#A70 INTEGERS 26 (2026)

ON THE ASYMPTOTIC FORMULA FOR THE GCD-SUM FUNCTION

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Received: 11/18/25, Accepted: 5/11/26, Published: 6/8/26

Abstract

For $n \in \mathbb{N}$, we define the gcd-sum function $P(n) := \sum_{k=1}^{n} gcd(k, n)$. We study the discrete higher moments of the function $A(n) = \frac{P(n)}{n}$ in the paper and obtain asymptotic formulae for the sums $\sum_{n \le x} A^k(n)$ with improved error term for certain integers k.

1. Introduction

In 1933, Pillai [16] introduced the gcd-sum function

$$P(n) := \sum_{k=1}^{n} gcd(k,n) = n \sum_{d|n} \frac{\varphi(d)}{d},$$

where φ is Euler's function. This arithmetical function is studied by many authors (see [1], [3], [4], [5], [6], [7], [10], [17], [18], [19]). Define $A(n) := \frac{P(n)}{n}$ for any integer $n \ge 1$. In 2010, Tóth [18] considered the summatory function $\sum_{n \le x} A^2(n)$ and established the asymptotic formula

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DOI: 10.5281/zenodo.20597548

$$\sum_{n\le x} A^2(n) = xP_3(\log x) + O(x^{\frac{1}{2}+\epsilon}),$$

where $P_3(\log x)$ is a polynomial of degree 3 in $\log x$.

Later, in 2013, Zhang and Zhai [19] gave an asymptotic formula for the summatory function $\sum_{n\le x} A^k(n)$ for any integer $k \ge 2$, which is

$$\sum_{n\le x} A^k(n) = xQ^{2^k-1}(\log x) + O(x^{\beta_k+\epsilon}), \tag{1}$$

(For $\ell \in \mathbb{Z}^+$, $Q_\ell(y)$ denotes a polynomial of degree ℓ in y). The β_k's in the error term are given by

$$\beta_2 = \frac{1}{2},\ \beta_3 = \frac{5}{8},\ \beta_4 = \frac{7}{9},\ \beta_5 = \frac{31}{36},\ \beta_6 = \frac{207}{224},\quad \text{and}\quad \beta_k = 1 - \frac{1}{50.2^{\frac{2}{3}k}}\quad \text{for}\quad k \ge 7.$$

It is not difficult to see that

$$\begin{aligned}\sum_{n\le x} A(n) &= \sum_{n\le x}\sum_{d|n} \frac{\varphi(d)}{d} = \sum_{d\le x} \frac{\varphi(d)}{d} \sum_{q\le \frac{x}{d}} 1\\ &= \sum_{d\le x} \frac{\varphi(d)}{d}\Big\{\frac{x}{d} + O(1)\Big\}\\ &= x\sum_{d\le x} \frac{\varphi(d)}{d^2} + O\Big(\sum_{d\le x} \frac{\varphi(d)}{d}\Big)\\ &= cx\log x + O(x).\end{aligned}$$

Throughout the paper, $C_1,\ C_2,\ldots$ with or without suffixes, denote effective positive constants and may depend only on k, and need not be the same at each occurrence. Similarly, the number ϵ is any small positive constant, the value of which need not be the same in different occurrences. The aim of this paper is to improve the error term in 1 for $k = 2$, $k = 3$, and $7 \le k \le 11$. Indeed, we prove the following theorems.

Theorem 1. *For $k = 2$ and $x \ge x_0$, where x_0 is sufficiently large, we have*

$$\sum_{n\le x} A^2(n) = M_2(x) + O(x^{\frac{1}{2}} \exp(C_2(\log\log x)^2)).$$

Theorem 2. *For $k = 3$ and $x \ge x_0$, where x_0 is sufficiently large, we have*

$$\sum_{n\le x} A^3(n) = M_3(x) + O(x^{\frac{5}{8}} \exp(C_3(\log\log x)^2)).$$

Theorem 3. *For $k = 4$ and $x \ge x_0$, where x_0 is sufficiently large, we have*

$$\sum_{n\le x} A^4(n) = M_4(x) + O(x^{\frac{157}{199}+\epsilon}).$$

Theorem 4. *Let $k \geq 5$ be any fixed integer and $x \geq x_0$, where x_0 is sufficiently large, we have*

$$\sum_{n \leq x} A^k(n) = M_k(x) + O_k(x^{1+2\epsilon-\frac{42}{13.2^k-9}}).$$

Remark 1. Clearly, the exponents of the error terms appearing in Theorems 1 and 2 are improvements of the exponents β_2, β_3 (appearing in (1)), respectively. We also notice that $1 - \frac{42}{13(2^k)-9} < 1 - \frac{1}{50(2^{\frac{2}{3}k})}$, provided $\frac{1}{50(2^{\frac{2}{3}k})} < \frac{42}{13(2^k)-9}$. It is enough to secure the inequality $13(2^k) - 9 < 13(2^k) < 2100(2^{\frac{2}{3}k})$ (i.e., $2^{\frac{k}{3}} \leq \frac{2100}{13}$). By taking the logarithm, we see that, Theorem 4 is an improvement as long as $7 \leq k \leq 22$. Note that $k \leq \left[\frac{3\log(\frac{2100}{13})}{\log 2}\right] = 22$. It should be noted that in Theorem 3, the exponent of the error term is $\frac{157}{199} = 0.7889 > \beta_4 = \frac{7}{9} = 0.7777....$

2. Preliminaries and Some Lemmas

To establish our main theorems, we need the following lemmas.

