



THE DIOPHANTINE EQUATION $Q_n^x + Q_{n+1}^x = y^2$

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Abstract

We show that the Diophantine equation $Q_n^x + Q_{n+1}^x = y^2$, where Q_n is the n -th Pell–Lucas number, has only one nonnegative integer solution.

1. Introduction

The Pell–Lucas numbers sequence $(Q_n)_n$, the sequence of interest in this article, is the linear recurrence sequence of integers given by

$$Q_0 = 1, Q_1 = 1, Q_{n+2} = 2Q_{n+1} + Q_n, (n \geq 0); \quad (1)$$

see (A001333) and [2]. The first few terms are 1, 1, 3, 7, 17, 41, 99, 239, 577, 1393... They are closely related to the Pell numbers sequence $(P_n)_n$ (see A000129),

$$P_0 = 0, P_1 = 1, P_{n+2} = 2P_{n+1} + P_n, (n \geq 0), \quad (2)$$

whose first few terms are 0, 1, 2, 5, 12, 29, 70, 169, 408, 985...

Several authors have investigated particular cases of exponential Diophantine equations of the form $a^x + b^y = z^2$. The special case where a and b are two consecutive Fibonacci numbers and $y = x$, that is,

$$F_n^x + F_{n+1}^x = z^2 \quad (3)$$

was recently studied by Richard Taclay [5] who elegantly found all of its nonnegative integer solutions.

Motivated by the complete resolution of the Diophantine Equation (3) involving consecutive Fibonacci numbers, it is natural to ask if a similar behavior occurs within the Pell–Lucas sequence. As both sequences are defined by analogous

linear recurrence relations, investigating the Pell–Lucas case provides a deeper understanding of how these structural similarities influence the existence of integer solutions.

In light of these findings, and relying on Taclay’s approach [5] using only elementary methods, we study the Diophantine equation

$$Q_n^x + Q_{n+1}^x = z^2, \tag{4}$$

where $(Q_n)_n$ is the Pell–Lucas sequence (1).

Furthermore, the elementary techniques utilized in this work may be adapted to investigate similar exponential Diophantine equations involving other linear recurrence sequences, such as the Jacobsthal sequence or the Jacobsthal-Lucas sequence.

2. Preliminaries

Throughout the text, we will need the following identities relating Pell and Pell-Lucas numbers; see pages 122, 123 and 148 in [2]:

$$Q_n + Q_{n+1} = 2P_{n+1} \tag{5}$$

$$P_n + Q_n = P_{n+1} \tag{6}$$

$$P_{2n} = 2P_n Q_n \tag{7}$$

$$Q_n^2 + Q_{n+1}^2 = 2P_{2n+1} \tag{8}$$

$$Q_{n+1}Q_{n-1} - Q_n^2 = 2(-1)^{n-1}, \quad n \geq 1 \tag{9}$$

$$Q_{2n} = 2Q_n^2 - (-1)^n \tag{10}$$

$$Q_{2n} = Q_n^2 + 2P_n^2 \tag{11}$$

Note that, since $Q_0 = Q_1 = 1$, Identity (5) implies that every term Q_n is odd. The following three auxiliary results are required for the proof of our main theorem.

Lemma 1 ([6]). *Let p be an odd prime, a, b, c, k integers with $\gcd(a, b) = 1$ and $k \geq 2$. If*

$$a^p + b^p = c^k,$$

then $a + b = d^k$ or $a + b = p^{k-1}d^k$, for some integer d .

Theorem 1 ([4]). *The equation $x^n + y^n = z^2$ has no nontrivial primitive solutions for $n \geq 4$.*

In Lemma 3.1 of [3], it is shown that the only square term in the associated Pell sequence is $Q_1 = 1$. This immediately implies the following lemma.

Lemma 2. *The only square terms in the Pell–Lucas sequence are $Q_0 = 1$ and $Q_1 = 1$.*

It is well known that $\gcd(P_n, P_{n+1}) = 1$; see [2, Corollary 8.1]. We now establish two additional results regarding coprimality.

Lemma 3. *For every nonnegative integer n , $\gcd(P_n, Q_n) = 1$.*

Proof. Let $d = \gcd(P_n, Q_n)$. Then, $d \mid P_n$ and $d \mid Q_n$ and so $d \mid P_n + Q_n$. Identity (6), $P_n + Q_n = P_{n+1}$, implies that $d \mid P_{n+1}$. But $\gcd(P_n, P_{n+1}) = 1$ which means that $d = 1$. □

Lemma 4. *For every nonnegative integer n , $\gcd(Q_n, Q_{n+1}) = 1$.*

Proof. Since $Q_{n-1} = Q_{n+1} - 2Q_n$, Identity (9) yields $Q_{n+1}Q_{n-1} - Q_n^2 = 2(-1)^{n-1}$ if and only if $Q_{n+1}^2 - 2Q_nQ_{n+1} - Q_n^2 = 2(-1)^{n-1}$. Let $d = \gcd(Q_n, Q_{n+1})$. Then, $d \mid Q_n$ and $d \mid Q_{n+1}$ and so d divides each term on the left-hand side of the above equation. Therefore, $d \mid \pm 2$. Since every Pell–Lucas number is odd, we conclude that $d = 1$. □

3. Main Result

We may now state and prove our main result.

