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NONEXISTENCE OF THE CONTINUED FRACTION OF A QUADRATIC IRRATIONAL NUMBER WITH A GIVEN SEQUENCE

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Abstract

It is well known that the continued fraction expansion of $\sqrt{d}$ (respectively, $(1+\sqrt{d})/2$) has the form $[a_0; \overline{a_1, \ldots, a_{l_d-1}, 2a_0}]$ (respectively, $[a'_0; \overline{a'_1, \ldots, a'_{l'_d-1}, 2a'_0 - 1}]$), where $a_1, \ldots, a_{l_d-1}$ (respectively, $a'_1, \ldots, a'_{l'_d-1}$) is a palindromic sequence of positive integers. In this paper, for every integer l (respectively, l') and palindromic sequence of positive integers $a_1, \ldots, a_{l-1}$ (respectively, $a'_1, \ldots, a'_{l'-1}$), we compute the number of cases where there does not exist a continued fraction of $\sqrt{d}$ (respectively, $(1+\sqrt{d})/2$) that has a periodic length of l (respectively, l') and a given palindromic sequence.

1. Introduction

For a positive square-free integer d, let t_d and u_d be positive integers such that

$$\epsilon_d = \frac{t_d + u_d\sqrt{d}}{z} > 1$$

is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$, where $z = 2$ if $d \equiv 1 \pmod 4$ and $z = 1$ otherwise. It is well known that there is a connection between the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$ and the continued fraction of $\sqrt{d}$ (or $(1+\sqrt{d})/2$) [1, 2, 3, 4, 7, 12, 13, 18, 19].

Let d be a non-square positive integer. We denote the continued fraction of $\sqrt{d}$

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by

$$\sqrt{d} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_{l_d} + \cfrac{1}{a_1 + \cdots}}}} = [a_0; \overline{a_1, \ldots, a_{l_d}}],$$

where l_d is the length of the period of the continued fraction expansion. Then the period is *palindromic*, that is, $a_{l_d - t} = a_t$ for $1 \leq t < l_d$ and $a_{l_d} = 2a_0$ (see [16], Satz 3.29). Similarly, we denote the continued fraction of $(1+\sqrt{d})/2$ by

$$(1+\sqrt{d})/2 = a'_0 + \cfrac{1}{a'_1 + \cfrac{1}{\cdots + \cfrac{1}{a'_{l'_d} + \cfrac{1}{a'_1 + \cdots}}}} = [a'_0; \overline{a'_1, \ldots, a'_{l'_d}}],$$

where l'_d is the length of the period of the continued fraction expansion. Then the continued fraction of $(1+\sqrt{d})/2$ has properties similar to the continued fraction of $\sqrt{d}$. In fact, the period is also palindromic and $a'_{l'_d} = 2a'_0 - 1$ (see [16], Satz 3.30).

On the other hand, for any given positive integer l (respectively l') and a palindromic sequence of positive integers $a_1, \ldots, a_{l-1}$ (respectively $a'_1, \ldots, a'_{l'-1}$), necessary and sufficient conditions for the existence of a positive integer d to have a continued fraction of the form of the continued fraction

$$\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l-1}, 2a_0}] \text{ (respectively } (1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'-1}, 2a'_0 - 1}])$$

are known ([4, 7, 11]). In this paper, we focus on counting the number of cases where continued fractions of this form do not exist.

