#A67 INTEGERS 26 (2026)

A NOTE ON TERAI'S CONJECTURE RELATED TO EXPONENTIAL DIOPHANTINE EQUATIONS

Kalyan Chakraborty
Department of Mathematics, SRM University AP, Neerukonda, Amaravati, Andhra Pradesh, India
kalyan.c@srmap.edu.in

Meenu Sharma
Department of Mathematics, The University of Hong Kong, Pokfulam, Hong Kong
sharmameenuphd@gmail.com

Received: 9/9/25, Revised: 12/22/25, Accepted: 5/6/26, Published: 6/8/26

Abstract

We study non-negative integer solutions (x, y, z) of the Diophantine equation $(10m^2+1)^x + (15m^2-1)^y = (5m)^z$, where m is a positive integer. This equation is related to Terai's conjecture, which concerns the uniqueness of integer solutions to certain exponential Diophantine equations. We provide a partial affirmation of Terai's conjecture by proving that the equation has the unique positive integer solution $(x, y, z) = (1, 1, 2)$ for all $m > 1$. The proof relies on the well-known primitive divisor theorem and several results from the theory of exponential Diophantine equations.

1. Introduction

Let a, b, c be fixed, pairwise relatively prime positive integers, and each greater than one. The exponential Diophantine equation

$$a^x + b^y = c^z \tag{1}$$

in positive integers x, y, z has been studied by many researchers (see [2], [10], [21]). It is known that Equation (1) has only finitely many positive integer solutions [6]. If (a, b, c) is a Pythagorean triple (i.e., positive integers satisfying $a^2 + b^2 = c^2$), then the following conjecture is well known.

Conjecture 1 (Jeśmanowicz's conjecture [8])**.** If (a, b, c) is Pythagorean triple, then (1) has the unique positive integer solution $(x, y, z) = (2, 2, 2)$.

DOI: 10.5281/zenodo.20597499

There exist many positive results for Jeśmanowicz's conjecture, and we refer to [12, 18, 20] for recent developments on this conjecture. Notably, Terai's conjecture [14] extends the ideas of Jeḿanowicz's conjecture to a more general setting.

Conjecture 2 (Terai's conjecture [14])**.** Let a, b, c, p, q, r be fixed natural numbers, where a, b, c are pairwise relatively prime integers and $a^p + b^q = c^r$ with $p, q, r \geq 2$. Then (1) has the unique solution $(x, y, z) = (p, q, r)$ with the exception of finitely many triples (a, b, c).

The exceptional cases in the above conjecture are listed explicitly in [17]. Terai's conjecture has been demonstrated to be valid in various special scenarios ([1], [3], [5], [9], [13], [16]), but in general, it remains unsolved. In 2024, Miyazaki and Pink proved the following result.

Theorem 1 ([19])**.** *Assume that at least one of a and b is congruent to 1 or -1 modulo c. Then there is at most one solution to Equation (1), except when (a, b, c) or (b, a, c) is one of $(3, 5, 2), (3, 13, 2), (2, 5, 3)$, and $(2, 7, 3)$.*

The main result (Theorem 2) of this paper provides a partial affirmation of the two conjectures stated above. Although Theorem 2 is a consequence of Theorem 1, our proof takes a different path. Like the argument in [19], our proof also ultimately depends on Baker's theory of linear forms in logarithms via the primitive divisor result recorded below as Theorem 7. Nevertheless, the structure of our proof is essentially distinct. This difference in method represents the main novelty of this work. It proposes a simpler approach, which might be useful in addressing other classes of Diophantine equations. We consider the exponential Diophantine equation

$$(10m^2 + 1)^x + (15m^2 - 1)^y = (5m)^z, \tag{2}$$

where m is a positive integer and we prove the following theorem.

Theorem 2. *Let m be a positive integer. Then (2) has the unique positive integer solution $(x, y, z) = (1, 1, 2)$ for all $m > 1$.*

Remark 1. It can be easily verified that the integers $10m^2 + 1, 15m^2 - 1$, and $5m$ are pairwise relatively prime.

