EXTENDED CONTINUANTS AND BI-INFINITE PERIODIC CONTINUED FRACTIONS

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Abstract

The simple continued fraction (SCF) of a real quadratic irrational α is eventually periodic; the SCF of its conjugate has periodic terms which are α's taken in reverse order and cyclically permuted. Viewing continued fraction convergents as ratios of solutions to an order-2 linear recurrence equation, we investigate ways in which the conjugate pair's two sequences of partial quotients can be joined to form a bi-infinite integer sequence called a bi-infinite periodic continued fraction or BpCF. We construct BpCF convergents using extended nth continuants, in which n can be negative, and we show that familiar continuant identities (and one which deserves wider recognition) hold for all integer values of their parameters.

1. Introduction

Consider the quadratic irrational conjugates and simple continued fractions (SCFs)

$$\begin{aligned}\alpha_0 &= \tfrac{1}{7}(7+2\sqrt{42}) = [\overline{2,1,5,1}] \\ \overline{\alpha}_0 &= \tfrac{1}{7}(7-2\sqrt{42}) = [-1,6,\overline{1,2,1,5}]\,,\end{aligned} \tag{1}$$

where the vincula span repeating terms. From Serret's work in the nineteenth century (cf. [7, Theorem III]), we know that $\overline{\alpha}_0$'s period is the reversed and cyclically permuted period of α_0. Let us introduce aspects of directionality in Equations (1) by writing $\overline{\alpha}_0$'s SCF in reverse order; by replacing the vincula with arrows; and by pairing square brackets with parentheses, like half-open intervals:

$$\begin{aligned}\alpha_0 &= [\overrightarrow{2,1,5,1}) \\ \overline{\alpha}_0 &= (\overleftarrow{2,1,5,1},2,1,6,-1]\,.\end{aligned} \tag{2}$$

A sequence of convergents is associated with each continued fraction:

$$\begin{aligned}\overrightarrow{\mathsf{C}}\,(\alpha_0) &= [\![2,3,\tfrac{17}{6},\tfrac{20}{7},\tfrac{57}{20},\ldots)\!) \\ \overleftarrow{\mathsf{C}}\,(\overline{\alpha}_0) &= (\!(\ldots,-\tfrac{23}{27},-\tfrac{17}{20},-\tfrac{6}{7},-\tfrac{5}{6},-1]\!]\;;\end{aligned} \tag{3}$$

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here $\overline{\alpha}_0$'s convergents are in reverse order, and the delimiters match those in (2) but are doubled, to distinguish these sequences of rationals from (2)'s sequences of integers.

In this paper we explore ways in which conjugate quadratic irrational SCFs can be joined to make a bi-infinite sequence called a bi-infinite periodic continued fraction or BpCF. From α_0 and $\overline{\alpha}_0$ we may construct, for instance,

$$\begin{aligned} \overleftrightarrow{B}_0 &= (\overleftarrow{2,1,5,1}, 2, 1\,; 6, 0, -1, 1 \,|\, \overrightarrow{2,1,5,1}) \\ \overleftrightarrow{B}_1 &= (\overleftarrow{-2,-1,-5,-1}, -2, -1\,; -6, 1, 0 \,|\, \overrightarrow{2,1,5,1})\,, \end{aligned} \tag{4}$$

where the vertical bar $|$ separates terms of negative index on the left from those of nonnegative index on the right; between the semicolon and the bar are special "silent" terms; and the 4-tuples under the arrows repeat in the indicated directions.

To create a BpCF's bi-infinite sequence of convergents, we extend the definition of the nth continuant to include all integers n, and we construct continuants from any finite sequence of consecutively indexed terms. Key roles are played by Herzog's taxonomy of quadratic irrationals' SCFs [8], and Bennett's remarkable but unheralded symmetric identity for continuants [1].

Throughout, we consider continued fraction convergents to be "ratio[s] of solutions to a second-order recurrence equation" [10], where the numerator's initial conditions are linearly independent of the denominator's. Although we find that the general BpCF lacks a unique, "canonical" form, we nevertheless show that $\overleftrightarrow{B}_1$ is a viable BpCF, while $\overleftrightarrow{B}_0$ is not.

2. Preliminaries

The inverse of a matrix M is $\mathrm{inv}(M)$. For bi-infinite integer sequences $(x_i)_{i\in\mathbb{Z}}$ we assume that a zeroth term has been chosen; a vertical bar $|$ is inserted between x_{-1} and x_0. The modifier *eastbound*, and right-pointing harpoon arrows, indicate entities and processes involving nonnegative indices increasing to the right; *westbound*, and left-pointing arrows, apply when the indices are negative and decreasing to the left. We write infinite west- and eastbound integer sequences as $(x_{-i}]_{i\geq 1}$ and $[x_i)_{i\geq 0}$, respectively; $(x_0,\ldots,x_n)$ is a finite sequence, and (x_i) is a generic integer sequence. For infinite sequences of rational numbers, the delimiters are doubled: $(\!(y_{-i}]\!]_{i\geq 1}$, $[\![y_i)\!)_{i\geq 0}$, $(\!(y_0,\ldots,y_n)\!)$, and $(\!(y_i)\!)$ are, respectively, westbound, eastbound, finite, and generic sequences of rationals y_i.

2.1. Simple Continued Fractions (SCFs) and Quadratic Irrationals

Given a nonzero $\gamma \in \mathbb{R}$, the nth convergent of $\gamma = [x_i)_{i\geq 0}$ is

$$c_n = x_0 + \cfrac{1}{x_1 + \cfrac{1}{\ddots + \cfrac{1}{x_n}}} = \frac{p_n}{q_n} .$$

The x_i, traditionally known as "partial quotients", will here be called *terms*. SCFs of real numbers are assumed to be *EA-generated*, that is, computed via the Euclidean algorithm. Thus, $x_0 \geq 1$ when $\gamma > 1$; $x_0 = 0$ when $0 < \gamma < 1$; and if $\gamma < 0$, the only negative term is x_0. Note that all EA-generated SCFs are eastbound. The eastbound sequences of convergents for α and $\overline{\alpha}$ are, respectively,

$$\overrightarrow{\mathsf{C}}(\alpha) = \llbracket c_i \rrparenthesis_{i\geq 0} \quad \text{and} \quad \overrightarrow{\mathsf{C}}(\overline{\alpha}) = \llbracket d_i \rrparenthesis_{i\geq 0} . \tag{5}$$

We assume $\overline{\alpha} < \alpha$; the latter's EA-generated SCF is

$$\alpha = [a_0, a_1, \ldots, a_{r-1}, \overrightarrow{b_0, b_1, \ldots, b_{s-1}}) ,$$

where the a_i and b_i are positive integers, except $a_0 = 0$ if $0 < \alpha < 1$ and $a_0 < 0$ if $\alpha < 0$. The a_i are α's *nonperiodics*; there are r of them for $r \geq 1$, and none if $r = 0$. The b_i are α's *periodics*. The s-periodic conjugate $\overline{\alpha}$ has $\dot{r}$ nonperiodics and eastbound SCF

$$\overline{\alpha} = [\dot{a}_0, \dot{a}_1, \ldots, \dot{a}_{\dot{r}-1}, \overrightarrow{\dot{b}_0, \dot{b}_1, \ldots, \dot{b}_{s-1}}) .$$

2.2. Raw Rationals

It will be convenient to represent an ordered pair of integers (u, v) as an unreduced fraction $\frac{u}{v}$, called a *raw rational*. By this protocol, $\frac{-u}{v}$ and $\frac{u}{-v}$ are distinct from each other, but have the same value; likewise, $\frac{-u}{-v}$ and $\frac{u}{v}$ share the same value but are notationally (and, as lattice points in $\mathbb{Z}^2$, are geometrically) distinct. The value of a raw rational $x = \frac{u}{v}$ is $\mu(x)$, where $\mu : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Q}$ is a surjection such that

$$\mu(x) = \begin{cases} |u| \div |v| & \text{if } \operatorname{sgn}(u) = \operatorname{sgn}(v), \\ -(|u| \div |v|) & \text{if } \operatorname{sgn}(u) = -\operatorname{sgn}(v), \\ 0 & \text{if } u = 0 \text{ and } v \neq 0, \\ \text{undefined} & \text{if } v = 0 . \end{cases}$$

If $X = \llparenthesis x_i \rrparenthesis$ is a sequence of raw rationals, then $\mu(X) = \llparenthesis \mu(x_i) \rrparenthesis$. Considered as unusually-notated vector endpoints, raw rationals admit of scalar multiplication: if x is a raw rational and $c \in \mathbb{R}$, then $cx = \frac{cu}{cv}$, and $\mu(cx) = \mu(x)$.