2. Preliminaries

In this section, we start with the basic properties of a continued fraction. Similar discussion and the proofs for the properties can be seen in many excellent books and papers such as [5, 6, 8, 9, 14, 15, 17]. First, we obtain positive integers p_n, q_n from partial quotients $a_0, a_1, \ldots, a_n$ of the continued fraction of $\sqrt{d}$ by using recurrence relations:

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-2} \ (n \geq 1), \\ q_{-1} &= 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \ (n \geq 1), \\ r_{-1} &= 1, \quad r_0 = 0, \quad r_n = a_n r_{n-1} + r_{n-2} \ (n \geq 1). \end{aligned} \tag{1}$$

Note that

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n], \quad \lim_{n\to\infty} \frac{p_n}{q_n} = \sqrt{d},$$

and

$$\frac{q_n}{r_n} = [a_1 a_2, \ldots, a_n].$$

We can easily prove the following recurrence relations for the sequences $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$:

$$q_n r_{n-1} - r_n q_{n-1} = (-1)^n, \tag{2}$$

$$q_n r_{n-2} - r_n q_{n-2} = (-1)^{n-1} a_n,$$

$$p_n - a_0 q_n = r_n.$$

We can also give a similar expression for the continued fraction expansion of $(1+\sqrt{d})/2$. For any positive integer l (respectively l') and a palindromic sequence of positive integers $a_1, \ldots, a_{l-1}$ (respectively $a'_1, \ldots, a'_{l'-1}$), necessary and sufficient conditions for the existence of a positive integer d having the form of the continued fraction expansion

$$\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l-1}, 2a_0}] \text{ (respectively } (1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'-1}, 2a'_0 - 1}])$$

are known as follows.

Proposition 1 ([7])**.** *There exists a positive integer d having the form of the continued fraction $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l-1}, 2a_0}]$ if and only if one of the following two cases holds*:

(i) q_{l-1} *is odd;*

(ii) *both q_{l-1} and r_{l-2} are even, and q_{l-2} is odd.*

Proposition 2 ([10])**.** *We define p'_i/q'_i by the i-th convergent of the continued fraction of $[a'_0; a'_1, \ldots, a'_n]$ and q'_i/r'_i the i-th convergent of the continued fraction of $[a'_1, \ldots, a'_n]$. Then there exists a positive integer d having the form of the continued fraction $(1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'-1}, 2a'_0 - 1}]$ if and only if one of the following two cases holds*:

(i) $q'_{l'-1}$ *is odd;*

(ii) *both $q'_{l'-2}$ and $r'_{l'-2}$ are odd, and $q'_{l'-1}$ is even.*

The following theorems give conditions to immediately check from a given sequence $a_1, \ldots, a_{l-1}$ (respectively $a'_1, \ldots, a'_{l'-1}$) whether the continued fraction of $\sqrt{d}$ (respectively $(1+\sqrt{d})/2$) with that sequence as partial quotients exists.

Theorem 1 ([11])**.** *Let l be a fixed positive integer and $a_1, \ldots, a_{l-1}$ symmetric positive integers. For a positive nonsquare integer d, the following statements hold.*

(i) *If l is odd, there exists a_0 such that $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l-1}, 2a_0}]$ if and only if q_{l-1} is odd.*

(ii) *If l is even, there exists a_0 such that $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l-1}, 2a_0}]$ if and only if both $a_{l/2}$ and q_{l-1} are odd or $a_{l/2}$ is even.*

Theorem 2 ([11])**.** *Let l be a fixed positive integer and $a'_1, \ldots, a'_{l'-1}$ symmetric positive integers. For a positive nonsquare integer d, the following statements hold.*

(i) *If l' is odd, there exists a_0 such that $(1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'-1}, 2a'_0 - 1}]$.*

(ii) *If l' is even, then $(1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'-1}, 2a'_0 - 1}]$ if and only if $a'_{l'/2}$ is odd.*

In this paper, for every integer l (respectively l') and palindromic sequence of positive integers $a_1, \ldots, a_{l-1}$ (respectively $a'_1, \ldots, a'_{l'-1}$), we compute the number of cases where there does not exist a continued fraction of $\sqrt{d}$ (respectively $(1+\sqrt{d})/2$) that has a periodic length of l (respectively l') and a given palindromic sequence.

3. Main Theorems

We will use the following results to prove the main theorem.