The proof of Theorem 2 is presented in Section 3. We proceed by considering the parity of m. When m is even, we utilize congruence theory, and when m is odd, we reformulate Equation (2) and apply results from Section 2, which include the class number of imaginary quadratic fields, the characteristic number of solutions, and primitive divisors of Lucas numbers.

The organization of this paper is as follows. In Section 2, we provide a review of several key results that will be instrumental in our study. Section 3 presents the proof of Theorem 2. Section 4 provides a summary of the results.

2. Preliminaries

We begin by recalling some definitions and known results that will be needed later. *The Fibonacci numbers* F_n are defined by the recurrence

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \ \text{for} \ n > 1.$$

Definition 1. Let α and β be algebraic integers. A pair (α, β) is called a *Lucas pair* if the sum $\alpha + \beta$ and the product $\alpha\beta$ are nonzero, relatively prime integers, and $\frac{\alpha}{\beta}$ is not a root of unity.

Definition 2. For any Lucas pair (α, β), the *Lucas numbers* $L_n(\alpha, \beta)$ are defined recursively by

$$L_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n = 0, 1, 2, \ldots$$

Definition 3. The *primitive divisors* of the Lucas number $L_n(\alpha, \beta)$ are the prime numbers p such that

$$p \mid L_n(\alpha, \beta) \quad \text{and} \quad p \nmid (\alpha - \beta)^2 L_1(\alpha, \beta) \ldots L_{n-1}(\alpha, \beta).$$

Theorem 3 ([7], Theorems 11.4.3, 12.10.1, and 12.14.3)**.** *Let D be any positive integer and $h(-4D)$ be the class number of positive binary quadratic forms of discriminant $-4D$. Then*

$$h(-4D) < \frac{4}{\pi}\sqrt{D}\log(2e\sqrt{D}).$$

Theorem 4 ([11], Theorems 1 and 2)**.** *Let D and k be relatively prime positive integers such that $D > 1$ and k is odd. If the equation*

$$u^2 + Dv^2 = k^w$$

has a solution (u, v, w), where $u, v, w \in \mathbb{Z}, \gcd(u, v) = 1, w > 0$, then any solution of the above equation can be expressed as

$$u + v\sqrt{-D} = \lambda_1(u_1 + \lambda_2 v_1\sqrt{-D})^t,$$

$$w = w_1 t, \quad t \in \mathbb{N},$$

where $\lambda_1, \lambda_2 \in \{\pm 1\}$ and u_1, v_1, w_1 are positive integers satisfying ${u_1}^2 + D{v_1}^2 = k^{w_1}$, $\gcd(u_1, v_1) = 1$, and $h(-4D) \equiv 0 \pmod{w_1}$.

Lemma 1 ([11], Lemma 1)**.** *Let D_1, D_2 be relatively prime positive integers greater than 1, and let k be an odd integer greater than 1 with $\min\{D_1, D_2, k\} > 1$. Let (X, Y, z) be a fixed solution of the equation*

$$D_1X^2 + D_2Y^2 = k^z, \quad \text{where } \gcd(X, Y) = 1, \quad z \geq 1, \quad X, Y, z \in \mathbb{Z}. \tag{3}$$

Then there exists a unique positive integer L such that

$$L = D_1\alpha X + D_2\beta Y, \quad 1 \le L < k,$$

where α, β are integers with $\beta X - \alpha Y = 1$.

Definition 4. The positive integer L (defined in Lemma 1) is called the *characteristic number* of the solution (X, Y, z) and is denoted by $\langle X, Y, z\rangle$.