In the second half of the paper we will want to compare pairs of sets of raw rational numbers. The following definitions use raw rationals $x_1 = \frac{u_1}{v_1}$ and $x_2 = \frac{u_2}{v_2}$, and sets X and Y of raw rationals.

Definition 1. Raw rationals x_1 and x_2 are *strictly equivalent*, written $x_1 \doteq x_2$, if $u_1 = u_2$ and $v_1 = v_2$. For *strictly equivalent sets* X and Y, written $X \doteq Y$, there exists a bijection $\theta : X \to Y$, such that for all $x \in X$, $x \doteq \theta(x)$.

Definition 2. Raw rationals x_1 and x_2 are *value equivalent* if $x_1 \not\doteq x_2$, but $\mu(x_1) = \mu(x_2)$. For *value equivalent sets* X and Y, there exists a bijection $\phi : X \to Y$, such that for all $x \in X$, $\mu(x) = \mu(\phi(x))$.

Any expression below incorporating a horizontal fraction bar is to be considered a raw rational; the notation x/y indicates an ordinary rational number.[1]

3. Extended Continuants

For $n \geq 0$, an nth convergent's numerator p_n is a function, the nth *continuant*, of the $(n+1)$-tuple $(x_0, x_1, \ldots, x_n)$. We abbreviate a common continuant notation (cf. [4]) as

$$p_n = K_n(x_0, x_1, \ldots, x_n) = K_n(\mathsf{x})_0^n \ ,$$

where n is the continuant's *order*. Similarly, the nth convergent's denominator q_n is the $(n-1)$st continuant of the n-tuple $(x_1, x_2, \ldots, x_n)$, written

$$q_n = K_{n-1}(\mathsf{x})_1^n.$$

In these expressions the orders of continuants and the indices of their input sequences exhibit a fixed relationship, which we choose to prioritize.

Continuant a, b, c Rule. In $K_a(\mathsf{x})_b^c$ we require that $a + b = c$.

Any one of the indices a, b, c can be recovered from the other two, but the use of all three helps in the checking of complicated continuant expressions.[2]

Of the several ways to compute $K_n(\mathsf{x})_0^n$, at least two involve matrices. Using 2×2 matrices, one has (c.f. [2, Lemma 2.8])

$$\vec{M}_{0,n} = \prod_{i=0}^{n} \begin{bmatrix} x_i & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} K_n(\mathsf{x})_0^n & K_{n-1}(\mathsf{x})_0^{n-1} \\ K_{n-1}(\mathsf{x})_1^n & K_{n-2}(\mathsf{x})_1^{n-1} \end{bmatrix} . \qquad (6)$$

[1] Most fractions in this paper will have been convergents to an SCF at some point, and their numerators' and denominators' magnitudes will be relatively prime; as raw rationals, only the signs on top and bottom remain uncanceled or unsimplified.

[2] That the a, b, c rule need not hold in general is due to the fact that a continuant is unchanged if its input sequence is reversed. Our work does not require such reversals.

In the right-most matrix, the first column contains the numerator and denominator of eastbound convergent c_n, while the second column contains those of c_{n-1}.

Alternatively, $K_n(\mathsf{x})_0^n = \det \overrightarrow{D}_{0,n}$, where

$$\overrightarrow{D}_{0,n} = \begin{bmatrix} x_0 & 1 & 0 & \cdots & 0 \\ -1 & x_1 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & x_{n-1} & 1 \\ 0 & \cdots & 0 & -1 & x_n \end{bmatrix} \tag{7}$$

is tridiagonal (cf. [5]). Recursive definitions for p_n and q_n, $n \geq 0$, are

$$\begin{aligned} p_{-2} &= 0\,, \quad & p_{-1} &= 1\,, \quad & p_n &= x_n p_{n-1} + p_{n-2} \\ q_{-2} &= 1\,, \quad & q_{-1} &= 0\,, \quad & q_n &= x_n q_{n-1} + q_{n-2}\,. \end{aligned}$$

In our notation, p_n becomes

$$p_n = K_n(\mathsf{x})_0^n = \begin{cases} 0 & \text{if } n = -2, \\ 1 & \text{if } n = -1, \\ x_n K_{n-1}(\mathsf{x})_0^{n-1} + K_{n-2}(\mathsf{x})_0^{n-2} & \text{if } n \geq 0. \end{cases} \tag{8}$$

In [4, pp. 302–303], recurrence (8) is called "right-adjusting" (designated here as RA). An equivalent, "left-adjusting" (LA) recurrence [4, Equation 6.132] has the same initial conditions and

$$K_n(\mathsf{x})_0^n = x_0 K_{n-1}(\mathsf{x})_1^n + K_{n-2}(\mathsf{x})_2^n\,, \quad n \geq 0\,. \tag{9}$$

3.1. Extended nth Continuants

Given $n \geq 0$, $h \in \mathbb{Z}$, and a bi-infinite sequence of integers $(x_i)_{i \in \mathbb{Z}}$, the extended nth continuant is a function of consecutive terms $(x_h, x_{h+1}, \ldots, x_{h+n})$, and is abbreviated as

$$K_n(x_h, x_{h+1}, \ldots, x_{h+n}) = K_n(\mathsf{x})_h^{h+n}\,.$$

The extended continuant is derived from Equations (8) and (9) by what amounts to a relabeling process: substitute x'_{i+h} for x_i, $i \in (0, 1, \ldots, n)$, then replace x' with x. Because (x_i) is bi-infinite, we are not concerned that h or $h+n$ may be negative.

Definition 3. Given an integer $n \geq -2$, $h \in \mathbb{Z}$, and a bi-infinite integer sequence $(x_i)_{i \in \mathbb{Z}}$, the *extended nth continuant* is

$$K_n(\mathsf{x})_h^{h+n} = \begin{cases} 0 & \text{if } n = -2, \\ 1 & \text{if } n = -1, \\ x_{h+n} K_{n-1}(\mathsf{x})_h^{h+n-1} + K_{n-2}(\mathsf{x})_h^{h+n-2} & \text{if } n \geq 2 \text{ (RA)}, \\ x_h K_{n-1}(\mathsf{x})_{h+1}^{h+n} + K_{n-2}(\mathsf{x})_{h+2}^{h+n} & \text{if } n \geq 2 \text{ (LA)}. \end{cases}$$

The extended nth continuant's matrix formulations include the 2×2 construction

$$\vec{M}_{h,h+n} = \prod_{i=h}^{h+n} \begin{bmatrix} x_i & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} K_n(\mathsf{x})_h^{h+n} & K_{n-1}(\mathsf{x})_h^{h+n-1} \\ K_{n-1}(\mathsf{x})_{h+1}^{h+n} & K_{n-2}(\mathsf{x})_{h+1}^{h+n-1} \end{bmatrix} ,$$

and the determinant $\det \vec{D}_{h,h+n}$, where $\vec{D}_{h,h+n}$ is tridiagonal like the matrix in (7), but with diagonal elements $(x_h, x_{h+1}, \ldots, x_{h+n})$.

3.2. Index Shifts and Order Shifts

The substitution $x_i = x'_{i+h}$, $i \in (0, 1, \ldots, n)$, and mapping $x' \to x$ amounts to a translation of the indices, which we call an *index shift*, denoted by $i \mapsto i + h$. The same idea applies to the continuant orders in Definition 3. Given $d \geq 0$, one can make the substitution $n = n' + d$ throughout the RA version, for instance, to obtain

$$K_{n'+d}(\mathsf{x})_h^{h+n'+d} = x_{h+n'+d} K_{n'+d-1}(\mathsf{x})_h^{h+n'+d-1} + K_{n'+d-2}(\mathsf{x})_h^{h+n'+d-2} ;$$

one can then replace n' with n. In both the RA and LA cases the a, b, c rule is satisfied in every continuant. The process of substitution in the order, followed by symbol replacement, is an *order shift*, written $n \mapsto n + d$.

Order and index shifts must be applied to all continuants and input sequences in an equation.