Lemma 1 ([11], Theorem 8.4 and 8.8)**.** *For any $\epsilon > 0$, we have*

$$\int_0^T \left|\zeta\left(\frac{5}{7} + it\right)\right|^{12} dt \ll_\epsilon \ T^{1+\epsilon},$$

uniformly for $T \geq 1$.

Lemma 2 ([9])**.** *For T sufficiently large, we have*

$$\int_{\frac{T}{2}}^T \left|\zeta\Big(\frac{5}{8} + it\Big)\right|^8 dt \ll T(\log T)^{38},$$

uniformly.

Lemma 3 ([12])**.** *For any $\epsilon > 0$, we have*

$$\int_1^T |\zeta(\sigma + it)|^{12}\, dt = T\sum_{n=1}^{\infty} \frac{d_6^2(n)}{n^{2\sigma}} + O(T^{\frac{9-7\sigma}{4}+\epsilon}),$$

for any fixed σ satisfying $\frac{5}{7} < \sigma < 1$ and $|t| \geq 1$.

Lemma 4 ([9])**.** *For T sufficiently large, we have*

$$\int_1^T \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^{12} dt \ll T^2(\log T)^{17},$$

uniformly.

Lemma 5 ([13])**.** *For T sufficiently large, we have*

$$\int_{10}^{T} \frac{|\zeta(\frac{1}{2}+it)|^4}{|\zeta(1+2it)|}\, dt \ll T(\log T)^5,$$

uniformly.

Lemma 6 ([14], [2])**.** *For any $\epsilon > 0$, we have*

$$\zeta(\sigma+it) \ll_{\epsilon} (1+|t|)^{max\{\frac{13}{42}(1-\sigma),0\}+\epsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1+\epsilon$ and $|t| \geq 1$.

Lemma 7 ([15])**.** *For $U \geq U_0$, where U_0 is sufficiently large, there exists a point $T^* \in (U,\, 2U)$ such that*

$$\max_{\sigma \geq \frac{1}{2}} |\zeta(\sigma \pm iT^*)| \;\leq\; exp(C(\log\log U)^2) \ll U^{\epsilon}.$$

Lemma 8 ([10])**.** *Let s be a complex variable with $\Re(s) > 1$. Then we have*

$$\sum_{n=1}^{\infty} \frac{A^k(n)}{n^s} = \zeta^{2^k}(s) G_k(s),$$

where $G_k(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is a Dirichlet series which is absolutely convergent for $\Re(s) > \frac{1}{2}$. Moreover, we have

$$G_k(s) = \frac{\tilde{G}_{k,2}(s)}{\zeta^{4^k-3^k-\binom{2^k}{2}}(2s)}, \tag{2}$$

where

$$\begin{aligned}\tilde{G}_{k,2}(s) = \prod_p \Big(1-\frac{1}{p^{2s}}\Big)^{3^k-4^k+\binom{2^k}{2}} \Big(1 &+ \frac{3^k-4^k+\binom{2^k}{2}}{p^{2s}} - \frac{k2^{k-1}}{p^{s+1}} + \cdots + (-1)^k \frac{1}{p^{s+k}} \\ &+ \frac{4^k-6^k+\binom{2^k}{2}2^k-\binom{2^k}{3}}{p^{3s}} + \frac{k2^{k-1}-2k3^{k-1}}{p^{2s+1}} + \dots \\ &+ \frac{2k.2^k3^{k-1}-3k.4^{k-1}-k.2^{k-1}\binom{2^k}{2}}{p^{3s+1}} + \dots\Big),\end{aligned} \tag{3}$$

which is absolutely convergent for $\Re(s) > \frac{1}{3}$.

Now we discuss the structure of the L-function $G_k(s)$.

3. Structure of L-function $G_k(s)$

Let

$$F_k(s) := \sum_{n=1}^{\infty} \frac{A^k(n)}{n^s}.$$

For $k \geq 2$, by Lemma 8, we can express $F_k(s)$ as

$$F_k(s) = \frac{\zeta^{2^k}(s)}{\zeta^{4^k-3^k-\binom{2^k}{2}}(2s)} \tilde{G}_{k,2}(s).$$

By [10], $\tilde{G}_{k,2}(s)$ absolutely converges in the half-plane $\Re(s) > \frac{1}{3}$. From Equation (2), we have

$$G_k(s) = \frac{\tilde{G}_{k,2}(s)}{\zeta^{4^k-3^k-\binom{2^k}{2}}(2s)}.$$

By induction, we can express $G_k(s)$ as

$$G_k(s) = \frac{\tilde{G}_{k,n}(s)}{\zeta^{4^k-3^k-\binom{2^k}{2}}(2s)\zeta^{\ell_3}(3s)\zeta^{\ell_4}(4s)\cdots\zeta^{\ell_n}(ns)}.$$

From Equation (3), we see that

$$\ell_m = \sum_{r=0}^{2^k}(-1)^r\binom{2^k}{r}((m+1)-r)^k \qquad 2 \leq m \leq n.$$

Theorem 5. *For any fixed integer $k \geq 2$, $\ell_m = 0$ for $m \geq 2^k - 1$. Therefore, $\tilde{G}_{k,n}(s)$ converges absolutely for $\sigma > \frac{1}{n}$ for any fixed $k \geq 2$. In particular, for $k = 2$ we have*

$$F_2(s) = \frac{\zeta^4(s)}{\zeta(2s)}\tilde{G}_{2,2}(s),$$

where $\tilde{G}_{2,2}(s)$ is a Dirichlet series that converges absolutely in the half-plane $\Re(s) > \frac{1}{n}$ for any positive integer n.