Lemma 2 ([11], Lemma 6)**.** *If $\langle X, Y, z\rangle = L$, then $D_1X \equiv -LY \pmod k$.*

Definition 5. If (X_0, Y_0, z_0) is a solution of (3) with $\langle X_0, Y_0, z_0\rangle = L_0$, then the set of all solutions (X, Y, z) with $\langle X, Y, z\rangle \equiv \pm L_0 \pmod k$ is called a *solution class* of (3) and is denoted by $S(L_0)$.

Theorem 5 ([11], Theorems 1 and 2)**.** *For any fixed solution class $S(L_0)$ of (3), there exists a unique solution $(X_0, Y_0, z_0) \in S(L_0)$ such that $X_0 \ge 1$, $Y_0 \ge 1$ and $z_0 \le z$, where z runs through all the solutions $(X, Y, z) \in S(L_0)$. The solution (X_0, Y_0, z_0) is called the least solution of $S(L_0)$. Every solution $(X, Y, z) \in S(L_0)$ can be expressed as*

$$X\sqrt{D_1} + Y\sqrt{-D_2} = \lambda_1(X_0\sqrt{D_1} + \lambda_2 Y_0\sqrt{-D_2})^t, \qquad \lambda_1, \lambda_2 \in \{\pm 1\},$$

$$z = z_0 t, \qquad 2 \nmid t, \quad t \in \mathbb{N}.$$

Theorem 6 ([23], Theorem 2)**.** *Let (X_0, Y_0, z_0) be the least solution of $S(L_0)$. If Equation (3) has a solution $(X, Y, z) \in S(L_0)$ satisfying $X \ge 1$ and $Y = 1$, then $Y_0 = 1$. Furthermore, if $(X, z) \ne (X_0, z_0)$, then one of the following conditions is satisfied:*

1. $D_1{X_0}^2 = \frac{1}{4}(k^{z_0} \pm 1), D_2 = \frac{1}{4}(3k^{z_0} \pm 1), (X, z) = (X_0|D_1{X_0}^2 - 3D_2|, 3z_0)$,
2. $D_1{X_0}^2 = \frac{1}{4}F_{3r+3e}, D_2 = \frac{1}{4}L_{3r}$, *and* $k^{z_0} = F_{3r+e}, (X, z) = (X_0|{D_1}^2{X_0}^4 - 10D_1D_2{X_0}^2 + 5{D_2}^2|, 5z_0)$, *where r is a positive integer, $e \in \{1, -1\}$, and F_n is the nth Fibonacci number.*

The primitive divisor results presented in [22] serve as powerful tools for solving exponential Diophantine equations.

Theorem 7 ([22])**.** *If $n > 30$, then $L_n(\alpha, \beta)$ has primitive divisors.*

Theorem 8 ([15])**.** *If $4 < n \le 30$ and $n \ne 6$, then up to equivalence, $L_n(\alpha, \beta)$ has a primitive divisor except for the following parameters (e, f), where $(\alpha, \beta) = (\frac{e+\sqrt{f}}{2}, \frac{e-\sqrt{f}}{2})$:*

- $(1, 5), (1, -7), (2, -40), (1, -11), (1, -15), (12, -76)$ *or* $(12, -1364)$ *if* $n = 5$,

- $(1, -7)$ *or* $(1, -19)$ *if* $n = 7$,
- $(1, -7)$ *or* $(2, -24)$ *if* $n = 8$,
- $(2, -8), (5, -3)$ *or* $(5, -47)$ *if* $n = 10$,
- $(1, -5), (1, -7), (1, -11), (2, -56), (1, -15)$ *or* $(1, -19)$ *if* $n = 12$,
- $(1, -7)$ *if* $n = 13, 18$ *or* 30.

Definition 6. Two Lucas pairs (α_1, β_1) and (α_2, β_2) are *equivalent* if $\alpha_1 = \pm\alpha_2$ and $\beta_1 = \pm\beta_2$, with the same sign in both components.

3. Proof of Theorem 2

The proof is divided into two cases based on the parity of m.