3.3. Extended $-n$th Continuants

Like Fibonacci numbers, Chebyshev polynomials, and other sequences generated by order-2 linear recurrence formulas, extended nth continuants can be defined for negative n by solving the defining recurrence equation for the term of least index. Doing so with the RA and LA recurrences in Definition 3, and applying the order shift $n \mapsto n + 2$ (along with the index shift $h \mapsto h - 2$ in the LA case), we have

$$K_n(\mathsf{x})_h^{h+n} = \begin{cases} -x_{h+n+2} K_{n+1}(\mathsf{x})_h^{h+n+1} + K_{n+2}(\mathsf{x})_h^{h+n+2} & \text{(RA)} \\ -x_{h-2} K_{n+1}(\mathsf{x})_{h-1}^{h+n} + K_{n+2}(\mathsf{x})_{h-2}^{h+n} & \text{(LA)} \end{cases} ,$$

both of which accommodate negative n. In the LA case,

$$K_{-n}(\mathsf{x})_h^{h-n} = -x_{h-2} K_{-n+1}(\mathsf{x})_{h-1}^{h-n} + K_{-n+2}(\mathsf{x})_{h-2}^{h-n}$$

for $n \geq 0$, so that, for instance,

$$\begin{aligned} K_{-1}(\mathsf{x})_h^{h-1} &= -x_{h-2} K_0(\mathsf{x})_{h-1}^{h-1} + K_1(\mathsf{x})_{h-2}^{h-1} \\ &= -x_{h-2}(x_{h-1}) + (1 + x_{h-2} x_{h-1}) = 1 \\ K_{-2}(\mathsf{x})_h^{h-2} &= -x_{h-2} K_{-1}(\mathsf{x})_{h-1}^{h-2} + K_0(\mathsf{x})_{h-2}^{h-2} = -x_{h-2}(1) + x_{h-2} = 0 \,. \end{aligned} \tag{10}$$

Definition 4. Given an integer $n \geq 1$, $h \in \mathbb{Z}$, and a bi-infinite integer sequence $(x_i)_{i\in\mathbb{Z}}$, the *extended* $-n$*th continuant* is

$$K_{-n}(\mathsf{x})_h^{h-n} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n = 2, \\ -x_{h-n+2}K_{-n+1}(\mathsf{x})_h^{h-n+1} + K_{-n+2}(\mathsf{x})_h^{h-n+2} & \text{if } n \geq 3 \text{ (RA)}, \\ -x_{h-2}K_{-n+1}(\mathsf{x})_{h-1}^{h-n} + K_{-n+2}(\mathsf{x})_{h-2}^{h-n} & \text{if } n \geq 3 \text{ (LA)}. \end{cases}$$

Note that $K_{-2}(\mathsf{x})_h^{h-2} = 0$ and $K_{-1}(\mathsf{x})_h^{h-1} = 1$ in each of Definitions 3 and 4. The following identities may be proved directly from Definition 4.

Lemma 1. *For* $h \in \mathbb{Z}$,

$$K_{-3}(\mathsf{x})_h^{h-3} = 1\,, \qquad K_{-4}(\mathsf{x})_h^{h-4} = -x_{h-2}\,,$$
$$K_{-5}(\mathsf{x})_h^{h-5} = 1 + x_{h-3}x_{h-2}\,, \qquad K_{-6}(\mathsf{x})_h^{h-6} = -(x_{h-4} + x_{h-2} + x_{h-4}x_{h-3}x_{h-2})\,.$$

The matrix product formula (6) has an analog for $-n$th continuants.

Proposition 1. *For* $n \geq 1$,

$$\overleftarrow{M}_{-n,-1} = \prod_{i=n}^{1} \begin{bmatrix} 0 & 1 \\ 1 & -x_{-i} \end{bmatrix} = \begin{bmatrix} K_{-n-1}(\mathsf{x})_0^{-n-1} & K_{-n-2}(\mathsf{x})_1^{-n-1} \\ K_{-n-2}(\mathsf{x})_0^{-n-2} & K_{-n-3}(\mathsf{x})_1^{-n-2} \end{bmatrix}. \tag{11}$$

As n increases, the product expands westward; matrix-form term x_{-n-1} is multiplied on the left of $\overleftarrow{M}_{-n,-1}$. Following Equation (6) we observed that the first and second columns of the eastbound matrix product $\overrightarrow{M}_{0,n}$ contain the numerators and denominators of c_n and c_{n-1}, respectively. If we call $⟅c_{-i}⟧_{i\geq 1}$ the sequence of westbound convergents associated with $(x_i)_{i\in\mathbb{Z}}$, then the first and second rows of the westbound product $\overleftarrow{M}_{-n,-1}$ comprise the numerators and denominators of c_{-n-1} and c_{-n-2}, respectively.

Proof of Proposition 1. Take $n = 2$ as the initial value for induction. From Lemma 1, we verify

$$\overleftarrow{M}_{-2,-1} = \prod_{i=2}^{1} \begin{bmatrix} 0 & 1 \\ 1 & -x_{-i} \end{bmatrix} = \begin{bmatrix} 1 & -x_{-1} \\ -x_{-2} & 1 + x_{-2}x_{-1} \end{bmatrix} = \begin{bmatrix} K_{-3}(\mathsf{x})_0^{-3} & K_{-4}(\mathsf{x})_1^{-3} \\ K_{-4}(\mathsf{x})_0^{-4} & K_{-5}(\mathsf{x})_1^{-4} \end{bmatrix}.$$

Now assuming the validity of formula (11) for $2 \leq n \leq N$, we consider the case $n = N + 1$:

$$\overleftarrow{M}_{-N-1,-1} = \prod_{i=N+1}^{1} \begin{bmatrix} 0 & 1 \\ 1 & -x_{-i} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -x_{-N-1} \end{bmatrix} \begin{bmatrix} K_{-N-1}(\mathsf{x})_0^{-N-1} & K_{-N-2}(\mathsf{x})_1^{-N-1} \\ K_{-N-2}(\mathsf{x})_0^{-N-2} & K_{-N-3}(\mathsf{x})_1^{-N-2} \end{bmatrix}$$

$$= \begin{bmatrix} K_{-N-2}(\mathsf{x})_0^{-N-2} & K_{-N-3}(\mathsf{x})_1^{-N-2} \\ -x_{-N-1}K_{-N-2}(\mathsf{x})_0^{-N-2} + K_{-N-1}(\mathsf{x})_0^{-N-1} & -x_{-N-1}K_{-N-3}(\mathsf{x})_1^{-N-2} + K_{-N-2}(\mathsf{x})_1^{-N-1} \end{bmatrix}$$

$$= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}.$$

By the RA version of Definition 4 with $h = 0$,

$$K_{-N-3}(\mathsf{x})_0^{-N-3} = -x_{-N-1}K_{-N-2}(\mathsf{x})_0^{-N-2} + K_{-N-1}(\mathsf{x})_0^{-N-1} = m_{21} \ .$$

Similarly, using $h = 1$,

$$K_{-N-4}(\mathsf{x})_1^{-N-3} = -x_{-N-1}K_{-N-3}(\mathsf{x})_1^{-N-2} + K_{-N-2}(\mathsf{x})_1^{-N-1} = m_{22} \ .$$

It follows that

$$\overleftarrow{M}_{-N-1,-1} = \begin{bmatrix} K_{-N-2}(\mathsf{x})_0^{-N-2} & K_{-N-3}(\mathsf{x})_1^{-N-2} \\ K_{-N-3}(\mathsf{x})_0^{-N-3} & K_{-N-4}(\mathsf{x})_1^{-N-3} \end{bmatrix} \ ,$$

which confirms the induction hypothesis for $N + 1$. □

In Section 3.5 we show how the $-n$th continuant may be computed as the determinant of a tridiagonal matrix.

3.4. Examples of Extended Continuants

From Definition 3 we have, for instance,

$$K_1(\mathsf{x})_0^1 = 1 + x_0x_1 = p_1 \ , \quad K_1(\mathsf{x})_5^6 = 1 + x_5x_6 \ , \quad K_1(\mathsf{x})_{-4}^{-3} = 1 + x_{-4}x_{-3} \ ,$$
$$K_2(\mathsf{b})_3^5 = b_3 + b_5 + b_3b_4b_5 \ , \qquad K_2(\mathsf{z})_{m-2}^m = z_{m-2} + z_m + z_{m-2}z_{m-1}z_m \ ,$$

while Definition 4 produces

$$K_{-5}(\mathsf{x})_7^2 = 1 + x_4x_5 \ , \quad K_{-6}(\mathsf{x})_3^{-3} = -(x_{-1} + x_1 + x_{-1}x_0x_1) \ ,$$
$$K_{-7}(\mathsf{t})_{-3}^{-10} = 1 + t_{-8}t_{-7} + t_{-8}t_{-5} + t_{-6}t_{-5} + t_{-8}t_{-7}t_{-6}t_{-5} \ .$$

Per the a, b, c rule (Section 3), in all cases $b \le c$ when $a > 0$, and $c \le b$ when $a < 0$.