In order to prove the theorem stated above, we first require the following lemmas. Similar results related to this lemma can be found in [8, Sec. 1.6].

Lemma 9. *Let $n \in \mathbb{N}$ and $0 \leq \ell < n$. Then*

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^{\ell} = 0.$$

Proof. Let

$$f(x) = (1-x)^n = \sum_{r=0}^{n} \binom{n}{r} (-x)^r.$$

By taking $x = 1$, we have $\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$. Now,

$$\begin{aligned} f_1(x) :=& x\frac{df}{dx} = \sum_{r=0}^{n} \binom{n}{r} (-x)^r \, r, \\ f_2(x) :=& x\frac{df_1}{dx} = \sum_{r=0}^{n} \binom{n}{r} (-x)^r \, r^2, \\ &\vdots \\ f_i(x) :=& x\frac{df_{i-1}}{dx} = \sum_{r=0}^{n} \binom{n}{r} (-x)^r \, r^i. \end{aligned}$$

For each $i \le n-1$, $(1-x)$ divides $x\frac{df_i}{dx}$, and hence $f_i(1) = 0$ for all $i \le n-1$. Therefore, we have

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \, r^\ell = 0, \qquad \text{for all} \;\; \ell < n.$$

$\square$

Corollary 1. *Let $n \in \mathbb{N}$ and $0 \le \ell < n$. Then*

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \, (r-1)^\ell = 0.$$

Proof. Since $(r-1)^\ell$ is a polynomial of degree ℓ in r, using Lemma 9 we obtain

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \, (r-1)^\ell = 0.$$

$\square$

Proof of Theorem 5. First, we consider the case $n = 2^k - 1$. In this case, we have

$$\begin{aligned} \sum_{r=0}^{2^k-1} (-1)^r \binom{2^k}{r} (2^k - r)^k &= \sum_{r=0}^{2^k-1} \left[(-1)^r \binom{2^k}{r} \Big(\sum_{l=0}^{k} \binom{k}{l} (-1)^l (2^k)^{k-l} r^l \Big) \right] \\ &= \sum_{l=0}^{k} \left[(-1)^l (2^k)^{k-l} \binom{k}{l} \Big(\sum_{r=0}^{2^k-1} (-1)^r \binom{2^k}{r} r^l \Big) \right]. \end{aligned} \tag{4}$$

By Lemma 9, we have

$$\sum_{r=0}^{2^k}(-1)^r\binom{2^k}{r}r^l = 0.$$

Separating the last term, we obtain

$$\sum_{r=0}^{2^k-1}(-1)^r\binom{2^k}{r}r^l = -(2^k)^l.$$

Using this relation in Equation (4), we get

$$\begin{aligned}\sum_{r=0}^{2^k-1}(-1)^r\binom{2^k}{r}(2^k-r)^k &= \sum_{l=0}^{k}(-1)^l(2^k)^{k-l}\binom{k}{l} \times -(2^k)^l\\ &= -(2^k)^k\sum_{l=0}^{k}(-1)^l\binom{k}{l}\\ &= 0.\end{aligned}$$

Now, when $n \geq 2^k$, we have

$$\begin{aligned}&\sum_{r=0}^{2^k}(-1)^r\binom{2^k}{r}((n+1)-r)^k\\ &\qquad= \sum_{r=0}^{2^k}\left[(-1)^r\binom{2^k}{r}\Big(\sum_{l=0}^{k}\binom{k}{l}(-1)^l n^{k-l}(r-1)^l\Big)\right]\\ &\qquad= \sum_{l=0}^{k}\left[(-1)^l n^{k-l}\binom{k}{l}\Big(\sum_{r=0}^{2^k}(-1)^r\binom{2^k}{r}(r-1)^l\Big)\right].\end{aligned} \tag{5}$$

By Corollary 1, we have

$$\sum_{r=0}^{2^k}(-1)^r\binom{2^k}{r}(r-1)^l = 0.$$

Using this identity in Equation (5), we get

$$\sum_{r=0}^{2^k}(-1)^r\binom{2^k}{r}((n+1)-r)^k = 0.$$

Hence, $\ell_n = 0$ for all $n \geq 2^k - 1$. □

4. Proof of the Main Theorems

Proof of Theorem 1. We choose $T = \pm T_1^{\star} \in (U, 2U)$ such that Lemma 7 is satisfied. Applying Perron's formula to $F_2(s)$, we obtain

$$\sum_{n\leq x} A^2(n) = \frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} F_2(s)\frac{x^s}{s}\,ds + O\Big(\frac{x(\log x)^2}{T}\Big).$$