Proof of Theorem 2. We now prove Theorem 2. The proof is divided into two cases based on the parity of m.
`Case 1:` m is even. If $z \leq 2$, then $(x, y, z) = (1, 1, 2)$ is clearly the only one solution of (2). For a contradiction, assume $z > 2$. Then taking Equation (2) modulo m^2, we obtain $1 + (-1)^y \equiv 0 \pmod{m^2}$. Clearly y is odd since $m^2 > 2$. Reducing Equation (2) modulo $5m^3$, yields

$$\begin{aligned} 1 + 10m^2x + (-1) + 15m^2y &\equiv 0 \pmod{5m^3}, \\ 2x + 3y &\equiv 0 \pmod{m}. \end{aligned}$$

Now y is odd and so $2x + 3y$ is an odd number, but m is even. We conclude that Equation (2) has no positive integer solutions when $z > 2$. Thus, when m is even, (2) has a unique positive integer solution $(1, 1, 2)$.
`Case 2:` m is odd. This is more involved, and the rest of the manuscript deals with this case. Let (x, y, z) be a solution of (2). Clearly $(x, y, z) = (1, 1, 2)$ is a solution of (2). Taking Equation (2) modulo m^2 (since $m > 1$), we conclude that y is odd by the same argument as in Case 1. We now consider two subcases based on the parity of x.
`Subcase (a):` x is odd. We rewrite (2) as

$$(10m^2 + 1)X^2 + (15m^2 - 1)Y^2 = (5m)^Z, \qquad \text{where } X, Y, Z \in \mathbb{Z}, \ \ Z > 0. \tag{4}$$

Since (x, y, z) is a solution of (2), we see that

$$(X, Y, Z) = ((10m^2 + 1)^{\frac{x-1}{2}}, (15m^2 - 1)^{\frac{y-1}{2}}, z) \tag{5}$$

is a solution of (4). Let

$$L =< (10m^2+1)^{\frac{x-1}{2}}, (15m^2-1)^{\frac{y-1}{2}}, z >$$

be a characteristic number of the solution given in (5). Then from Lemma 2, we get

$$(10m^2+1)^{\frac{x+1}{2}} \equiv -L(15m^2-1)^{\frac{y-1}{2}} \pmod{5m}, \tag{6}$$

$$L \equiv \pm 1 \pmod{5m}. \tag{7}$$

Let $L_0 =< 1, 1, 2 >$ be the characteristic number of the solution $(X_0, Y_0, Z_0) = (1, 1, 2)$ of (4). Thus, Lemma 2 implies

$$(10m^2+1) \times 1 \equiv -L_0 \times 1 \pmod{5m}, \tag{8}$$

$$L_0 \equiv -1 \pmod{5m}. \tag{9}$$

From (6) and (8), we get $L \equiv \pm L_0 \pmod{5m}$, which implies that the solution $(X_0, Y_0, Z_0) = (1, 1, 2)$ and the solution in (5) are in the same solution class $S(L_0)$ of Equation (4). Since the solution $(X, Y, Z) = (1, 1, 2)$ is evidently the smallest possible solution in $S(L_0)$, by Theorem 5, we get

$$X\sqrt{10m^2+1} + Y\sqrt{1-15m^2} = \lambda_1(X_0\sqrt{10m^2+1} + \lambda_2 Y_0\sqrt{1-15m^2})^t, \tag{10}$$

where $\lambda_1, \lambda_2 \in \{1, -1\}$ and $z = Z_0 t = 2t$ with $2 \nmid t, t \in \mathbb{N}$.