Theorem 2. *For any nonnegative integers (n, x, y) , the Diophantine equation*

$$Q_n^x + Q_{n+1}^x = y^2,$$

where Q_n is the n -th Pell–Lucas number, has only one solution, namely $(n, x, y) = (1, 1, 2)$.

Proof. In the case $n = 0$, the equation becomes $Q_0^x + Q_1^x = y^2$ if and only if $y^2 = 2$, which has no integer solutions. From now on we consider $n > 0$. The proof is by exhaustion of cases. From Theorem 1, we only have to consider $x \in \{0, 1, 2, 3\}$.

Case 1: $x = 0, n \geq 1$. The equation again yields $y^2 = 2$, which has no solutions in integers.

Case 2: $x = 1, n \geq 1$. Applying Identity (5), $Q_n + Q_{n+1} = 2P_{n+1}$, Equation (4) reduces to

$$2P_{n+1} = y^2. \tag{12}$$

Taking $n = 1$ we get $2P_2 = 4 = y^2$, which has the nonnegative solution $y = 2$. Therefore, $(n, x, y) = (1, 1, 2)$ is a solution of Equation (4). Write $m = n + 1$. We now look for solutions when $m \geq 3$. From $2P_m = y^2$, it follows that y must be even. Setting $y = 2t$, Equation (12) reduces to

$$P_m = 2t^2. \tag{13}$$

Suppose m is odd and $m \geq 3$. Let us consider Equation (13) modulo 8. On the right-hand side, we have: if $t = 2k$ is even, then $P_m = 2(2k)^2 = 8k^2 \equiv 0 \pmod{8}$, and if $t = 2k + 1$ is odd, then $P_m = 2(2k + 1)^2 = 8k^2 + 8k + 2 \equiv 2 \pmod{8}$. However, for odd m , a simple observation shows that the Pell numbers satisfy

$$P_m \equiv 1 \pmod{8} \text{ or } P_m \equiv 5 \pmod{8}.$$

Comparing these residues with those of the right-hand side of Equation (13), we conclude that no solutions exist for odd $m \geq 3$.

Now, Suppose $m = 2k$ is even and $k \geq 2$. By Identity (7), $P_{2k} = 2P_kQ_k$, and we have $P_{2k} = 2P_kQ_k = 2t^2$, that is,

$$P_kQ_k = t^2. \tag{14}$$

By Lemma 3, $\gcd(P_k, Q_k) = 1$. Hence, Equation (14) is equivalent to $P_k = r_k^2$ and $Q_k = s_k^2$, for some integers r_k, s_k . But the only Pell numbers which are perfect squares are $P_0 = 0, P_1 = 1$, and $P_7 = 169$; see [1]. On the other hand, $Q_0 = 1, Q_1 = 1$ and $Q_7 = 239$. Since we are considering indices $k \geq 2$ and $Q_7 = 239$ is not a perfect square, there are no solutions.

Case 3: $x = 2, n \geq 1$. Identity (8), $Q_n^2 + Q_{n+1}^2 = 2P_{2n+1}$, leads to the following form of Equation (4):

$$2P_{2n+1} = y^2,$$

that is, $P_m = 2t^2$ with $m = 2n + 1 \geq 3$. This is precisely Case 2 with m odd. Therefore, no integer solutions exist when $x = 2$.

Case 4: $x = 3, n \geq 1$. When $x = 3$ Equation (4) becomes

$$Q_n^3 + Q_{n+1}^3 = y^2. \tag{15}$$

Since $\gcd(Q_n, Q_{n+1}) = 1$ (Lemma 4), we conclude from Lemma 1 that either $Q_n + Q_{n+1} = d^2$ or $Q_n + Q_{n+1} = 3d^2$. Note that $Q_n + Q_{n+1} = d^2$ is the same as in case 2, which has a solution only when $n = 1$. However, for $n = 1$,

$$Q_1^3 + Q_2^3 = 28 = y^2$$

has no integer solutions.