We move the line of integration to $\Re(s) = \frac{1}{2}$. Let R be the rectangle with the vertices $\frac{1}{2} \pm iT$ and $1 + \frac{1}{\log x} \pm iT$. In the rectangle R, the function $F_2(s)\frac{x^s}{s}$ has a pole of multiplicity 4 at $s = 1$. Thus, by using Cauchy's residue theorem on R, we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} F_2(s)\frac{x^s}{s}\,ds = M_2(x) + \frac{1}{2\pi i}\Bigg\{ &-\int_{1+\frac{1}{\log x}+iT}^{\frac{1}{2}+iT} - \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} \\ &-\int_{\frac{1}{2}-iT}^{1+\frac{1}{\log x}-iT}\Bigg\} F_2(s)\frac{x^s}{s}\,ds,\end{aligned}$$

where $M_2(x)$ is the main term, given by the residue of $F_2(s)\frac{x^s}{s}$ at $s = 1$.

Now we begin by estimating the horizontal contribution $C_{H,2}$ along the line from $\frac{1}{2} + iT$ to $1 + \frac{1}{\log x} + iT$. We assume that $T \geq 10$, and it follows that

$$\begin{aligned}|C_{H,2}| &\ll \Big|\int_{\frac{1}{2}+iT}^{1+\frac{1}{\log x}+iT} F_2(s)\frac{x^s}{s}\,ds\Big| \\ &\ll \int_{\frac{1}{2}+iT}^{1+\frac{1}{\log x}+iT} \Big|\frac{\zeta^4(s)}{\zeta(2s)}\tilde{G}_{2,2}(s)\frac{x^s}{s}\Big|\,ds \\ &\ll \int_{\frac{1}{2}+iT}^{1+\frac{1}{\log x}+iT} \frac{|\zeta(s)|^4}{|\zeta(2s)|}|\tilde{G}_{2,2}(s)|\frac{|x^s|}{|s|}\,ds.\end{aligned}$$

For $\sigma > \frac{1}{2}$, $|\tilde{G}_{2,2}(s)|$ is bounded, and $\frac{1}{|\zeta(2s)|} \ll \log(|T| + 10)$. Moreover, by Lemma 7, for $\frac{1}{2} \leq \sigma \leq 1 + \frac{1}{\log x}$, we have

$$|\zeta(s)|^4 \ll \exp(C_2(\log\log T)^2).$$

Therefore,

$$|C_{H,2}| \ll (\log T).\exp(C_2(\log\log T)^2)\int_{\frac{1}{2}}^{1+\frac{1}{\log x}} \frac{|x^{\sigma+iT}|}{|\sigma + iT|}\,d\sigma,$$

from which it follows that

$$\begin{aligned}|C_{H,2}| &\ll (\log T).\exp(C_2(\log\log T)^2)\frac{x^{1+\frac{1}{\log x}}}{T}\\ &\ll \frac{xx^{\frac{1}{\log x}}}{T}\exp(C_2(\log\log T)^2)\\ &\ll \frac{x}{T}\exp(C_2(\log\log x)^2),\end{aligned}$$

since our choice of T is going to be a power of x.

Now we estimate the left vertical contribution $C_{V,2}$ of $F_2(s)\frac{x^s}{s}$ in absolute value, that is,

$$\begin{aligned}|C_{V,2}| &\leq \Big|\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} F_2(s)\frac{x^s}{s}\,ds\Big|\\ &\ll \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\frac{|\zeta(s)|^4}{|\zeta(2s)|}\frac{|x^s|}{|s|}\,ds\\ &\ll \int_{-T}^{T}\frac{|\zeta(\frac{1}{2}+it)|^4}{|\zeta(1+2it)|}\frac{x^{\frac{1}{2}}}{t}\,dt\\ &\ll x^{\frac{1}{2}}+x^{\frac{1}{2}}\int_{10}^{T}\frac{|\zeta(\frac{1}{2}+it)|^4}{|\zeta(1+2it)|}\frac{1}{t}\,dt.\end{aligned}$$

Using Lemma 5, we get

$$|C_{V,2}| \ll x^{\frac{1}{2}}+x^{\frac{1}{2}}(\log T)^5.$$

Therefore, the total contributions of $F_2(s)\frac{x^s}{s}$ (i.e., the sum of the horizontal and vertical contribution and $O\Big(\frac{x(\log x)^2}{T}\Big)$), is given by

$$\begin{aligned}|C_{H,2}|+|C_{V,2}|+O\Big(\frac{x(\log x)^2}{T}\Big) \ll \frac{x}{T}&\exp(C_2(\log\log x)^2)+x^{\frac{1}{2}}(\log T)^5\\ &+O\Big(\frac{x(\log x)^2}{T}\Big).\end{aligned}$$

We choose $T=x^{\frac{1}{2}}$ to obtain an optimal estimate, so that

$$\begin{aligned}|C_{H,2}|+|C_{V,2}|+O\Big(\frac{x(\log x)^2}{T}\Big) = O(x^{\frac{1}{2}}&\exp(C_2(\log\log x)^2))+O(x^{\frac{1}{2}}(\log x^{\frac{1}{2}})^5)\\ &+O(x^{\frac{1}{2}}(\log x)^2).\end{aligned}$$