3.5. Relationships Between Extended nth and $-n$th Continuants

Again like Fibonacci and Chebyshev sequences, extended $-n$th continuants are related to those of nonnegative order by sign changes and index and order shifts.

Theorem 1. *For $h, n \in \mathbb{Z}$,*

$$K_{-n}(\mathsf{x})_h^{h-n} = (-1)^{n+1}K_{n-4}(\mathsf{x})_{h-n+2}^{h-2} \ . \tag{12}$$

In particular, for $h = 0$ and $h = 1$,

$$K_{-n}(\mathsf{x})_0^{-n} = (-1)^{n+1}K_{n-4}(\mathsf{x})_{-n+2}^{-2}$$
$$K_{-n}(\mathsf{x})_1^{-n+1} = (-1)^{n+1}K_{n-4}(\mathsf{x})_{-n+3}^{-1} \ .$$

Proof. The proof is by induction, and we provide details of several index shifts. Suppose first that $n \geq 0$, and take $n = 0$ as the initial case. By Definition 3, the left-hand side of (12) is

$$K_0(\mathsf{x})_h^h = x_h \ . \tag{13}$$

On the right-hand side, we evaluate $K_{-4}(\mathsf{x})_{h+2}^{h-2}$ using the substitution $h = h' - 2$, the RA formula in Definition 4, and identities from Lemma 1:

$$\begin{aligned} K_{-4}(\mathsf{x})_{h+2}^{h-2} = K_{-4}(\mathsf{x})_{h'}^{h'-4} &= -x_{h'-2}K_{-3}(\mathsf{x})_{h'}^{h'-3} + K_{-2}(\mathsf{x})_{h'}^{h'-2} \\ &= -x_{h'-2}(1) + 0 = -x_h \ . \end{aligned} \tag{14}$$

Equations (13) and (14) verify the initial induction step:

$$K_0(\mathsf{x})_h^h = (-1)^{0+1}K_{-4}(\mathsf{x})_{h+2}^{h-2} \ .$$

Now assume the validity of identity (12) for $0 \leq n \leq N$. Begin with Definition 4's LA recurrence for $n = N + 1$:

$$K_{-(N+1)}(\mathsf{x})_h^{h-(N+1)} = -x_{h-2}K_{-(N+1)+1}(\mathsf{x})_{h-1}^{h-(N+1)} + K_{-(N+1)+2}(\mathsf{x})_{h-2}^{h-(N+1)} \ .$$

Substitute $h = h' + 2$, then apply the induction hypothesis (12), followed by the RA version of Definition 3:

$$\begin{aligned} K_{-(N+1)}(\mathsf{x})_{h'+2}^{h'-N+1} &= -x_{h'}K_{-N}(\mathsf{x})_{h'+1}^{h'-N+1} + K_{-(N-1)}(\mathsf{x})_{h'}^{h'-N+1} \\ &= -x_{h'}\left((-1)^{N+1}K_{N-4}(\mathsf{x})_{h'-N+3}^{h'-1}\right) + (-1)^N K_{N-5}(\mathsf{x})_{h'-N+3}^{h'-2} \\ &= (-1)^{N+2}\, K_{N-3}(\mathsf{x})_{h'-N+3}^{h'} \ . \end{aligned}$$

Finally, replace h' with $h - 2$ to obtain the desired

$$K_{-(N+1)}(\mathsf{x})_h^{h-(N+1)} = (-1)^{(N+1)+1}K_{(N+1)-4}(\mathsf{x})_{h-N+1}^{h-2} \ .$$

The induction hypothesis becomes

$$K_n(\mathsf{x})_h^{h+n} = (-1)^{-n+1}K_{-n-4}(\mathsf{x})_{h+n+2}^{h-2}$$

for $n < 0$, and is proved similarly. □

Index and order shifts $i \mapsto i+h$ and $n \mapsto n+d$, respectively, can now be justified for all $d, h, i, n \in \mathbb{Z}$.

We may also interpret an extended $-n$th continuant as the determinant of a tridiagonal matrix. For $h \in \mathbb{Z}$ and $n \geq 1$, let $\overrightarrow{D}_{h-n+2,h-2}$ be tridiagonal like (7) but with main diagonal $(\mathsf{x})_{h-n+2}^{h-2}$ comprising $n - 4$ terms. By Theorem 1,

$$\begin{aligned} (-1)^{n+1} \det \overrightarrow{D}_{h-n+2,h-2} &= (-1)^{n+1}K_{n-4}(\mathsf{x})_{h-n+2}^{h-2} \\ &= K_{-n}(\mathsf{x})_h^{h-n} \ . \end{aligned}$$

In this way, $(-1)^{n+1} \det \overrightarrow{D}_{h-n+2,h-2}$ represents a continuant of order $-n$.

3.6. A Symmetric Continuant Identity and its Corollaries

An exceptionally general continuant identity is derived in a little-known 1939 paper by Bennett.[3]

Theorem 2 ([1])**.** *If k, ℓ, m, n are nonnegative integers, then*

$$\begin{aligned}(-1)^{k+\ell} & K_{n-k-2}(\mathsf{x})_k^{n-2} \cdot K_{m-\ell-2}(\mathsf{x})_\ell^{m-2} \\ & + (-1)^{\ell+m} K_{n-\ell-2}(\mathsf{x})_\ell^{n-2} \cdot K_{k-m-2}(\mathsf{x})_m^{k-2} \\ & + (-1)^{m+k} K_{n-m-2}(\mathsf{x})_m^{n-2} \cdot K_{\ell-k-2}(\mathsf{x})_k^{\ell-2} = 0\ .\end{aligned} \tag{15}$$

As Bennett remarks of his original formulation, Equation (15) "repeats itself identically for all permutations of [the four ordinals] and is thus wholly symmetric in regard to [them], and entirely independent of any arithmetical inequalities affecting them."[4] It can also be further generalized.

Theorem 3. *Theorem 2 holds for all integers k, ℓ, m, n.*

Proof. We basically recap Bennett's original proof in a bi-infinite setting. Define a horizontally and vertically bi-infinite matrix $\mathbf{F} = (f_{i,j})_{i\in\mathbb{Z},j\in\mathbb{Z}}$, by

$$f_{i,j} = (-1)^i K_{j-i-2}(\mathsf{x})_i^{j-2}\ . \tag{16}$$

(We envision $\mathbf{F}$'s row indices increasing from $-\infty$ above to $+\infty$ below.) Next create $\mathbf{F}' = (f'_{i,j})$ by negating $\mathbf{F}$'s odd-indexed rows, to get

$$f'_{i,j} = \begin{cases} (-1)^i K_{j-i-2}(\mathsf{x})_i^{j-2} & \text{if } i \text{ is even,} \\ (-1)^{i+1} K_{j-i-2}(\mathsf{x})_i^{j-2} & \text{if } i \text{ is odd.} \end{cases} \tag{17}$$

Using Theorem 1, the elements of $\mathbf{F}'$ can be re-expressed as

$$f'_{i,j} = \begin{cases} (-1)^{j+1} K_{i-j-2}(\mathsf{x})_j^{i-2} & \text{if } i \text{ is even,} \\ (-1)^{j} K_{i-j-2}(\mathsf{x})_j^{i-2} & \text{if } i \text{ is odd.} \end{cases} \tag{18}$$

From (17) and (18) we have $f'_{i,j} = -f'_{j,i}$, i.e., $\mathbf{F}'$ is skew-symmetric. Furthermore,

$$\begin{aligned} |f'_{i,i-1}| &= \left|K_{-3}(\mathsf{x})_i^{i-3}\right| = 1 \\ |f'_{i,i}| &= \left|K_{-2}(\mathsf{x})_i^{i-2}\right| = 0 \\ |f'_{i,i+1}| &= \left|K_{-1}(\mathsf{x})_i^{i-1}\right| = 1\ , \end{aligned}$$

[3] As of this writing, MathSciNet reports that [1] has never been cited. It is written in a narrative style, more like a lecture's notes than as a research article; there are no formally stated theorems. As an historical aside, in 1890 G. T. Bennett was Cambridge's Senior Wrangler, the male student whose score on the Mathematical Tripos exams was highest; and at that time, though both sexes could take the exams, men received rankings and women did not. Nonetheless, Philippa Fawcett's score was 13% higher than Bennett's, and this caused something of a sensation around the world. Cf. [3], [6].

