 &\quad \times ((v_1^2 + v_2^2)u_1 - 4u_1v_1v_2 - (v_1^2 + v_2^2)u_2) \\
 &\quad \times ((v_1^2 + v_2^2)u_1 + 4u_2v_1v_2 + (v_1^2 + v_2^2)u_2) \\
 &\quad \times ((v_1^2 + v_2^2)u_1 - 4u_2v_1v_2 + (v_1^2 + v_2^2)u_2), \\
 b_4 &= -(v_1^2 - v_2^2)^2((v_1^2 + v_2^2)u_1 + 4u_1v_1v_2 + (v_1^2 + v_2^2)u_2) \\
 &\quad \times ((v_1^2 + v_2^2)u_1 - 4u_1v_1v_2 + (v_1^2 + v_2^2)u_2) \\
 &\quad \times ((v_1^2 + v_2^2)u_1 + 4u_2v_1v_2 - (v_1^2 + v_2^2)u_2) \\
 &\quad \times ((v_1^2 + v_2^2)u_1 - 4u_2v_1v_2 - (v_1^2 + v_2^2)u_2),
 \end{aligned} \tag{5.1}$$

then all 16 sums $a_i + b_j, i = 1, \dots, 4, j = 1, \dots, 4$, are perfect squares.

Proof. As discussed above, we will obtain the two quadruples $\{a_i : i = 1, \dots, 4\}$ and $\{b_i : j = 1, \dots, 4\}$ with all 16 sums $a_i + b_j$ being perfect squares by just constructing, in parametric terms, two 3×3 semi-magic matrices E and G with squared entries such that the conditions (2.5) are satisfied. As before, we will use Euler’s matrix M to construct the desired semi-magic matrices E and G .

We take the matrix E to be the matrix M itself, that is, $E = M$. To construct the matrix G , we replace p and q in the matrix M by mp and mq , respectively, where m is some nonzero rational number, and then divide each entry of the matrix thus obtained by m^2 , whereby we get the semi-magic matrix

$$G = \begin{bmatrix} g_{11}^2 & (2qr + 2ps)^2 & (2qs - 2pr)^2 \\ (2qr - 2ps)^2 & g_{22}^2 & 4(m^2pq + rs)^2/m^2 \\ (2qs + 2pr)^2 & 4(m^2pq - rs)^2/m^2 & g_{33}^2 \end{bmatrix} \tag{5.2}$$

where

$$\begin{aligned}
 g_{11} &= (m^2p^2 + m^2q^2 - r^2 - s^2)/m, \\
 g_{22} &= (m^2p^2 - m^2q^2 + r^2 - s^2)/m, \\
 g_{33} &= (m^2p^2 - m^2q^2 - r^2 + s^2)/m.
 \end{aligned} \tag{5.3}$$

The entries of the matrices E and G satisfy the relations (2.5), and using the relations (2.2) and (2.8), we get two quadruples $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ such that except for the two sums $a_3 + b_4$ and $a_4 + b_3$, all other 14 sums $a_i + b_j$ are perfect squares. The values of $a_i, b_i, i = 1, \dots, 4$, are given in terms of arbitrary parameters m, p, q, r , and s by

$$\begin{aligned} a_1 &= 4(pr - qs)^2, & a_2 &= 4(pr + qs)^2, \\ a_3 &= (p^2 - q^2 + r^2 - s^2)^2, & a_4 &= (m^2p^2 - m^2q^2 + r^2 - s^2)^2/m^2, \\ b_1 &= 0, & b_2 &= -4(p^2 - q^2)(r^2 - s^2), \\ b_3 &= 4(p^2 - s^2)(q^2 - r^2), & b_4 &= 4(m^2p^2 - s^2)(m^2q^2 - r^2)/m^2. \end{aligned} \tag{5.4}$$

To make the two sums $a_3 + b_4$ and $a_4 + b_3$ also perfect squares, we impose the auxiliary condition $a_3 + b_4 = a_4 + b_3$ which, on using the relations (5.4), reduces to

$$(m^2 - 1)(mp^2 + mq^2 + r^2 + s^2)(mp^2 + mq^2 - r^2 - s^2) = 0. \tag{5.5}$$

A solution of (5.5), obtained by equating to 0 the last factor on the left-hand side of Equation (5.5), is given by

$$\begin{aligned} p &= u_1v_1 + u_2v_2, & q &= u_1v_2 - u_2v_1, \\ r &= t(u_1v_1 - u_2v_2), & s &= t(u_1v_2 + u_2v_1), \\ m &= t^2. \end{aligned} \tag{5.6}$$

where $t, u_i, v_i, i = 1, 2$, are arbitrary parameters.