After expanding the right-hand side of (10) and equating the coefficients of $\sqrt{1-15m^2}$, we get

$$(15m^2-1)^{\frac{y-1}{2}} = \lambda_1\lambda_2 \sum_{i=0}^{\frac{t-1}{2}} \binom{t}{2i+1} (10m^2+1)^{\frac{t-1}{2}-i}(1-15m^2)^i. \tag{11}$$

We claim that $y = 1$. We prove this claim by contradiction. Let $y > 1$. Then from (11), we have

$$0 \equiv \lambda_1\lambda_2 t(10m^2+1)^{\frac{t-1}{2}} \pmod{(15m^2-1)},$$

$$0 \equiv \pm t(10m^2+1)^{\frac{t-1}{2}} \pmod{(15m^2-1)}.$$

Since m is odd, we get $2 \mid t(10m^2+1)^{\frac{t-1}{2}}$, which is a contradiction. Hence, $y = 1$. From (5), we have $Y = (15m^2-1)^{\frac{y-1}{2}} = 1$. Since $(X_0, Y_0, Z_0) = (1, 1, 2)$ is the least solution of $S(L_0)$, by Theorem 6, we have either

$$10m^2+1 = \frac{1}{4}((5m)^2 \pm 1) \tag{12}$$

or

$$F_{3r+e} = (5m)^2, \tag{13}$$

where $e = \pm 1$. From (12), we get $4(10m^2 + 1) = (5^2m^2 \pm 1)$. This implies that $4 = \pm 1 \pmod{m^2}$, which is a contradiction. Furthermore, from (13), we get $F_{12} = 12^2 = (5m)^2$, because the only Fibonacci number greater than 1 that is a perfect square is $F_{12} = 12^2$ [4]. Thus, $5m = 12$, which is also not possible because m is odd. Hence, by Theorem 6, we get

$$(X, Z) = ((10m^2 + 1)^{\frac{x-1}{2}}, z) = (X_0, Z_0) = (1, 2).$$

`Subcase (b):` x is even. We rewrite (2) as

$$U^2 + (15m^2 - 1)V^2 = (5m)^W, \quad \text{where } \gcd(U, V) = 1,\ W > 0. \tag{14}$$

Hence, Equation (14) has a solution

$$(U, V, W) = ((10m^2 + 1)^{\frac{x}{2}}, (15m^2 - 1)^{\frac{y-1}{2}}, z).$$

Thus, from Theorem 4, we have

$$(10m^2 + 1)^{\frac{x}{2}} + (15m^2 - 1)^{\frac{y-1}{2}}\sqrt{1 - 15m^2} = \lambda_1(U_0 + \lambda_2 V_0\sqrt{1 - 15m^2})^t \tag{15}$$

with $\lambda_1, \lambda_2 \in \{1, -1\}$ and

$$z = W_0 t, \quad t \in \mathbb{N}. \tag{16}$$

Here, U_0, V_0, W_0 are positive integers that satisfy

$${U_0}^2 + (15m^2 - 1){V_0}^2 = (5m)^{W_0}, \quad \text{where } \gcd(U_0, V_0) = 1, \tag{17}$$

$$h(-4(15m^2 - 1)) \equiv 0 \pmod{W_0}. \tag{18}$$

Suppose $2 \mid t$ and let

$$u + v\sqrt{1 - 15m^2} = (U_0 + \lambda_2 V_0\sqrt{1 - 15m^2})^{\frac{t}{2}}. \tag{19}$$

Taking the norm on both sides of (19) in $\mathbb{Q}(\sqrt{1 - 15m^2})$ and using (17), we obtain

$$u^2 + (15m^2 - 1)v^2 = (5m)^{\frac{W_0 t}{2}} = (5m)^{\frac{z}{2}}. \tag{20}$$

Substituting (19) into (15), we get

$$(10m^2 + 1)^{\frac{x}{2}} + (15m^2 - 1)^{\frac{y-1}{2}}\sqrt{1 - 15m^2} = \lambda_1(u + v\sqrt{1 - 15m^2})^2.$$