Hence,

$$\sum_{n\leq x}A^2(n)=M_2(x)+O(x^{\frac{1}{2}}\exp(C_2(\log\log x)^2)).$$

This proves Theorem 1. □

Proof of Theorem 2. We choose $T = \pm T_2^\star \in (U, 2U)$ satisfying Lemma 7. By applying Perron's formula to $F_3(s)$, we get

$$\sum_{n\leq x} A^3(n) = \frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} F_3(s)\frac{x^s}{s}\,ds + O\Big(\frac{x(\log x)^2}{T}\Big).$$

Now, we shift the line of integration to $\Re(s) = \frac{5}{8}$. Let R be the rectangle with the vertices $\frac{5}{8} \pm iT$ and $1 + \frac{1}{\log x} \pm iT$. In the rectangle R, $F_3(s)\frac{x^s}{s}$ has a pole of multiplicity 8 at $s = 1$. Now, by using Cauchy's residue theorem on the rectangle R, we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} F_3(s)\frac{x^s}{s}\,ds = M_3(x) + \frac{1}{2\pi i}\Bigg\{ &-\int_{1+\frac{1}{\log x}+iT}^{\frac{5}{8}+iT} - \int_{\frac{5}{8}+iT}^{\frac{5}{8}-iT} \\ &- \int_{\frac{5}{8}-iT}^{1+\frac{1}{\log x}-iT}\Bigg\} F_3(s)\frac{x^s}{s}\,ds,\end{aligned}$$

where $M_3(x)$ is the main term (i.e., the residue of $F_3(s)\frac{x^s}{s}$) at $s = 1$.

First, we estimate the horizontal contribution $C_{H,3}$ of $F_3(s)\frac{x^s}{s}$ (i.e., sum of the contributions of the line integrals $\int_{1+\frac{1}{\log x}+iT}^{\frac{5}{8}+iT} F_3(s)\frac{x^s}{s}\,ds$ and $\int_{\frac{5}{8}-iT}^{1+\frac{1}{\log x}-iT} F_3(s)\frac{x^s}{s}\,ds$ in absolute value) is given by

$$\begin{aligned}|C_{H,3}| &\ll \Big|\int_{\frac{5}{8}+iT}^{1+\frac{1}{\log x}+iT} F_3(s)\frac{x^s}{s}\,ds\Big| \\ &\ll \int_{\frac{5}{8}+iT}^{1+\frac{1}{\log x}+iT} \Big|\frac{\zeta^8(s)}{\zeta^9(2s)}\tilde{G}_{3,2}(s)\frac{x^s}{s}\,ds\Big| \\ &\leq \int_{\frac{5}{8}+iT}^{1+\frac{1}{\log x}+iT} \frac{|\zeta(s)|^8}{|\zeta(2s)|^9}|\tilde{G}_{3,2}(s)|\frac{|x^s|}{|s|}\,ds.\end{aligned}$$

For $\sigma > \frac{5}{8}$, $|\tilde{G}_{3,2}(s)|$ and $\frac{1}{|\zeta(2s)|^9}$ are bounded, and hence

$$\begin{aligned}|C_{H,3}| &\ll \int_{\frac{5}{8}}^{1+\frac{1}{\log x}} |\zeta(\sigma+iT)|^8\frac{x^\sigma}{T}\,d\sigma \\ &\ll \frac{x^{1+\frac{1}{\log x}}}{T}\int_{\frac{5}{8}}^{1+\frac{1}{\log x}} |\zeta(\sigma+iT)|^8\,d\sigma.\end{aligned}$$

By using Lemma 7, we get

$$\begin{aligned}|C_{H,3}| &\ll \frac{x\,x^{\frac{1}{\log x}}}{T}\exp(C_3(\log\log T)^2) \\ &\ll \frac{x}{T}\exp(C_3(\log\log x)^2),\end{aligned}$$

as our choice of T is going to be a power of x.

Now we estimate the vertical contribution $C_{V,3}$ in absolute value, that is,

$$\begin{aligned}|C_{V,3}| &\leq \Big| \int_{\frac{5}{8}-iT}^{\frac{5}{8}+iT} F_3(s)\frac{x^s}{s}\,ds\Big| \\ &\ll \int_{\frac{5}{8}-iT}^{\frac{5}{8}+iT} \frac{|\zeta(s)|^8}{|\zeta(2s)|^9}|\tilde{G}_{3,2}(s)|\frac{|x^s|}{|s|}\,ds \\ &\ll \int_{-T}^{T} \frac{|\zeta(\frac{5}{8}+it)|^8}{|\zeta(\frac{5}{4}+2it)|^9}|\tilde{G}_{3,2}(\frac{5}{8}+it)|\frac{|x^{\frac{5}{8}+it}|}{|\frac{5}{8}+it|}\,dt \\ &\ll x^{\frac{5}{8}} + x^{\frac{5}{8}} \int_{\frac{T}{2}}^{T} \frac{|\zeta(\frac{5}{8}+it)|^8}{|\zeta(\frac{5}{4}+2it)|^9}\,|\tilde{G}_{3,2}(\frac{5}{8}+iT)|\,\frac{1}{t}\,dt.\end{aligned}$$