The value of the sum $a_3 + b_4$, denoted by $\phi(t, u_1, u_2, v_1, v_2)$, may now be worked out using the relations (5.4) and (5.6), and is given by

$$\begin{aligned} \phi(t, u_1, u_2, v_1, v_2) &= (t^2 - 1)((t^2 - 1)(v_1^2 + v_2^2)^2u_1^4 - 16(t^2 + 1)v_1v_2(v_1^2 - v_2^2)u_1^3u_2 \\ &\quad + 2(t^2 - 1)(v_1^2 + v_2^2)^2u_1^2u_2^2 + 16(t^2 + 1)v_1v_2(v_1^2 - v_2^2)u_1u_2^3 \\ &\quad + (t^2 - 1)(v_1^2 + v_2^2)^2u_2^4). \end{aligned}$$

Similarly, the value of the sum $a_4 + b_3$ may be worked out and it is given by $\phi(t, u_1, u_2, -v_1, v_2)$.

To make $\phi(t, u_1, u_2, v_1, v_2)$ a perfect square, we computed its discriminant d with respect to u_1 and equated it to 0. The discriminant of $\phi(t, u_1, u_2, -v_1, v_2)$ with respect to u_1 is also d , and the value of d is given by

$$\begin{aligned} d &= 262144(t^2 + 1)^2(t^2 - 1)^6u_2^{12}v_1^2v_2^2(v_1^2 - v_2^2)^2(t(v_1^2 - 2v_1v_2 - v_2^2) - v_1^2 - 2v_1v_2 + v_2^2)^2 \\ &\quad \times (t(v_1^2 - 2v_1v_2 - v_2^2) + v_1^2 + 2v_1v_2 - v_2^2)^2(t(v_1^2 + 2v_1v_2 - v_2^2) \\ &\quad - v_1^2 + 2v_1v_2 + v_2^2)^2(t(v_1^2 + 2v_1v_2 - v_2^2) + v_1^2 - 2v_1v_2 - v_2^2)^2. \end{aligned}$$

Equating the factor $(t(v_1^2 - 2v_1v_2 - v_2^2) - v_1^2 - 2v_1v_2 + v_2^2)^2$ to 0, we get

$$t = (v_1^2 + 2v_1v_2 - v_2^2)/(v_1^2 - 2v_1v_2 - v_2^2), \tag{5.7}$$

and with this value of t , it is readily verified that both $\phi(t, u_1, u_2, v_1, v_2)$ and $\phi(t, u_1, u_2, -v_1, v_2)$ become squares, that is, both the remaining sums $a_3 + b_4$ and $a_4 + b_3$ become perfect squares.

Thus, on substituting the values of m, p, q, r, s , and t given by (5.6) and (5.7), respectively, in the relations (5.4), we get two quadruples $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ such that all 16 sums $a_i + b_j, i = 1, \dots, 4, j = 1, \dots, 4$, are squares. With appropriate scaling, we now get the two quadruples defined by (5.1) as stated in the theorem. \square

As a numerical example, on taking $(u_1, u_2, v_1, v_2) = (3, 1, 4, 3)$, Theorem 3 yields, with appropriate scaling, the two quadruples,

$$A = \{44782864, 340218025, 1738222864, 2777290000\},$$

$$B = \{0, 25777136, 1222007600, 1719217136\},$$

such that all 16 elements of the sumset $A + B$ are perfect squares.

5.2. More Examples of Pairs of Quadruples

We will now apply Lemma 2 to prove the following theorem, which gives two new pairs of quadruples A and B such that all 16 elements of the sumset $A + B$ are squares.

Theorem 4. *If a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are rational numbers, defined in terms of arbitrary rational parameters m, q , and r by*