[4] What Bennett called "precontinuants" are in fact continuants of negative order; thus, the continuant orders in (15) can be negative.

which is to say that the subdiagonal of $\mathbf{F}'$ consists of 1s whose signs alternate, the diagonal comprises a bi-infinite sequence of zeroes, and the superdiagonal is the negative of the diagonally-reflected subdiagonal.

Both Definitions 3 and 4 express a continuant as a linear combination of two other continuants. For any $i, j, k, \ell, m, n \in \mathbb{Z}$, it follows that in

$$G_0 = \begin{bmatrix} f'_{i,\ell} & f'_{i,m} & f'_{i,n} \\ f'_{j,\ell} & f'_{j,m} & f'_{j,n} \\ f'_{k,\ell} & f'_{k,m} & f'_{k,n} \end{bmatrix} ,$$

any row can be expressed as a linear combination of the other two, and therefore $\det G_0 = 0$. In particular, if $i = \ell$ and $j = m$, G_0 becomes

$$G_1 = \begin{bmatrix} 0 & f'_{\ell,m} & f'_{\ell,n} \\ f'_{m,\ell} & 0 & f'_{m,n} \\ f'_{k,\ell} & f'_{k,m} & f'_{k,n} \end{bmatrix} ,$$

whose determinant is

$$\det G_1 = -f'_{k,n} f'_{\ell,m} f'_{m,\ell} + f'_{k,m} f'_{\ell,n} f'_{m,\ell} + f'_{k,\ell} f'_{\ell,m} f'_{m,n} = 0 .$$

The further substitutions $f'_{m,\ell} = -f'_{\ell,m}$ and $f'_{k,m} = -f'_{m,k}$ yield

$$\det G_1 = \pm f'_{\ell,m} \left(f'_{k,n} f'_{\ell,m} + f'_{\ell,n} f'_{m,k} + f'_{k,\ell} f'_{m,n} \right) = 0 . \tag{19}$$

The sign of $f'_{\ell,m}$ in (19) is determined as follows:

1. if all of k, ℓ, m are even, then all elements of G_1 are of the form defined in (16) for the original matrix $\mathbf{F}$, and the sign of $f'_{\ell,m}$ is positive;

2. if two of k, ℓ, m are even, then one row of G_1 is from an odd-indexed row in $\mathbf{F}'$, its elements are the negatives of the form (16), and the sign of $f'_{\ell,m}$ is negative;

3. if one of k, ℓ, m is even, then two rows of G_1 are from odd-indexed rows in $\mathbf{F}'$, their elements are the negatives of (16), and the sign of $f'_{\ell,m}$ is positive;

4. if all of k, ℓ, m are odd, then all elements of G_1 are the negatives of (16)'s form, and the sign of $f'_{\ell,m}$ is negative.

This analysis shows that the effects of odd row indices on the signs of G_1's elements are accounted for without altering the signs in the second factor in (19), which, as a zero of the equation, gives the desired identity (15). □

Bennett also showed that Euler's general continuant identity (cf. [4, Equation 6.134]) is a corollary of Theorem 2. More generally, it is a corollary of Theorem 3.

Corollary 1. *For $h, k, \ell, m \in \mathbb{Z}$,*

$$K_{\ell+m}(\mathsf{x})_h^{\ell+m+h} \cdot K_{k-1}(\mathsf{x})_{\ell+h+1}^{k+\ell+h} - K_{m-1}(\mathsf{x})_{\ell+h+1}^{\ell+m+h} \cdot K_{k+\ell}(\mathsf{x})_h^{k+\ell+h} = (-1)^k K_{m-k-2}(\mathsf{x})_{k+\ell+h+2}^{\ell+m+h} \cdot K_{\ell-1}(\mathsf{x})_h^{\ell+h-1} \ .$$

Proof. The case $h = 0$ follows from Theorem 3 by substituting $k = 0$, $\ell = \ell' + 1$, $m = \ell' + k' + 2$, and $n = \ell' + m' + 2$. The direct result is

$$\begin{aligned} &(-1)^{\ell'+1} K_{\ell'+m'}(\mathsf{x})_0^{\ell'+m'} \cdot K_{k'-1}(\mathsf{x})_{\ell'+1}^{k'+\ell'} \\ &\quad + (-1)^{k'+2\ell'+3} K_{m'-1}(\mathsf{x})_{\ell'+1}^{\ell'+m'} \cdot K_{-k'-\ell'-4}(\mathsf{x})_{k'+\ell'+2}^{-2} \\ &\quad + (-1)^{k'+\ell'+2} K_{-k'-m'-2}(\mathsf{x})_{k'+\ell'+2}^{\ell'+m'} \cdot K_{\ell'-1}(\mathsf{x})_0^{\ell'-1} = 0 \ . \end{aligned} \tag{20}$$

The fourth continuant in (20) can be re-expressed using Theorem 1:

$$K_{-k'-\ell'-4}(\mathsf{x})_{k'+\ell'+2}^{-2} = (-1)^{-k'-\ell'-3} K_{k'+\ell'}(\mathsf{x})_0^{k'+\ell'} \ .$$

Making this substitution in (20) and simplifying the exponents of -1 completes the proof for $h = 0$. The index shift $i \mapsto i + h$ in every input sequence merely relabels those inputs.[5] □

A special case of Euler's identity, in turn, is the following classical result, which is also the determinant of the matrices $\overrightarrow{M}_{0,n}$ in (6) and $\overleftarrow{M}_{-n,-1}$ in (11).

Corollary 2. *For all $n \in \mathbb{Z}$,*

$$K_n(\mathsf{x})_0^n \cdot K_{n-2}(\mathsf{x})_1^{n-1} - K_{n-1}(\mathsf{x})_1^n \cdot K_{n-1}(\mathsf{x})_0^{n-1} = (-1)^{n+1} \ . \tag{21}$$

4. Bi-infinite Periodic Continued Fractions

Throughout this section we employ the eastbound expansions

$$\alpha = [y_i)_{i \geq 0} \quad \text{and} \quad \overline{\alpha} = [z_i)_{i \geq 0} \ , \tag{22}$$

and recall from (5) that

$$\overrightarrow{\mathsf{C}}(\alpha) = [\![c_i)\!)_{i \geq 0} \quad \text{and} \quad \overrightarrow{\mathsf{C}}(\overline{\alpha}) = [\![d_i)\!)_{i \geq 0} \ .$$

Definition 5. A BpCF is a bi-infinite integer sequence $\overleftrightarrow{B} = (x_i)_{i \in \mathbb{Z}}$ such that

1. the west- and eastbound terms are eventually s-periodic; that is, there exist integers $s \geq 1$, $i_0 \geq 1$, and $i_1 \geq 0$ such that for $j \in (0, 1, \ldots, s-1)$ and for all $m \geq 0$,
$$x_{-j-i_0-ms} = x_{-j-i_0}$$
$$x_{j+i_1+ms} = x_{j+i_1} \ ,$$
where the west- and eastbound periodics need not be identical s-tuples;

[5] Ustinov's proof [9] of Euler's continuant identity uses a 4×4 version of $\mathbf{F}'$ defined in (17).

2. there exist a bi-infinite sequence of raw rational convergents

$$\mathsf{C}(\overleftrightarrow{B}) = \left(\!\left(k_n \right)\!\right) = \left(\!\!\left(\frac{K_n(\mathsf{x})_0^n}{K_{n-1}(\mathsf{x})_1^n} \right)\!\!\right)_{n\in\mathbb{Z}}$$

and conjugate quadratic irrationals α and $\overline{\alpha}$ such that $\lim_{i\to\infty} k_i = \alpha$ and $\lim_{i\to\infty} k_{-i} = \overline{\alpha}$. The notation $\overleftrightarrow{B} = \overleftrightarrow{B}(\overline{\alpha}, \alpha)$ indicates these limits.[6]

4.1. Strict Versus Value Equivalence of Convergents

The sequence

$$\overleftarrow{\mathsf{C}}(\overline{\alpha}_0) = \left(\!\left(\dots, -\tfrac{23}{27}, -\tfrac{17}{20}, -\tfrac{6}{7}, -\tfrac{5}{6}, -1\right]\!\right]$$

from (3) actually shows just the values of the westbound convergents for the introductory example $\overline{\alpha}_0$. The corresponding sequence of raw rationals could be

$$\begin{aligned} &\left(\!\left(\dots, \tfrac{-23}{27}, \tfrac{-17}{20}, \tfrac{-6}{7}, \tfrac{-5}{6}, \tfrac{-1}{1}\right]\!\right] &&\text{(negative numerators)} \\ &\left(\!\left(\dots, \tfrac{23}{-27}, \tfrac{-17}{20}, \tfrac{6}{-7}, \tfrac{-5}{6}, \tfrac{1}{-1}\right]\!\right] &&\text{(num. and denom. signs alternate)}\ , \end{aligned} \tag{23}$$

the negatives of these, or one of uncountably many other value-equivalent sequences. When comparing a BpCF's westbound with $\overline{\alpha}$'s eastbound convergents, we will limit ourselves to the case where the two sequences are strictly equivalent.