Proposition 3 ([11])**.** *For $0 \le i \le l-2$, we have the following recurrence relations:*

(i) $q_{l-1} = q_i q_{l-1-i} + q_{i-1} q_{l-2-i}$,

(ii) $r_{l-2} = r_i r_{l-1-i} + r_{i-1} r_{l-2-i}$.

Depending on the parity of l, the following results can be directly obtained.

Corollary 1 ([11])**.** *If l is even, for $0 \le i \le l-2$, we have the following equations for parity:*

(i) $q_{l-1} \equiv a_{l/2} q_{l/2-1} (\text{mod } 2)$,

(ii) $r_{l-2} \equiv a_{l/2} r_{l/2-1} (\text{mod } 2)$.

Using Proposition 3, we can also obtain a formula similar to Corollary 1 as follows, when l is odd.

Corollary 2. *If l is odd, for $0 \le i \le l-2$, we have the following equations for parity:*

(i) $q_{l-1} \equiv q_{(l-1)/2} + q_{(l-3)/2} (\text{mod } 2)$,

(ii) $r_{l-2} \equiv r_{(l-1)/2} + r_{(l-3)/2} (\text{mod } 2)$.

Proof. Substituting $i = (l-1)/2$ in the recurrence relations of Proposition 3, we have $q_{l-1} \equiv q^2_{(l-1)/2} + q^2_{(l-3)/2} \pmod 2$ and $r_{l-2} \equiv r^2_{(l-1)/2} + r^2_{(l-3)/2} \pmod 2$. Therefore, the recurrence relations for Corollary 2 can be obtained. □

We now are ready to state main theorems.

Theorem 3. *Let $l \geq 3$ be odd. For $(a_1, a_2, \ldots, a_{(l-1)/2})$ in $\mathbb{Z}_2^{(l-1)/2}$ and a positive nonsquare integer d, let $A_{(l-1)/2}$ be the number of cases where there does not exist an a_0 satisfying*

$$\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{(l-1)/2}, a_{(l+1)/2}, \ldots, a_{l-1}, 2a_0}]\}.$$

Then

$$A_1 = 1,\ A_2 = 1,\ \text{and } A_{(l-1)/2} = A_{(l-3)/2} + 2A_{(l-5)/2} \text{ for } l \geq 7.$$

Therefore, $A_{(l-1)/2}$ is $\frac{1}{3}\left(2^{(l-1)/2} - (-1)^{(l-1)/2}\right)$.

Proof. If l is odd, by Theorem 1 (i), there does not exist a positive integer d having the continued fraction expansion of the form $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l-1}, 2a_0}]$ if and only if q_{l-1} is even. The initial terms can be determined by direct computation. If $l = 3$, such an a_0 does not exist precisely when a_1 is odd, yielding $A_1 = 1$. If $l = 5$, this occurs precisely when a_1 is odd and a_2 is even, yielding $A_2 = 1$. Now, suppose $l \geq 7$. If q_{l-1} is even, then $q_{(l-1)/2} + q_{(l-3)/2} \equiv 0 \pmod 2$ by Corollary 2. Since $q_{(l-1)/2}$ and $q_{(l-3)/2}$ are coprime by Equation (2), they cannot both be even. This implies that both $q_{(l-1)/2}$ and $q_{(l-3)/2}$ are odd. On the other hand, it follows from Equation (1) that $q_{(l-1)/2} = a_{(l-1)/2}q_{(l-3)/2} + q_{(l-5)/2}$. This implies that $a_{(l-1)/2} + q_{(l-5)/2} \equiv 1 \pmod 2$. Now, we consider the parity conditions on $a_{(l-1)/2}$ and $q_{(l-5)/2}$. By repeating the above process, we obtain the following. If $a_{(l-1)/2} \equiv 0 \pmod 2$ and $q_{(l-5)/2} \equiv 1 \pmod 2$, then the number of such cases corresponding to $A_{(l-1)/2}$ is $A_{(l-3)/2}$. If $a_{(l-1)/2} \equiv 1 \pmod 2$ and $q_{(l-5)/2} \equiv 0 \pmod 2$, then the number of such cases corresponding to $A_{(l-1)/2}$ is $2A_{(l-5)/2}$. Finally, the desired formula for $A_{(l-1)/2}$ is obtained from this recurrence relation. This completes the proof. □