As $\frac{5}{8} > \frac{1}{3}$, both $|\tilde{G}_{3,2}(\frac{5}{8}+iT)|$ and $\frac{1}{|\zeta\left(\frac{5}{4}+2it\right)|^9}$ are bounded. Hence, we have

$$|C_{V,3}| \ll x^{\frac{5}{8}} + x^{\frac{5}{8}} \int_{\frac{T}{2}}^{T} \frac{|\zeta(\frac{5}{8}+it)|^8}{t}\,dt.$$

By using Lemma 2 and partial integration, we obtain

$$|C_{V,3}| \ll x^{\frac{5}{8}}(\log x)^{39}.$$

Therefore, in total, we have

$$\begin{aligned}|C_{H,3}| + |C_{V,3}| + O\Big(\frac{x(\log x)^2}{T}\Big) &\ll \frac{x}{T}\exp(C_3(\log\log x)^2) + x^{\frac{5}{8}}(\log x)^{39} \\ &\quad + O\Big(\frac{x(\log x)^2}{T}\Big).\end{aligned}$$

To get an optimal estimate, we choose $T = x^{\frac{3}{8}}$, so that we obtain

$$\begin{aligned}|C_{H,3}| + |C_{V,3}| + O\Big(\frac{x(\log x)^2}{T}\Big) &= O(x^{\frac{5}{8}}\exp(C_3(\log\log x)^2)) \\ &\quad + O(x^{\frac{5}{8}}(\log x)^{39}) + O(x^{\frac{5}{8}}(\log x)^2).\end{aligned}$$

Hence,

$$\sum_{n\leq x} A^3(n) = M_3(x) + O(x^{\frac{5}{8}}\exp(C_3(\log\log x)^2)).$$

This proves Theorem 2. □

Proof of Theorem 3. As before, we choose $T = \pm T_3^\star \in (U, 2U)$ satisfying Lemma 7. By applying Perron's formula to $F_4(s)$, we get

$$\sum_{n\leq x} A^4(n) = \frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} F_4(s)\frac{x^s}{s}\,ds + O\Big(\frac{x(\log x)^2}{T}\Big).$$

Now we move the line of integration to $\Re(s) = \frac{5}{7}$. Let R be the rectangle with vertices $\frac{5}{7} \pm iT$ and $1 + \frac{1}{\log x} \pm iT$. We note that the function $F_4(s)\frac{x^s}{s}$ has a pole of multiplicity 16 at $s = 1$. Thus, by using Cauchy's residue theorem on the rectangle R, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} F_4(s)\frac{x^s}{s}\,ds = M_4(x) + \frac{1}{2\pi i}\Bigg\{ &-\int_{1+\frac{1}{\log x}+iT}^{\frac{5}{7}+iT} - \int_{\frac{5}{7}+iT}^{\frac{5}{7}-iT} \\ &- \int_{\frac{5}{7}-iT}^{1+\frac{1}{\log x}-iT}\Bigg\} F_4(s)\frac{x^s}{s}\,ds,\end{aligned}$$

where $M_4(x)$ is the main term (i.e., the residue of $F_4(s)\frac{x^s}{s}$ at $s = 1$).

Now we estimate the horizontal contribution $C_{H,4}$ in absolute value:

$$\begin{aligned}|C_{H,4}| &\ll \Big|\int_{\frac{5}{7}+iT}^{1+\frac{1}{\log x}+iT} F_4(s)\frac{x^s}{s}\,ds\Big| \\ &\le \int_{\frac{5}{7}+iT}^{1+\frac{1}{\log x}+iT} \frac{|\zeta(s)|^{16}}{|\zeta(2s)|^{55}}|\tilde{G}_{4,2}(s)|\frac{|x^s|}{|s|}\,ds.\end{aligned}$$

We note that $|\tilde{G}_{4,2}(s)|$ and $\frac{1}{|\zeta(2s)|^{55}}$ are bounded for $\sigma \ge \frac{5}{7}$. Hence, we have

$$\begin{aligned}|C_{H,4}| &\ll \int_{\frac{5}{7}+iT}^{1+\frac{1}{\log x}+iT} |\zeta(s)|^{16}\frac{|x^s|}{|s|}\,ds \\ &\ll \int_{\frac{5}{7}}^{1+\frac{1}{\log x}} |\zeta(\sigma+iT)|^{16}\,\frac{x^\sigma}{T}\,d\sigma \\ &\ll \frac{x\,x^{\frac{1}{\log x}}}{T}\int_{\frac{5}{7}}^{1+\frac{1}{\log x}} |\zeta(\sigma+iT)|^{16}\,d\sigma.\end{aligned}$$

By Lemma 7, we have

$$\begin{aligned}|C_{H,4}| &\ll \frac{x\,x^{\frac{1}{\log x}}}{T}(\exp(C(\log\log T))^2)^{16} \\ &\ll \frac{x}{T}\exp(C_4(\log\log x)^2),\end{aligned}$$

as our choice of T is going to be a power of x.