4.2. Silent Convergents and Silent Terms

While the eastbound side of $\overleftrightarrow{B}(\overline{\alpha}, \alpha)$ is simply α's SCF, the westbound side faces some construction challenges. First, the convergents must include $k_{-2} = \frac{0}{1}$ and $k_{-1} = \frac{1}{0}$; and second, the terms x_{-2} and x_{-1} need not be specified (as demonstrated in Equations (10) with $h = 0$). We give special status to such convergents and their corresponding terms.

Definition 6. Given $\overleftrightarrow{B}(\overline{\alpha}, \alpha)$ and an integer $\sigma \geq 0$, suppose that $k_{-i-\sigma-1} \doteq d_i$ for all $i \geq 0$. Then $k_{-\sigma}, \dots, k_{-2}, k_{-1}$ are *silent convergents*; $x_{-\sigma}, \dots, x_{-2}, x_{-1}$ are *silent terms*; and $\overleftrightarrow{B}$ is a σ-*silent* BpCF.

In the introduction's sample BpCFs (Equations (4)), the silent terms are those delimited on the left by the semicolon and on the right by the vertical bar. The 4-silent $\overleftrightarrow{B_0}$ has silent convergents $\left(\!\left(k_{-4}, k_{-3}, k_{-2}, k_{-1}\right)\!\right) = \left(\!\left(\frac{1}{0}, \frac{1}{-1}, \frac{0}{1}, \frac{1}{0}\right)\!\right)$, while $\overleftrightarrow{B_1}$ is 3-silent, with $\left(\!\left(k_{-3}, k_{-2}, k_{-1}\right)\!\right) = \left(\!\left(\frac{1}{0}, \frac{0}{1}, \frac{1}{0}\right)\!\right)$.

4.3. The Issue of Uniqueness

As we attempt to build BpCFs using the eastbound SCF terms of $\overline{\alpha}$ and α, we bear in mind the question, "Are these bi-infinite expansions unique?" The answer—in

[6]The periodicity defined in item (1) guarantees that the convergents' west- and eastbound limits are quadratic irrationals, but if these limits are to be conjugates, we must say so explicitly.

some cases dependent on the properties of $\overline{\alpha}$—will inform the discussion in Section 6.4 about a canonical BpCF form.

5. σ-Silent BpCFs for σ Even

Theorem 4. *For $j \geq 0$ and $\sigma = 2j$, there exists no σ-silent $\overleftrightarrow{B}(\overline{\alpha}, \alpha)$.*

Proof. We show that there exist no strictly equivalent sequences $(\!(k_{-i-\sigma-1}]\!]$ and $[\![d_i)\!)$ when $\sigma \geq 0$ is even. For $\sigma = 0$ and $i = 0$, this failure is immediate: $k_{-1} = \frac{1}{0} \not\approx \frac{z_0}{1} = d_0$. (Value equivalence also fails if d_0 is represented as $\frac{-z_0}{1}$ or $\frac{z_0}{-1}$.)

Now suppose $\sigma = 2j \geq 2$ and $i = 0$ in Definition 6, and consider $= k_{-\sigma-1}$ and d_0. From remarks following Proposition 1, we can schematically represent $\overleftrightarrow{B}$'s westbound matrix product $\overleftarrow{M}_{-\sigma,-1}$, and $\overline{\alpha}$'s eastbound product $\overrightarrow{M}_{0,1}$, as

$$\overleftarrow{M}_{-\sigma,-1} = \left[\begin{array}{c} k_{-\sigma-1} \\ \hline k_{-\sigma-2} \end{array}\right] \quad \text{and} \quad \overrightarrow{M}_{0,1} = \left[\begin{array}{c|c} d_1 & d_0 \end{array}\right] .$$

Let

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ,$$

so that

$$V = T \ \mathrm{inv}\big(\overleftarrow{M}_{-\sigma,-1}\big)\, U = \left[\begin{array}{c|c} k_{-\sigma-2} & k_{-\sigma-1} \end{array}\right] .$$

By Corollary 2, $\det \overrightarrow{M}_{0,1} = 1$ and $\det \mathrm{inv}(\overleftarrow{M}_{-\sigma,-1}) = 1/\det \overleftarrow{M}_{-\sigma,-1} = 1$, since an even σ implies

$$\begin{aligned} \det \overleftarrow{M}_{-\sigma,-1} &= K_{-\sigma-1}(\mathsf{x})_0^{-\sigma-1} \cdot K_{-\sigma-3}(\mathsf{x})_1^{-\sigma-2} - K_{-\sigma-2}(\mathsf{x})_1^{-\sigma-1} \cdot K_{-\sigma-2}(\mathsf{x})_0^{-\sigma-2} \\ &= (-1)^{\sigma} = 1 . \end{aligned}$$

However, $\det T = 1$ and $\det U = -1$, so $\det V = -1$. Therefore, $\overrightarrow{M}_{0,1}$ and V can be neither similar nor equal. (The argument fails if σ is odd, because then $\det \mathrm{inv}(\overleftarrow{M}_{-\sigma,-1}) = -1$.) In general, this method shows that $(\!(k_{-2m-\sigma-1}]\!]$ and $[\![d_{2m})\!)$ cannot be strictly equivalent for $m \geq 0$. □

For example, we have in fact seen the 4-silent $\overleftrightarrow{B}_0$'s westbound raw rational convergents, and $\overline{\alpha}_0$'s eastbound convergents, respectively, in the expressions (23),

$$\begin{aligned} (\!(k_{-n-5}]\!]_{n\geq 0} &= (\!(\ldots, \tfrac{23}{-27}, \tfrac{-17}{20}, \tfrac{6}{-7}, \tfrac{-5}{6}, \tfrac{1}{-1}]\!] \\ [\![d_n)\!)_{n\geq 0} &= [\![\tfrac{-1}{1}, \tfrac{-5}{6}, \tfrac{-6}{7}, \tfrac{-17}{20}, \tfrac{-23}{27}, \ldots)\!) . \end{aligned}$$

For every $m \geq 0$, the raw rationals k_{-2m-5} and d_{2m} have opposite sign patterns; these convergents are value equivalent, but not strictly equivalent.

6. σ-Silent BpCFs for σ Odd

6.1. The Case $\sigma = 1$

Theorem 5 (1-silent BpCFs)**.** *Let α and $\overline{\alpha}$ have the eastbound SCFs given in (22). If $\overleftrightarrow{B}(\overline{\alpha},\alpha) = (x_i)_{i\in\mathbb{Z}}$ is such that $\big(\!\big(k_{-i-2}\big]\!\big] \doteq \big[\!\big[d_i\big)\!\big)$ and $\big[\!\big[k_i\big)\!\big) \doteq \big[\!\big[c_i\big)\!\big)$ for $i \geq 0$, then*

$$x_n = y_n \quad \text{for } n \geq 0\,,$$
$$x_{-n} = -z_n \quad \text{for } n \geq 1\,,$$

$k_{-1} = \frac{1}{0}$ is $\overleftrightarrow{B}$'s unique silent convergent, $x_{-1} = -z_1$ is its unique silent term, and $0 < \overline{\alpha} < 1$.