$$\begin{aligned} a_1 &= 16m^2(m^4 - 1)^2((5m^4 - 2m^2 + 1)q^2 - 2(m^4 - 6m^2 + 1)qr \\ &\quad + (m^4 - 2m^2 + 5)r^2)^2, \\ a_2 &= 16m^2(m^4 - 1)^2((5m^4 - 2m^2 + 1)q^2 - (m^4 - 2m^2 + 5)r^2)^2, \\ a_3 &= 16m^2(m^4 - 1)^2((7m^4 - 2m^2 - 1)q^2 - (2m^4 - 12m^2 + 2)qr \\ &\quad - (m^4 + 2m^2 - 7)r^2)^2, \\ a_4 &= (m^2 - 1)^2((3m^8 + 40m^6 - 26m^4 - 1)q^2 + 2(m^4 - 6m^2 + 1)^2qr \\ &\quad - (m^8 + 26m^4 - 40m^2 - 3)r^2)^2, \\ b_1 &= 0, \\ b_2 &= -4m^2((m^4 + 6m^2 - 3)q + (m^4 - 2m^2 + 5)r) \\ &\quad \times ((3m^4 - 6m^2 - 1)q - (m^4 - 2m^2 + 5)r) \\ &\quad \times ((5m^4 - 2m^2 + 1)q - (3m^4 - 6m^2 - 1)r) \\ &\quad \times ((5m^4 - 2m^2 + 1)q + (m^4 + 6m^2 - 3)r), \\ b_3 &= -128m^2(m^4 - 1)^3(q^2 - r^2)(3m^2q - m^2r - q + 3r)(m^2q + r), \\ b_4 &= 16(m^2 + 1)^2(m^2 - 1)^3(m^2q^2 - r^2) \\ &\quad \times ((m^4 + 6m^3 - 2m - 1)q - (m^4 + 2m^3 - 6m - 1)r) \\ &\quad \times ((m^4 - 6m^3 + 2m - 1)q - (m^4 - 2m^3 + 6m - 1)r), \end{aligned} \tag{5.8}$$

or

$$\begin{aligned}
 a_1 &= 16m^2(m^4 - 1)^2((m^6 - 2m^4 + 5m^2)q^2 + (2m^5 - 12m^3 + 2m)qr \\
 &\quad + (5m^4 - 2m^2 + 1)r^2)^2, \\
 a_2 &= 16m^2(m^4 - 1)^2((m^6 - 2m^4 + 5m^2)q^2 - (5m^4 - 2m^2 + 1)r^2)^2, \\
 a_3 &= (m^2 - 1)^2((m^{10} + 26m^6 - 40m^4 - 3m^2)q^2 \\
 &\quad + (2m^9 - 24m^7 + 76m^5 - 24m^3 + 2m)qr \\
 &\quad - (3m^8 + 40m^6 - 26m^4 - 1)r^2)^2, \\
 a_4 &= 16m^2(m^4 - 1)^2((m^6 + 2m^4 - 7m^2)q^2 \\
 &\quad - (2m^5 - 12m^3 + 2m)qr - (7m^4 - 2m^2 - 1)r^2)^2, \\
 b_1 &= 0, \\
 b_2 &= -4m^2((m^5 + 6m^3 - 3m)q - (5m^4 - 2m^2 + 1)r) \\
 &\quad \times ((3m^5 - 6m^3 - m)q + (5m^4 - 2m^2 + 1)r) \\
 &\quad \times ((m^5 - 2m^3 + 5m)q + (3m^4 - 6m^2 - 1)r) \\
 &\quad \times ((m^5 - 2m^3 + 5m)q - (m^4 + 6m^2 - 3)r), \\
 b_3 &= -16m^2(m^2 + 1)^2(m^2 - 1)^3(q^2 - r^2) \\
 &\quad \times ((m^5 - 2m^4 + 6m^2 - m)q + (m^4 - 6m^3 + 2m - 1)r) \\
 &\quad \times ((m^5 + 2m^4 - 6m^2 - m)q + (m^4 + 6m^3 - 2m - 1)r), \\
 b_4 &= 128m^3(m^4 - 1)^3(mr - q)(m^2q^2 - r^2)((m^3 - 3m)q + (3m^2 - 1)r),
 \end{aligned} \tag{5.9}$$

then all 16 sums $a_i + b_j, i = 1, \dots, 4, j = 1, \dots, 4$, are squares.

Proof. In Section 5.1 we have already obtained matrices E and G that satisfy the relations (2.5). We will now construct a matrix H such that the relations (2.6) and (2.7) are satisfied, and we will then apply Lemma 2 to obtain the quadruples $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ such that all 16 sums $a_i + b_j, i = 1, \dots, 4, j = 1, \dots, 4$, are squares.