The estimate of the vertical contribution $C_{V,4}$ in absolute value is given by

$$\begin{aligned}|C_{V,4}| &\leq \Big|\int_{\frac{5}{7}-iT}^{\frac{5}{7}+iT} F_4(s)\frac{x^s}{s}\,ds\Big|\\ &\ll \Big|\int_{\frac{5}{7}-iT}^{\frac{5}{7}+iT} \frac{\zeta^{16}(s)}{\zeta^{55}(2s)}\tilde{G}_{4,2}(s)\frac{x^s}{s}\,ds\Big|\\ &\ll \int_{-T}^{T}\frac{|\zeta(\frac{5}{7}+it)|^{16}}{|\zeta(\frac{10}{7}+2iT)|^{55}}|\tilde{G}_{4,2}(\frac{5}{7}+it)|\frac{|x^{\frac{5}{7}+it}|}{|\frac{5}{7}+it|}\,dt\\ &\ll \int_{-T}^{T}\frac{|\zeta(\frac{5}{7}+it)|^{16}}{|\zeta(\frac{10}{7}+2iT)|^{55}}|\tilde{G}_{4,2}(\frac{5}{7}+it)|\frac{x^{\frac{5}{7}}}{t}\,dt.\end{aligned}$$

As $|\tilde{G}_{4,2}(\frac{5}{7}+it)|$ and $\frac{1}{|\zeta(2s)|^{55}}$ are bounded for $\sigma > \frac{1}{3}$ and for $t \in \mathbb{R}$, we have

$$\begin{aligned}|C_{V,4}| &\ll \int_{-T}^{T}\frac{|\zeta(\frac{5}{7}+it)|^{16}}{t}x^{\frac{5}{7}}\,dt\\ &\ll x^{\frac{5}{7}} + x^{\frac{5}{7}}\int_0^T \frac{|\zeta(\frac{5}{7}+it)|^{16}}{t}\,dt\\ &\leq x^{\frac{5}{7}} + x^{\frac{5}{7}}\int_0^T \frac{|\zeta(\frac{5}{7}+it)|^{12}}{t}|\zeta(\frac{5}{7}+it)|^4\,dt, \quad |T| \geq 1.\end{aligned}$$

By using Lemmas 1 and 6, together with partial integration, we have

$$\begin{aligned}|C_{V,4}| &\ll x^{\frac{5}{7}}(1+T)^{\frac{52}{147}+\epsilon}\int_0^T \frac{|\zeta(\frac{5}{7}+it)|^{12}}{t}\,dt\\ &\ll_\epsilon x^{\frac{5}{7}}T^{\frac{52}{147}+2\epsilon}.\end{aligned}$$

The total contribution is given by

$$\begin{aligned}|C_{H,4}| + |C_{V,4}| + O\Big(\frac{x(\log x)^2}{T}\Big) \ll \frac{x}{T}\exp(C_4(\log\log x)^2) + x^{\frac{5}{7}}T^{\frac{52}{147}+2\epsilon}\\ + O\Big(\frac{x(\log x)^2}{T}\Big).\end{aligned}$$

To get an optimal value, we choose $T = x^{\frac{42}{199}}$ so that

$$|C_{H,4}| + |C_{V,4}| + O\Big(\frac{x(\log x)^2}{T}\Big) \ll x^{\frac{157}{199}+2\epsilon}.$$

Hence,

$$\sum_{n\leq x} A^4(n) = M_4(x) + O(x^{\frac{157}{199}+2\epsilon}).$$

This proves Theorem 3. $\square$

Proof of Theorem 4. Again, our choice of $T = \pm T_4^\star \in (U, 2U)$ satisfies Lemma 7. By applying Perron's formula to $F_k(s)$, we get

$$\sum_{n\le x} A^k(n) = \frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} F_k(s)\frac{x^s}{s}\,ds + O\Big(\frac{x(\log x)^2}{T}\Big).$$

Now we shift the line of integration to $\Re(s) = \frac{5}{7}$. Let R be the rectangle with vertices $\frac{5}{7} \pm iT$ and $1 + \frac{1}{\log x} \pm iT$. In R, $F_k(s)\frac{x^s}{s}$ has a pole of multiplicity 2^k at $s = 1$. Now, by applying Cauchy's residue theorem on the rectangle R, we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} F_k(s)\frac{x^s}{s}\,ds = M_k(x) + \frac{1}{2\pi i}\Bigg\{ &-\int_{1+\frac{1}{\log x}+iT}^{\frac{5}{7}+iT} - \int_{\frac{5}{7}+iT}^{\frac{5}{7}-iT} \\ &- \int_{\frac{5}{7}-iT}^{1+\frac{1}{\log x}-iT}\Bigg\} F_k(s)\frac{x^s}{s}\,ds,\end{aligned}$$

where $M_k(x)$ is the main term (i.e., $Res(F_k(s)\frac{x^s}{s})$ at $s = 1$).