Let us consider the case $Q_n + Q_{n+1} = 3d^2$. By Identity (5), this is equivalent to

$$2P_{n+1} = 3d^2. \tag{16}$$

Since $d = 2t$ must be even, Equation (16) can be written as

$$P_m = 6t^2, \quad m \geq 2, \tag{17}$$

where $m = n + 1$. Applying modulo 3, the right-hand side of (17) is

$$P_m = 6t^2 \equiv 0 \pmod{3}. \tag{18}$$

Next, we examine the Pell sequence modulo 3:

$$P_0 = 0, \quad P_1 = 1, \quad P_2 = 2, \quad P_3 \equiv 2 \pmod{3}, \quad P_4 \equiv 0 \pmod{3}, \quad P_5 \equiv 2 \pmod{3}$$

$$P_6 \equiv 1 \pmod{3}, \quad P_7 \equiv 1 \pmod{3}, \quad P_8 \equiv 0 \pmod{3}, \quad P_9 \equiv 1 \pmod{3}, \dots$$

By induction we may easily establish that $P_m \equiv 0 \pmod{3}$ only when $m \equiv 0 \pmod{4}$. Therefore, Equation (18) holds when $m = 4k$, with $k \geq 1$. We now prove that $P_{4k} = 6t^2$ has no solutions for integers $k \geq 1$ and $t \geq 1$ ($t = 0$ is also impossible for $k \geq 1$).

From Identity (7), we have $P_{4k} = 2P_{2k}Q_{2k} = 2(2P_kQ_k)Q_{2k} = 4P_kQ_kQ_{2k}$. Hence Equation (17) reduces to

$$2P_kQ_kQ_{2k} = 3t^2, \tag{19}$$

with $k \geq 1, t \geq 1$.

We claim that P_k, Q_k, Q_{2k} are pairwise coprime. In fact, by Lemma 3, $\gcd(P_k, Q_k) = 1$. Since $Q_{2k} = 2Q_k^2 - (-1)^k$ (Identity (10)), any common divisor of Q_k and Q_{2k} must divide ± 1 . Finally, $\gcd(P_k, Q_{2k}) = 1$ follows from Identity (11), $Q_{2k} = Q_k^2 + 2P_k^2$, because any common divisor of P_k and Q_{2k} must also divide Q_k .

From the above discussion, 3 must divide precisely one of P_k, Q_k , or Q_{2k} .

Subcase(a): Suppose $3 \mid P_k$. Then $k \equiv 0 \pmod{4}$, that is, $k = 4s$. Since $k \geq 1$ necessarily $s > 0$. Then, Equation (19) can be written as $2P_{4s}Q_{4s}Q_{8s} = 3t^2$. By coprimality and the fact that Q_n is odd for every n , the only possibility is that

$$P_{4s} = 6\alpha_s^2, \quad Q_{4s} = \beta_s^2, \quad Q_{8s} = \gamma_s^2;$$

for some $\alpha_s, \beta_s, \gamma_s \in \mathbb{N}$. Therefore, Q_{4s} and Q_{8s} are perfect squares. That is, $4s, 8s \in \{0, 1\}$ which implies $s = 0$. This is impossible because $s \geq 1$.

Subcase(b): Suppose $3 \mid Q_k$. Then $k \equiv 2 \pmod{4}$. In fact

$$Q_{k+2} = 2Q_{k+1} + Q_k \equiv -Q_{k+1} + Q_k \pmod{3}.$$

A straightforward computation and induction show that the sequence $Q_k \pmod{3}$ is periodic with period 8. Specifically, the values in one period are

$$Q_k \pmod{3} : 1, 1, 0, 1, 2, 2, 0, 2.$$

It follows that $Q_k \equiv 0 \pmod{3}$ if and only if $k = 8i + 2$ or $k = 8i + 6$. This condition may be summarized as $Q_k \equiv 0 \pmod{3}$ if and only if $k \equiv 2 \pmod{4}$. We may then write $2P_{4s+2}Q_{4s+2}Q_{8s+4} = 3t^2$. By an argument analogous to subcase (a), we conclude that

$$P_{4s+2} = 2\alpha_s^2, Q_{4s+2} = 3\beta_s^2, Q_{8s+4} = \gamma_s^2.$$

Hence, Q_{8s+4} is a perfect square, i.e., $8s + 4 \in \{0, 1\}$ which is impossible.

Subcase(c): Suppose $3 \mid Q_{2k}$. As in subcase (b), this happens when $2k \equiv 2 \pmod{4}$, that is, $k = 2s + 1$ is an odd number and $k \geq 1$.

We have $2P_{2s+1}Q_{2s+1}Q_{4s+2} = 3t^2$. The only possibility is that

$$P_{2s+1} = 2\alpha_s^2, Q_{2s+1} = \beta_s^2, Q_{8s+4} = 3\gamma_s^2.$$

Hence, Q_{2s+1} is a perfect square, i.e., $2s + 1 \in \{0, 1\}$ which implies $s = 0$. The equation $2P_1Q_1Q_2 = 3t^2$ if and only if $t^2 = 2$ has no integer solutions. Thus, the result follows. \square

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