We will construct the matrix H by replacing p, q, r, s , in Euler’s matrix M by

$$\begin{aligned}
 &((n + 1)p - (n - 1)s)/2, \quad ((n + 1)q - (n - 1)r)/2, \\
 &((-n + 1)q + (n + 1)r)/2, \quad ((-n + 1)p + (n + 1)s)/2,
 \end{aligned}$$

respectively, where n is an arbitrary parameter, and dividing the entries of the resulting matrix by n^2 , whereby we get the following semi-magic matrix:

$$H = \begin{bmatrix} (p^2 + q^2 - r^2 - s^2)^2 & h_{12}^2 & (2qs - 2pr)^2 \\ h_{21}^2 & (p^2 - q^2 + r^2 - s^2)^2 & h_{23}^2 \\ h_{31}^2 & (2rs - 2pq)^2 & h_{33}^2 \end{bmatrix} \tag{5.10}$$

where

$$\begin{aligned}
 h_{12} &= ((p^2 - 2ps + q^2 - 2qr + r^2 + s^2)n^2 \\
 &\quad - p^2 - 2ps - q^2 - 2qr - r^2 - s^2)/(2n), \\
 h_{21} &= ((p - q + r - s)(p + q - r - s)n^2 \\
 &\quad - (p + q + r + s)(p - q - r + s))/(2n), \\
 h_{23} &= ((q - r)(p - s)n^2 + (q + r)(p + s))/n, \\
 h_{31} &= ((q - r)(p - s)n^2 - (q + r)(p + s))/n, \\
 h_{33} &= ((p - q + r - s)(p + q - r - s)n^2 \\
 &\quad + (p + q + r + s)(p - q - r + s))/(2n).
 \end{aligned}
 \tag{5.11}$$

It is readily seen that the entries of the matrices E and H satisfy the relations (2.6).

We now impose the conditions (2.7), that is, $g_{11}^2 = h_{12}^2$ and $g_{22}^2 = h_{31}^2$ where the values of g_{11}, g_{22} and h_{12}, h_{31} are defined by (5.3) and (5.11), respectively. These conditions will be satisfied if $g_{11} = \pm h_{12}$ and $g_{22} = \pm h_{31}$, that is, we get four possible pairs of equations, and each of these pairs of equations can be solved to get a matrix H that satisfies all the conditions of Lemma 2, and we can thus obtain the desired pairs of quadruples.

The simplest pairs of quadruples are obtained by solving the pair of equations $g_{11} = -h_{12}$ and $g_{22} = -h_{31}$, which may be written as

$$\begin{aligned}
 (m^2p^2 + m^2q^2 - r^2 - s^2)/m &= -\{(p^2 - 2ps + q^2 - 2qr + r^2 + s^2)n^2 \\
 &\quad - p^2 - 2ps - q^2 - 2qr - r^2 - s^2\}/(2n),
 \end{aligned}
 \tag{5.12}$$

and

$$(m^2p^2 - m^2q^2 + r^2 - s^2)/m = -\{(q - r)(p - s)n^2 - (q + r)(p + s)\}/n,
 \tag{5.13}$$

respectively.

Taking the difference of Equations (5.12) and (5.13), we get

$$\begin{aligned}
 2(mq - r)(mq + r)/m &= -((n(p - q + r - s) + p - q - r + s) \\
 &\quad \times (n(p - q + r - s) - p + q + r - s))/(2n).
 \end{aligned}
 \tag{5.14}$$

Thus, there exists a rational number t such that

$$2n(mq - r) = -t(n(p - q + r - s) + p - q - r + s)m,
 \tag{5.15}$$

and

$$2t(mq + r) = n(p - q + r - s) - p + q + r - s.
 \tag{5.16}$$

Equations (5.15) and (5.16) may be considered as two linear equations in the variables p and s , and on solving them, we get,

$$\begin{aligned}
 p &= -(m(n - 1)(mq + r)t^2 - 2qmnt + n(n + 1)(mq - r))/(2mnt), \\
 s &= -(m(n + 1)(mq + r)t^2 - 2rmnt + n(n - 1)(mq - r))/(2mnt).
 \end{aligned}
 \tag{5.17}$$

On substituting the values of p and s given by (5.17) in Equation (5.12), transposing all terms to the left-hand side, and factorizing, we get

$$\begin{aligned} & \{(m-1)n^2 + (m^2t^2 + mt^2 - 4mt + m + 1)n - mt^2(m-1)\} \{(m+1)(mq-r)^2n^2 \\ & \quad + (m-1)(m^3q^2t^2 + 2m^2qrt^2 + mr^2t^2 + m^2q^2 - 2mqr + r^2)n \\ & \quad \quad - mt^2(m+1)(mq+r)^2\} = 0. \end{aligned} \tag{5.18}$$