Proof. For $n \geq 0$ we have the eastbound terms $x_n = y_n$ from $\big[\!\big[k_n\big)\!\big) \doteq \big[\!\big[c_n\big)\!\big)$. If $\big(\!\big(k_{-i-2}\big]\!\big] \doteq \big[\!\big[d_i\big)\!\big)$ for $i \geq 0$, then $k_{-2} = \frac{0}{1} \doteq \frac{z_0}{1} = d_0$ implies that $z_0 = 0$, and thus that $0 < \overline{\alpha} < 1$. To show that $x_{-n} = -z_n$, $n \geq 1$, begin an induction proof with $n = 1$. The denominators in the raw rationals $k_{-3} = \frac{1}{-x_{-1}} \doteq \frac{1+z_0z_1}{z_1} = d_1$ show that $x_{-1} = -z_1$; and x_{-1} is silent because $k_{-1} = \frac{1}{0}$. Now suppose there exists an N such that

$$k_{-i-2} = \frac{K_{-i-2}(\mathsf{x})_0^{-i-2}}{K_{-i-3}(\mathsf{x})_1^{-i-2}} \doteq \frac{K_i(\mathsf{z})_0^i}{K_{i-1}(\mathsf{z})_1^i} = d_i \tag{24}$$

for $1 \leq i \leq N$. In particular, from the numerators in (24) with $i = N$ and $i = N-1$, respectively, we have by Theorem 1,

$$\begin{aligned} K_{-N-2}(\mathsf{x})_0^{-N-2} &= (-1)^{N+1}K_{N-2}(\mathsf{x})_{-N}^{-2} = K_N(\mathsf{z})_0^N \\ K_{-N-1}(\mathsf{x})_0^{-N-1} &= (-1)^{N}K_{N-3}(\mathsf{x})_{-N+1}^{-2} = K_{N-1}(\mathsf{z})_0^{N-1}\,, \end{aligned} \tag{25}$$

Again via Theorem 1, we are to show that if

$$k_{-N-3} = \frac{K_{-N-3}(\mathsf{x})_0^{-N-3}}{K_{-N-4}(\mathsf{x})_1^{-N-3}} \doteq \frac{(-1)^N K_{N-1}(\mathsf{x})_{-N-1}^{-2}}{(-1)^{N+1}K_N(\mathsf{x})_{-N-1}^{-1}} \doteq \frac{K_{N+1}(\mathsf{z})_0^{N+1}}{K_N(\mathsf{z})_1^{N+1}} = d_{N+1}\,, \tag{26}$$

then $x_{-N-1} = -z_{N+1}$. Expand the second and third numerators of (26) using the LA and RA versions of Definition 3, respectively:

$$(-1)^N\big(x_{-N-1}K_{N-2}(\mathsf{x})_{-N}^{-2} + K_{N-3}(\mathsf{x})_{-N+1}^{-2}\big) = z_{N+1}K_N(\mathsf{z})_0^N + K_{N-1}(\mathsf{z})_0^{N-1}\,.$$

Substitutions from (25) give

$$\begin{aligned} (-1)^{2N}\big(-x_{-N-1}K_N(\mathsf{z})_0^N + K_{N-1}(\mathsf{z})_0^{N-1}\big) &= z_{N+1}K_N(\mathsf{z})_0^N + K_{N-1}(\mathsf{z})_0^{N-1} \\ -x_{-N-1} &= z_{N+1}\,. \end{aligned}$$

This also follows similarly starting with the denominators in (24). □

6.2. The Case $\sigma = 3$

Theorem 6 (3-silent BpCFs). *Let α and $\overline{\alpha}$ have the eastbound SCFs given in (22). If $\overleftrightarrow{B}(\overline{\alpha}, \alpha) = (x_i)_{i\in\mathbb{Z}}$ is such that $(\!(k_{-i-4}]\!] \doteq [\![d_i)\!)$ and $[\![k_i)\!) \doteq [\![c_i)\!)$ for $i \geq 0$, then for all $\overline{\alpha} \in \mathbb{R}$,*

$$x_n = y_n \qquad \text{if } n \geq 0\,,$$
$$x_{-n} = \begin{cases} 0 & \text{if } n = 1\,, \\ -z_{n-2} & \text{if } n \geq 2\,, \end{cases}$$

$(\!(k_{-3}, k_{-2}, k_{-1})\!) = (\!(\frac{1}{0}, \frac{0}{1}, \frac{1}{0})\!)$ are $\overleftrightarrow{B}$'s silent convergents, and $(x_{-3}, x_{-2}, x_{-1}) = (-z_1, -z_0, 0)$ are its silent terms.

Proof. The proof is similar to that for the 1-silent case. For $\overleftrightarrow{B}$'s eastbound terms, we have $x_n = y_n$ from $[\![k_n)\!) \doteq [\![c_n)\!)$ for $n \geq 0$. For the westbound terms, consider $k_{-i-4} \doteq d_i$ when $i = 0$:

$$k_{-4} = \frac{-x_{-2}}{1 + x_{-2}x_{-1}} \doteq \frac{z_0}{1} = d_0\ .$$

The numerators give $x_{-2} = -z_0$. The denominators then imply $z_0 x_{-1} = 0$, which presents a choice: 1) take $z_0 = 0$ and $x_{-1} \in \mathbb{Z}$, so that $\overline{\alpha}$ is in the open interval $(0, 1)$, or 2) take $x_{-1} = 0$ and $z_0 \in \mathbb{Z}$, so that $\overline{\alpha} \in \mathbb{R}$. Preferring the second option, we initialize an induction proof: since already $x_{-2} = -z_0$, assume there is an N such that $k_{-i-4} \doteq d_i$ for $2 \leq i \leq N$. We are to show that $x_{-N-3} = -z_{N+1}$, given (with Theorem 1 applied to k_{-N-5})

$$k_{-N-5} = \frac{K_{-N-5}(\mathsf{x})_0^{-N-5}}{K_{-N-6}(\mathsf{x})_1^{-N-5}} \doteq \frac{(-1)^N K_{N+1}(\mathsf{x})_{-N-3}^{-2}}{(-1)^{N+1} K_{N+2}(\mathsf{x})_{-N-3}^{-1}} \doteq \frac{K_{N+1}(\mathsf{z})_0^{N+1}}{K_N(\mathsf{z})_1^{N+1}} = d_{N+1}\ . \quad (27)$$

Expand the second and third numerators in (27) using the LA and RA versions of Definition 3, respectively:

$$(-1)^N \big(x_{-N-3} K_N(\mathsf{x})_{-N-2}^{-2} + K_{N-1}(\mathsf{x})_{-N-1}^{-2}\big) = z_{N+1} K_N(\mathsf{z})_0^N + K_{N-1}(\mathsf{z})_0^{N-1}\ . \quad (28)$$

By the induction hypothesis on the numerators of d_{N-1}, k_{-N-3}, d_N, and k_{-N-4}, with Theorem 1 applied,

$$K_{-N-4}(\mathsf{x})_0^{-N-4} = (-1)^{N+1} K_N(\mathsf{x})_{-N-2}^{-2} = -K_N(\mathsf{z})_0^N$$
$$K_{-N-3}(\mathsf{x})_0^{-N-3} = (-1)^N K_{N-1}(\mathsf{x})_{-N-1}^{-2} = K_{N-1}(\mathsf{z})_0^{N-1}\ .$$

Substituting these in (28) shows that $x_{-N-3} = -z_{N+1}$. Again, the denominators give the same result via a similar argument.

Lastly, the silent $k_{-3} = \frac{1}{0}$ follows from $K_{-3}(\mathsf{x})_0^{-3} = 1$ and $K_{-2}(\mathsf{x})_1^{-1} = 0$. ☐

6.3. The Case $\sigma = 2j+1, j \geq 2$

Theorem 7. *Let* α *and* $\overline{\alpha}$ *have the eastbound SCFs given in* (22), *and let* $\sigma = 2j+1 \geq 5$. *If* $\overleftrightarrow{B}(\overline{\alpha}, \alpha) = (x_i)_{i\in\mathbb{Z}}$ *is such that* $\big(\!\!\big(k_{-i-\sigma-1}\big]\!\!\big] \doteq \big[\!\!\big[d_i\big)\!\!\big)$ *and* $\big[\!\!\big[k_i\big)\!\!\big) \doteq \big[\!\!\big[c_i\big)\!\!\big)$ *for* $i \geq 0$, *then*

$$x_n = y_n \quad \text{if } n \geq 0,$$
$$x_{-n} = \begin{cases} \big(-1 + K_{\sigma-6}(\mathsf{x})_{-\sigma+4}^{-2} + z_0 K_{\sigma-5}(\mathsf{x})_{-\sigma+4}^{-1}\big)/x_{-\sigma+1} & \text{if } n = \sigma-2, \\ -K_{\sigma-5}(\mathsf{x})_{-\sigma+3}^{-2} - z_0 K_{\sigma-4}(\mathsf{x})_{-\sigma+3}^{-1} & \text{if } n = \sigma-1, \\ -K_{\sigma-3}(\mathsf{x})_{-\sigma+2}^{-1} - z_1 & \text{if } n = \sigma, \\ -z_{n-\sigma+1} & \text{if } n \geq \sigma+1, \end{cases} \tag{29}$$

$x_{-\sigma+3}, \ldots, x_{-1}$ *are arbitrary integers such that* $x_{-\sigma+1} \neq 0$, *and the silent terms are* $x_{-\sigma}, x_{-\sigma+1}, \ldots, x_{-1}$.