First, we calculate the horizontal contribution $C_{H,k}$, which is given by the line integrals $\int_{1+\frac{1}{\log x}+iT}^{\frac{5}{7}+iT} F_k(s)\frac{x^s}{s}\,ds$ and $\int_{\frac{5}{7}-iT}^{1+\frac{1}{\log x}-iT} F_k(s)\frac{x^s}{s}\,ds$. We have

$$\begin{aligned}|C_{H,k}| &\ll \Big|\int_{\frac{5}{7}+iT}^{1+\frac{1}{\log x}+iT} F_k(s)\frac{x^s}{s}\,ds\Big| \\ &\ll \int_{\frac{5}{7}+iT}^{1+\frac{1}{\log x}+iT} \Big|\frac{\zeta^{2^k}(s)}{\zeta^{4^k-3^k-\binom{2^k}{2}}(2s)}\tilde{G}_{k,2}(s)\frac{x^s}{s}\,ds\Big| \\ &\ll \int_{\frac{5}{7}+iT}^{1+\frac{1}{\log x}+iT} \frac{|\zeta(s)|^{2^k}}{|\zeta(2s)|^{4^k-3^k-\binom{2^k}{2}}}|\tilde{G}_{k,2}(s)|\frac{|x^s|}{|s|}\,ds.\end{aligned}$$

Since $|\tilde{G}_k(s)|$ and $|\zeta(2s)|^{4^k-3^k-\binom{2^k}{2}}$ are bounded for any $\sigma \ge \frac{5}{7}$, we get

$$\begin{aligned}|C_{H,k}| &\ll \int_{\frac{5}{7}}^{1+\frac{1}{\log x}} |\zeta(\sigma+iT)|^{2^k}\frac{|x^{\sigma+iT}|}{|\sigma+iT|}\,d\sigma \\ &\ll \int_{\frac{5}{7}}^{1+\frac{1}{\log x}} |\zeta(\sigma+iT)|^{2^k}\frac{x^{\sigma}}{T} \\ &\ll \frac{x\,x^{\frac{1}{\log x}}}{T}\int_{\frac{5}{7}}^{1+\frac{1}{\log x}} |\zeta(\sigma+iT)|^{2^k}\,d\sigma.\end{aligned}$$

By using Lemma 7, for sufficiently large T, we have

$$\begin{aligned}|C_{H,k}| &\ll \frac{x\,x^{\frac{1}{\log x}}}{T}\exp(2^kC(\log\log T)^2) \\ &\ll \frac{x}{T}\exp(C_k(\log\log x)^2),\end{aligned}$$

where C_k is an effective positive constant depending on k, since our choice of T is going to be a power of x.

The contribution of the left vertical line in absolute value is given by

$$\begin{aligned}|C_{V,k}| &\le \Big|\int_{\frac{5}{7}-iT}^{\frac{5}{7}+iT} F_k(s)\frac{x^s}{s}\,ds\Big|\\ &\le \int_{\frac{5}{7}-iT}^{\frac{5}{7}+iT}\left|\frac{\zeta^{2^k}(s)}{\zeta^{4^k-3^k-\binom{2^k}{2}}(2s)}\tilde{G}_{k,2}(s)\frac{x^s}{s}\,ds\right|\\ &\ll \int_{-T}^{T}\frac{|\zeta(\frac{5}{7}+it)|^{2^k}}{|\zeta(\frac{10}{7}+2it)|^{4^k-3^k-\binom{2^k}{2}}}|\tilde{G}_{k,2}(\frac{5}{7}+it)|\frac{|x^{\frac{5}{7}+it}|}{|\frac{5}{7}+it|}\,dt.\end{aligned}$$

Note that $|\tilde{G}_{k,2}(\frac{5}{7}+it)|$ and $|\zeta(\frac{10}{7}+2it)|^{4^k-3^k-\binom{2^k}{2}}$ are convergent for all $\sigma > \frac{5}{7}$. Thus, we have

$$|C_{V,k}| \ll x^{\frac{5}{7}} + x^{\frac{5}{7}}\int_0^T \frac{|\zeta(\frac{5}{7}+it)|^{12}}{t}|\zeta(\frac{5}{7}+it)|^{2^k-12}\,dt.$$

By using Lemmas 1 and 6, along with partial integration, we get

$$|C_{V,k}| \ll x^{\frac{5}{7}} + x^{\frac{5}{7}}T^{\frac{13}{147}(2^k-12)+2\epsilon}.$$

Therefore, in total we have

$$\begin{aligned}|C_{H,k}| + |C_{V,K}| + O\Big(\frac{x(\log x)^2}{T}\Big) &\ll \frac{x}{T}\exp(C_k(\log\log x)^2)\\ &\qquad + x^{\frac{5}{7}}T^{\frac{13}{147}(2^k-12)+2\epsilon} + O\Big(\frac{x(\log x)^2}{T}\Big).\end{aligned}$$

To get an optimal value, we choose $T = x^{\frac{42}{13.2^k-9}}$, so that

$$\begin{aligned}|C_{H,k}| + |C_{V,K}| + O\Big(\frac{x(\log x)^2}{T}\Big) &= O(x^{(1-\frac{42}{13.2^k-9})}\exp(2^kC(\log\log x)^2))\\ &\quad + O(x^{(1-\frac{42}{13.2^k-9}+2\epsilon)}) + O\Big(x^{(1-\frac{42}{13.2^k-9})}(\log x)^2\Big).\end{aligned}$$

Hence,

$$\sum_{n\le x}A^k(n) = M_k(x) + O_k(x^{1+2\epsilon-\frac{42}{13.2^k-9}}).$$

This proves Theorem 4. □

Acknowledgements. The authors are thankful to the anonymous referee for a thorough reading of the manuscript and giving some fruitful suggestions which enhanced the presentation. Additionally, Ekta Soni would like to express her thankfulness to CSIR Fellowship (ID No: 35112590) for its financial support.

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