To solve Equation (5.18), we simply equate the first factor to 0, that is, we will solve the equation

$$(m-1)n^2 + (m^2t^2 + mt^2 - 4mt + m + 1)n - mt^2(m-1) = 0. \tag{5.19}$$

Now, Equation (5.19) will have a rational solution for n if its discriminant d with respect to n is a perfect square, where

$$d = (m+1)^2m^2t^4 - 8m^2(m+1)t^3 + 6(m+1)^2mt^2 - 8m(m+1)t + (m+1)^2.$$

Since d is a quartic function of t , and the coefficient of t^4 as well as the constant term is 0, we can, following a method of Fermat (described by Dickson [1, p. 639]), readily find a value of t that makes d a perfect square, and we thus get $t = 2(m+1)/(m^2+1)$. With this value of t , the discriminant d becomes a perfect square, and Equation (5.19) can be solved to get two rational roots for n . We thus get the following two solutions of Equation (5.19):

$$(t, n) = (2(m+1)/(m^2+1), 4m(m^2-1)/(m^2+1)^2), \tag{5.20}$$

and

$$(t, n) = (2(m+1)/(m^2+1), -(m+1)/(m-1)). \tag{5.21}$$

Substituting the values of t and n given by (5.20) in the relations (5.17), we get

$$\begin{aligned} p &= -((m^4 - 6m^2 + 1)q - (m^4 - 2m^2 + 5)r)/(2(m^4 - 1)), \\ s &= -((5m^4 - 2m^2 + 1)q - (m^4 - 6m^2 + 1)r)/(2(m^4 - 1)). \end{aligned} \tag{5.22}$$

With the values of p and s given by (5.22), Equations (5.12) and (5.13) are satisfied, and hence we get three matrices $E = M$ (defined by (2.10)), along with G and H (defined by (5.2) and (5.10), respectively), satisfying all the conditions of Lemma 2. Now, on using the relations (2.2) and (2.8), we get two quadruples $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ such that all 16 sums $a_i + b_j, i = 1, \dots, 4, j = 1, \dots, 4$, are squares. With appropriate scaling, we get the pair of quadruples (5.8) given by the theorem.

Similarly, we can use the solution (5.21) of Equation (5.19) to obtain a new set of three matrices F, G , and H satisfying the conditions of Lemma 2 and we thus obtain the second pair of quadruples (5.9) given by the theorem. \square

As a numerical application of Theorem 4, on taking $(m, q, r) = (2, 1, 3)$ in the relations (5.8), we get two quadruples which, with appropriate scaling, yield the two quadruples,

$$\begin{aligned} A &= \{14400, 266256, 435600, 12110400\}, \\ B &= \{0, 104625, 223744, 12096000\}, \end{aligned}$$

such that all the 16 elements of the sumset $A + B$ are perfect squares.

While Theorem 4 gives two pairs of quadruples, in parametric terms, with the desired property, many more such examples may be obtained using Lemma 2. For instance, we could solve Equation (5.18) by equating to 0 the second factor on the left-hand side of (5.18) to get the matrix H . Further, as already mentioned, the conditions (2.7) may be satisfied by solving any of four possible pairs of equations arising from the conditions $g_{11} = \pm h_{12}$ and $g_{22} = \pm h_{31}$. In Theorem 4 we solved just one of these pairs of equations. Solving the remaining three pairs of equations also yields the desired matrix H and the pairs of quadruples. These solutions are, however, a little more cumbersome than the examples given by the theorem.

6. Some Open Problems

It would be interesting to find a quintuple $\{a_1, a_2, \dots, a_5\}$ and a quadruple $\{b_1, b_2, \dots, b_4\}$ such that all 20 sums $a_i + b_j$, $i = 1, \dots, 5$, $j = 1, \dots, 4$, are squares. Since we have obtained multi-parameter solutions for a quintuple and a triple as well as for a pair of quadruples with the desired property, it seems quite likely that there exist infinitely many examples of a quintuple $\{a_1, \dots, a_5\}$ and a quadruple $\{b_1, \dots, b_4\}$ such that all 20 sums $a_i + b_j$ are squares. It would be much more challenging to find two quintuples $\{a_1, \dots, a_5\}$ and $\{b_1, \dots, b_5\}$ such that all 25 sums $a_i + b_j$ are squares. We leave these problems for future investigations.

References

- [1] L. E. Dickson, *History of the Theory of Numbers, Vol. II*, Chelsea, New York, 1992.
- [2] R. K. Guy, *Unsolved Problems in Number Theory, Third Edition*, Springer, New York, 2004.