It follows that

$$(10m^2 + 1)^{\frac{x}{2}} = \lambda_1(u^2 - v^2(15m^2 - 1)), \tag{21}$$

$$(15m^2 - 1)^{\frac{y-1}{2}} = 2\lambda_1 uv. \tag{22}$$

Since $\gcd(10m^2+1, 15m^2-1) = 1$, from (21) and (22), we get $|u| = 1$. Hence, $|v| = \frac{1}{2}(15m^2-1)^{\frac{y-1}{2}}$. Substituting the values of $|u|$ and $|v|$ into (20), we obtain

$$1 + \frac{1}{4}(15m^2-1)^y = (5m)^{\frac{z}{2}}.$$

This implies $3 \equiv 0 \pmod{5m}$, a contradiction to the fact that $5m > 3$. So we conclude that $2 \nmid t$.

Let

$$\alpha = U_0 + V_0\sqrt{1-15m^2}, \quad \beta = U_0 - V_0\sqrt{1-15m^2}. \tag{23}$$

Moving further, we take the complex conjugation in (15) and get

$$(10m^2+1)^{\frac{x}{2}} - (15m^2-1)^{\frac{y-1}{2}}\sqrt{1-15m^2} = \lambda_1(U_0 - \lambda_2 V_0\sqrt{1-15m^2})^t. \tag{24}$$

Now, subtracting (24) from (15), we obtain

$$(15m^2-1)^{\frac{y-1}{2}} = V_0\left|\frac{\alpha^t - \beta^t}{\alpha - \beta}\right| = V_0|L_t(\alpha,\beta)|.$$

We will now check that $L_t(\alpha,\beta)$ is in fact a Lucas sequence. Clearly $\alpha + \beta = 2U_0$ and $\alpha - \beta = 2V_0\sqrt{1-15m^2}$ (by (23)). Also $\alpha\beta = (5m)^{W_0}$ (from (17)). Since $\gcd(U_0, V_0) = 1$, the integers $\alpha\beta = (5m)^{W_0}$ and $\alpha+\beta = 2U_0$ are also relatively prime (by (17)). Furthermore, $\frac{\alpha}{\beta} \neq \pm 1$, since ± 1 are the only units in $\mathbb{Q}(\sqrt{1-15m^2})$. Thus, $L_t(\alpha,\beta)$ is a Lucas sequence. From Theorems 7 and 8, we obtain $t \leq 30$. Moreover, if $4 < t \leq 30$ and $t \neq 6$, then the parameter

$$(e, f) = (2U_0, 4{V_0}^2(1-15m^2))$$

must be one of the parameters given in Theorem 8. But, none of them match with any such parameter. So it follows that $t \leq 3$. We will show that the case $t = 3$ is also not possible. If possible, let $t = 3$. Upon expanding the right-hand side of (15), we get

$$\begin{aligned}(U_0 + \lambda_2 V_0\sqrt{1-15m^2})^t =& {U_0}^3 + 3{U_0}^2\lambda_2 V_0\sqrt{1-15m^2} \\ &+ 3U_0{V_0}^2(1-15m^2) + \lambda_2{V_0}^3(1-15m^2)\sqrt{1-15m^2}.\end{aligned} \tag{25}$$

Combining (15) and (25) and equating coefficients on both sides, we obtain

$$(10m^2+1)^{\frac{x}{2}} = \lambda_1 U_0({U_0}^2 - 3(15m^2-1){V_0}^2) \tag{26}$$

and

$$(15m^2-1)^{\frac{y-1}{2}} = \lambda_1\lambda_2 V_0(3{U_0}^2 - (15m^2-1){V_0}^2). \tag{27}$$