Theorem 4. *Let $l \geq 4$ be even. For $(a_1, a_2, \ldots, a_{l/2})$ in $\mathbb{Z}_2^{l/2}$ and a positive nonsquare integer d, let $B_{(l-2)/2}$ be the number of cases where there does not exist an a_0 satisfying*

$$\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l/2}, \ldots, a_{l-1}, 2a_0}]\}.$$

Then

$$B_1 = 1,\ B_2 = 1,\ \text{and } B_{(l-2)/2} = B_{(l-4)/2} + 2B_{(l-6)/2} \text{ for } l \geq 8.$$

Therefore, $B_{(l-2)/2}$ is $\frac{1}{3}\left(2^{(l-2)/2} - (-1)^{(l-2)/2}\right)$.

Proof. If l is odd, by Theorem 1 (ii), there does not exist a positive integer d having the continued fraction expansion of the form $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l-1}, 2a_0}]$ if and only if $a_{l/2}$ is odd and q_{l-1} is even. If $l = 4$, such an a_0 does not exist precisely when a_1 is even and a_2 is odd, yielding $B_1 = 1$. When $l = 6$, this occurs precisely when a_1, a_2, and a_3 are odd, yielding $B_2 = 1$. Now, suppose $l \geq 8$. If $a_{l/2}$ is odd and q_{l-1} is even, then by Corollary 1, $q_{l/2-1}$ is even, which implies that $q_{l/2-2}$ is odd because $q_{l/2-1}$ and $q_{l/2-2}$ are relatively prime by Equation (2). On the other hand, $a_{l/2-1} + q_{l/2-3} \equiv 0 \pmod 2$ since $q_{l/2-1} = a_{l/2-1}q_{l/2-2} + q_{l/2-3}$ by Equation (1). Now, We examine the parity of $a_{l/2-1}$ and $q_{l/2-3}$. Applying the same argument as Theorem 3, we distinguish the following possibilities. If $a_{l/2-1} \equiv 1 \pmod 2$ and $q_{l/2-3} \equiv 1 \pmod 2$, then this contributes $B_{(l-2)/2}$ to $B_{(l-4)/2}$. On the other hand, if $a_{l/2-1} \equiv 0 \pmod 2$ and $q_{l/2-3} \equiv 0 \pmod 2$, then the contribution is $2B_{(l-6)/2}$. Therefore, we obtain the desired explicit formula for $B_{(l-2)/2}$ using this recurrence relation. This completes the proof. □

Example 1. Consider the case where $l = 7$. Let $a_1, a_2, \ldots, a_6$ form a palindromic sequence, meaning $a_1 = a_6$, $a_2 = a_5$, and $a_3 = a_4$. Using Proposition 1 and Theorem 1, one can directly calculate the number of the cases where there does not exist an a_0 satisfying

$$\sqrt{d} = [a_0; \overline{a_1, a_2, a_3, a_3, a_2, a_1, 2a_0}].$$

We can also see that $A_3 = 3$ by Theorem 3. The resulting list is shown in Table 1. The lists for $l = 9$ and $l = 11$ are also shown in Table 2 and 3, respectively.

a_1	a_2	a_3
0	0	1
0	1	1
1	0	0

Table 1: The case where $l = 7$

a_1	a_2	a_3	a_4
0	0	1	0
0	1	1	0
1	0	0	0
1	1	0	1
1	1	1	1

Table 2: The case where $l = 9$

a_1	a_2	a_3	a_4	a_5
0	0	0	0	1
0	0	0	1	1
0	0	1	0	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
1	0	0	0	0
1	0	1	0	1
1	0	1	1	1
1	1	0	1	0
1	1	1	1	0