From Equation (17), we see that $\gcd(3U_0, 15m^2-1) = 1$. Hence, from (27) we have

$$3{U_0}^2 - (15m^2-1){V_0}^2 = \pm 1. \tag{28}$$

By considering Equation (28) modulo 3, we conclude that only the positive sign can occur, i.e.,

$$3{U_0}^2 - (15m^2-1){V_0}^2 = 1. \tag{29}$$

Thus, from (29)

$$|V_0| = (15m^2-1)^{\frac{y-1}{2}}. \tag{30}$$

Substituting (29) into (26), we get

$$(10m^2+1)^{\frac{x}{2}} = \lambda_1 U_0({U_0}^2 - 3(15m^2-1)^y).$$

By reducing (29) and (30) modulo $3m$, we obtain

$$3{U_0}^2 - (-1)1 \equiv 1 \pmod{3m},$$

which means that ${U_0}^2 \equiv 0 \pmod{m}$. Reducing Equation (17) modulo m gives ${U_0}^2 \equiv 1 \pmod{m}$. These two congruences force $0 \equiv 1 \pmod{m}$, i.e., $m = 1$, contradicting $m > 1$. Therefore, the case $t = 3$ is not possible, forcing $t = 1$. Then $z = W_0 t = W_0$ (from (16)), and (18) gives $W_0 \leq h(-4(15m^2-1))$. Applying the upper bound from Theorem 3, we have

$$z < \frac{4}{\pi}\sqrt{15m^2-1}\log(2e\sqrt{15m^2-1}). \tag{31}$$

Assume that $z = 3$. Then at least one of x and y must be greater than 1. Let $x \geq 2$, which gives $(5m)^3 > (10m^2+1)^x \geq (10m^2+1)^2 > (10)^2m^4$. Hence, $5^3 > (10)^2m$, which leads to a contradiction since $m > 1$. Similarly, if $y \geq 2$, then the inequality $(5m)^3 \geq (15m^2-1)^2 + (10m^2+1)$ leads to $125m > 225m^2 - 20$, which is also a contradiction. Hence, $z \geq 4$. Reducing (2) modulo $(25m^4)$ yields

$$10m^2x + 15m^2y \equiv 0 \pmod{(25m^4)}.$$

So

$$2x + 3y \equiv 0 \pmod{5m^2},$$

and that implies

$$5m^2 \leq 2x + 3y. \tag{32}$$

Since

$$(10m^2+1)^x < (5m)^z \text{ and } (15m^2-1)^y < (5m)^z,$$

we deduce that $x < z$ and $y < z$. Hence, by (32), we have $m^2 < z$. Combining this with Inequality (31) yields

$$m^2 < z < \frac{4}{\pi}\sqrt{15m^2-1}\log(2e\sqrt{15m^2-1}).$$

Thus, $m \leq 32$. Since z is bounded, the same holds for x and y. Using Inequality (31) together with $x, y < z$, we wrote a brief Maple program to examine all potential solutions to Equation (2) in the range $2 \leq m \leq 32$. No positive integer solutions (m, x, y, z) were found for $z \geq 3$. This completes the proof. $\square$

4. Conclusions

In this study, we examined the equation

$$(am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z \tag{33}$$

with specific parameters $(a, b, c) = (10, 15, 5)$ and demonstrated that the corresponding Equation (2) has a unique solution $(x, y, z) = (1, 1, 2)$, when $m > 1$. These findings provide further support to Terai and Jeśmanowicz's conjectures. To expand on the results of this study, future research may explore Equation (33) in a more generalized case, where $2 \mid a$, $2 \nmid b$ with $a + b = c^2$.

Acknowledgements. This research was initiated during the second author's visit to Professor Benjamin Kane at the University of Hong Kong. The author is grateful for his support. The author also appreciates the hospitality and facilities provided by the University of Hong Kong during the visit period. The authors wish to thank Devendra Kumar Danodia for his assistance with the technical aspects of this paper. The authors are extremely grateful to anonymous referee for his/her valuable suggestions, which have led to substantial improvements in the manuscript.

References

[1] C. Bertok, The complete solution of the Diophantine equation $(4m^2+1)^x+(5m^2-1)^y = (3m)^z$, *Period. Math. Hung.* **72** (2016), 37–42.

[2] D. Wu, N. Deng, and P. Yuan, The exponential Diophantine equation $(3am^2 - 1)^x + (a(a-3)m^2 + 1)^y = (am)^z$, *Turk J. Math.* **43** (2019), 2561–2567.

[3] E. Kizildere, G. Soydan, and M. Le, A note on the ternary purely exponential Diophantine equation $A^x + B^y = C^z$ with $A + B = c^2$, *Stud. Sci. Math. Hung.* **57** (2020), 200–206.

[4] J. H. E. Cohn, On square Fibonacci numbers, *Fibonacci Quart.* **2** (1964), 109–113.

[5] J. Su and X. Li, The exponential Diophantine equation $(4m^2 + 1)^x + (5m^2 - 1)^y = (3m)^z$, *Abstr. Appl. Anal.* **2014** (1) (2014), 1–5.

[6] K. Mahler, Zur Approximation algebraischer Zahlen. I. (Über den größten Primteiler binärer Formen), *Math. Ann.* **107** (1933), 691–730.

[7] L. K. Hua, *Introduction to Number Theory*, Springer, Berlin, Germany, 1982.

[8] L. Jeśmanowicz, Some remarks on Pythagorean numbers, *Wiadom Mat.* **1** (1955/1956), 196–202.

[9] M. Alan, On the exponential Diophantine equation $(6m^2+1)^x+(3m^2-1)^y=(3m)^z$, *FUJMA* **5** (2022), 174–180.

[10] M. Le, R. Scott, and R. Styer, A survey on the ternary purely exponential Diophantine equation $a^x+b^y=c^z$, *Surv. Math. Appl.* **214** (2019), 109–140.

[11] M. Le, Some exponential Diophantine equations I: the equation $D_1x^2-D_2y^2=\lambda k^z$, *J. Number Theory* **55** (1995), 209–221.

[12] M. Ma and Y. G. Chen, Jeśmanowicz conjecture on Pythagorean triples, *Bull. Aust. Math. Soc.* **96** (1) (2017), 30–35.

[13] N. Terai and T. Hibino, On the exponential Diophantine equation $(3pm^2-1)^x+(p(p-3)m^2+1)^y=(pm)^z$, *Period. Math. Hung.* **74** (2017), 227–234.

[14] N. Terai, The Diophantine equation $a^x+b^y=c^z$, *Proc. Japan Acad. Ser. A Math. Sci.* **70** (1994), 22–26.

[15] P.M. Voutier, Primitive divisors of Lucas and Lehmer sequences, *Math. Comp.* **64** (1995), 869–888.

[16] R. Fu and H. Yang, On the exponential Diophantine equation $(am^2+1)^x+(bm^2-1)^y=(cm)^z$ with $c \mid m$, *Period. Math. Hung.* **75** (2017), 143–149.

[17] T. Miyazaki, Exceptional cases of Terai's conjecture on Diophantine equations, *Arch. Math.* **95** (2010), 519–527.

[18] T. Miyazaki, Generalizations of classical results on Jeśmanowicz' conjecture concerning Pythagorean triples, *J. Number Theory* **133** (2013), 583–595.

[19] T. Miyazaki and I. Pink, Number of solutions to a special type of unit equations in two unknowns II, *Res. Number Theory* **10** (2024), Paper No. 36, 41 pp.

[20] T. Miyazaki and N. Terai, On Jeśmanowicz conjecture concerning Pythagorean triples II, *Acta Math Hungar.* **147** (2015), 286–293.

[21] T. Negell, Sur une classe d' equations exponentielles, *Ark. Mat.* **3** (4) (1958), 569–582.

[22] Y. Bilu, G. Hanrot, and P.M. Voutier, Existence of primitive divisors of Lucas and Lehmer numbers (with Appendix By Mignotte), *J. Reine Angew. Math.* **539** (2001), 75–122.

[23] Y. Bugeaud and T. Shorey, On the number of solutions of the generalized Ramanujan-Nagell equation, *J. Reine Angew. Math.* **55** (2001), 55–74.