Table 3: The case where $l = 11$

Example 2. Consider the case where $l = 8$. Let a_1, a_2, $\ldots$, a_7 form a palindromic sequence, meaning $a_1 = a_7$, $a_2 = a_6$, and $a_3 = a_5$. Using Proposition 1 and Theorem 1, one can directly calculate the number of cases where there does not exist an a_0 satisfying

$$\sqrt{d} = [a_0; \overline{a_1, a_2, a_3, a_4, a_3, a_2, a_1, 2a_0}].$$

We can see that $B_3 = 3$ by Theorem 4. The resulting list is shown in Table 4. The lists for $l = 10$ and $l = 12$ are also shown in Table 5 and 6, respectively.

The case of $(1+\sqrt{d})/2$ is straightforward as shown by Theorem 2.

Theorem 5. *Let l' be an odd positive integer. For $(a'_1, a'_2, \ldots, a'_{(l'-1)/2})$ in $\mathbb{Z}_2^{(l'-1)/2}$ and a positive nonsquare integer d, let $A_{l'}$ be the number of cases where there does not exist an a'_0 satisfying*

$$(1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{(l-1)/2}, a'_{(l+1)/2}, \ldots, a'_{l-1}, 2a'_0 - 1}]\}.$$

Then $A_{l'} = 0$ for every odd positive integer l'.

Theorem 6. *Let l' be an even positive integer. For $(a'_1, a'_2, \ldots, a'_{l'/2})$ in $\mathbb{Z}_2^{l'/2}$ and a positive nonsquare integer d, let $B_{l'}$ be the number of cases where there does not exist an a'_0 satisfying*

$$(1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'/2}, \ldots, a'_{l'-1}, 2a_0 - 1}]\}.$$

Then $B_{l'} = 2^{l'/2-1}$ for every even positive integer l'.

a_1	a_2	a_3	a_4
0	0	0	1
0	1	0	1
1	0	1	1

Table 4: The case where $l = 8$

a_1	a_2	a_3	a_4	a_5
0	0	1	1	1
0	1	1	1	1
1	0	0	1	1
1	1	0	0	1
1	1	1	0	1

Table 5: The case where $l = 10$

a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	0	1
0	0	0	1	0	1
0	0	1	0	1	1
0	1	0	0	0	1
0	1	0	1	0	1
0	1	1	0	1	1
1	0	0	0	1	1
1	0	1	0	0	1
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	1

Table 6: The case where $l = 12$

Remark 1. If $l = 1$ (respectively $l = 2$), there does always exist an a_0 satisfying $\sqrt{d} = [a_0; \overline{2a_0}]$ (respectively $\sqrt{d} = [a_0; \overline{a_1, 2a_0}]$) by Theorem 1.

Remark 2. Let $k \geq 2$ be an integer. Then, applying $l = 2k - 1$ and $l = 2k$ to Theorem 3 and Theorem 4, respectively, reveals that they result in the same number of cases. This means that there is a one to one correspondence between the cases in Theorem 3 when $l = 2k-1$ and the cases in Theorem 4 when $l = 2k$. For example, by comparing Table 3 and Table 6, we can observe that the parities of $a_1, \dots, a_4$ are the same, whereas a_5 has the opposite parity in the two tables. This phenomenon can be explained in general as follows. In Theorem 3, the condition $a_{(l-1)/2} \equiv 0 \pmod 2$ and $q_{(l-5)/2} \equiv 1 \pmod 2$ corresponds to the condition in Theorem 4 that $a_{l/2-1} \equiv$

$1 \pmod 2$ and $q_{l/2-3} \equiv 1 \pmod 2$. Similarly, the condition $a_{(l-1)/2} \equiv 1 \pmod 2$ and $q_{(l-5)/2} \equiv 0 \pmod 2$ in Theorem 3 corresponds to $a_{l/2-1} \equiv 0 \pmod 2$ and $q_{l/2-3} \equiv 0 \pmod 2$ in Theorem 4.

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