Note that $x_{-\sigma+2}$ is not a raw rational but a rational function that requires a nonzero denominator $x_{-\sigma+1}$.

Proof. As in the 1- and 3-silent cases, $\big[\!\!\big[k_n\big)\!\!\big) \doteq \big[\!\!\big[c_n\big)\!\!\big)$ implies $x_n = y_n$ for $n \geq 0$. The assumption $\big(\!\!\big(k_{-i-\sigma-1}\big]\!\!\big] \doteq \big[\!\!\big[d_i\big)\!\!\big)$ for $i \geq 0$ means that when $i = 0$,

$$k_{-\sigma-1} = \frac{K_{-\sigma-1}(\mathsf{x})_0^{-\sigma-1}}{K_{-\sigma-2}(\mathsf{x})_1^{-\sigma-1}} \doteq \frac{-K_{\sigma-3}(\mathsf{x})_{-\sigma+1}^{-2}}{K_{\sigma-2}(\mathsf{x})_{-\sigma+1}^{-1}} \doteq \frac{z_0}{1} = d_0\ ; \tag{30}$$

the middle raw rational's parts follow from Theorem 1. To prove identity (29) when $n = \sigma-1$, use Euler's identity (Corollary 1) with $k = \sigma-3$, $\ell = 1$, $m = \sigma-4$, and $h = -\sigma+1$:

$$K_{\sigma-3}(\mathsf{x})_{-\sigma+1}^{-2} \cdot K_{\sigma-4}(\mathsf{x})_{-\sigma+3}^{-1} - K_{\sigma-5}(\mathsf{x})_{-\sigma+3}^{-2} \cdot K_{\sigma-2}(\mathsf{x})_{-\sigma+1}^{-1}$$
$$= (-1)^{\sigma-3} K_{-3}(\mathsf{x})_1^{-2} \cdot K_0(\mathsf{x})_{-\sigma+1}^{-\sigma+1}\ .$$

Substituting from (30)'s numerators and denominators, along with $K_0(\mathsf{x})_{-\sigma+1}^{-\sigma+1} = x_{-\sigma+1}$ and $K_{-3}(\mathsf{x})_1^{-2} = 1$, yields $x_{-\sigma+1} = -K_{\sigma-5}(\mathsf{x})_{-\sigma+3}^{-2} - z_0 K_{\sigma-4}(\mathsf{x})_{-\sigma+3}^{-1}$.

The case $n = \sigma-2$ is proved similarly, using $k = \sigma-1$, $\ell = -4$, $m = \sigma-2$, and $h = -\sigma+4$ in Corollary 1, the substitutions $-K_{\sigma-3}(\mathsf{x})_{-\sigma+1}^{-2} = z_0$ and $K_{\sigma-2}(\mathsf{x})_{-\sigma+1}^{-1} = 1$ from (30), and $K_{-5}(\mathsf{x})_{-\sigma+4}^{-\sigma-1} = K_1(\mathsf{x})_{-\sigma+1}^{-\sigma+2} = 1 + x_{-\sigma+2}x_{-\sigma+1}$ from Theorem 1.

For $n = \sigma$, take $\big(\!\!\big(k_{-i-\sigma-1}\big]\!\!\big] \doteq \big[\!\!\big[d_i\big)\!\!\big)$ with $i = 1$ to get

$$k_{-\sigma-2} = \frac{K_{-\sigma-2}(\mathsf{x})_0^{-\sigma-2}}{K_{-\sigma-3}(\mathsf{x})_1^{-\sigma-2}} \doteq \frac{K_{\sigma-2}(\mathsf{x})_{-\sigma}^{-2}}{-K_{\sigma-1}(\mathsf{x})_{-\sigma}^{-1}} \doteq \frac{1 + z_0 z_1}{z_1} = d_1\ , \tag{31}$$

where again Theorem 1 creates the middle raw rational. The denominators in (31) give

$$K_{\sigma-1}(\mathsf{x})_{-\sigma}^{-1} = -z_1 \ .$$

Expand the left-hand continuant above with Definition 3's LA recurrence:

$$x_{-\sigma} K_{\sigma-2}(\mathsf{x})_{-\sigma+1}^{-1} + K_{\sigma-3}(\mathsf{x})_{-\sigma+2}^{-1} = -z_1 \ .$$

Substituting $K_{\sigma-2}(\mathsf{x})_{-\sigma+1}^{-1} = 1$ from the denominators of (30) gives the desired formula for $x_{-\sigma}$.

Finally, again like the cases $\sigma = 1, 3$, an induction proof will show that if $(\!(k_{-i-\sigma-1}]\!] \doteq [\![d_i)\!)$ for $0 \le i \le N$, then $x_{-N-\sigma} = -z_{N+1}$. From the numerators of $k_{-N-\sigma-2}$ and d_{N+1} (the latter order- and index-shifted using Theorem 1), we have

$$K_{-N-\sigma-2}(\mathsf{x})_0^{-N-\sigma-2} = (-1)^N K_{N+\sigma-2}(\mathsf{x})_{-N-\sigma}^{-2} = K_{N+1}(\mathsf{z})_0^{N+1} \ .$$

Apply Definition 3's LA and RA recurrences to the second and third continuants above, respectively:

$$(-1)^N \left(x_{-N-\sigma} K_{N+\sigma-3}(\mathsf{x})_{-N-\sigma+1}^{-2} + K_{N+\sigma-4}(\mathsf{x})_{-N-\sigma+2}^{-2}\right) = z_{N+1} K_N(\mathsf{z})_0^N + K_{N-1}(\mathsf{z})_0^{N-1} \ . \quad (32)$$

By the induction hypothesis, the numerators of d_N and d_{N-1}, and the numerators of $k_{-N-\sigma-1}$ and $k_{-N-\sigma}$ modified by Theorem 1, satisfy

$$K_{-N-\sigma-1}(\mathsf{x})_0^{-N-\sigma-1} = (-1)^{N+1} K_{N+\sigma-3}(\mathsf{x})_{-N-\sigma+1}^{-2} = -K_N(\mathsf{z})_0^N$$
$$K_{-N-\sigma}(\mathsf{x})_0^{-N-\sigma} = (-1)^N K_{N+\sigma-4}(\mathsf{x})_{-N-\sigma+2}^{-2} = K_{N-1}(\mathsf{z})_0^{N-1} \ .$$

Substitution in (32) produces

$$(-1)^{2N} \left(-x_{-N-\sigma} K_N(\mathsf{z})_0^N + K_{N-1}(\mathsf{z})_0^{N-1}\right) = z_{N+1} K_N(\mathsf{z})_0^N + K_{N-1}(\mathsf{z})_0^{N-1}$$
$$x_{-N-\sigma} = -z_{N+1} \ .$$

□

Theorem 7 almost implies that when σ is odd and not less than 5, the silent terms of a σ-silent BpCF are not unique. What it does not say, however, is that the rational function for $x_{-\sigma+2}$ invariably produces an integer for all sequences of silent terms. But such sequences do exist. For instance, when $\sigma = 5$, the following result gives silent terms satisfying the relevant conditions in Theorem 7 and producing $x_{-\sigma+2} \in \mathbb{Z}$.

Proposition 2. *Let α and $\overline{\alpha}$ have the eastbound SCFs given in (22). For $\sigma = 5$, the sequences $(x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}) = (-1-z_1, 1, c, -1, 1)$ and $(1-z_1, -1, -c, 1, -1)$, where $c \in \mathbb{Z}$, both satisfy the requirements of Theorem 7.*

The proof is by direct substitution of $(c,-1,1)$ and $(-c,1,-1)$ for (x_{-3},x_{-2},x_{-1}) in the cases $n=-\sigma,-\sigma+1,-\sigma+2$ of (29).

We can exhibit similar sequences of silent terms which invariably yield $x_{-\sigma+2}\in\mathbb{Z}$ for all odd $\sigma\geq 7$.

Proposition 3. *Given $\sigma=2j+1\geq 7$, in Theorem 7 let $x_{-3}=x_0+c_1$, $x_{-2}=1, x_{-1}=-1$, and write $z_0=x_0+c_2$, where $c_1,c_2\in\mathbb{Z}$ and $c_1\neq c_2$. For $-\sigma+3\leq -m\leq -4$, let x_{-m} be as given in (33), where $i\geq 0$:*

$$\begin{array}{r|ccc} \sigma\setminus -m & -3i-6 & -3i-5 & -3i-4 \\ \hline \equiv 1 \bmod 6 & 1 & 1 & -1 \\ \equiv 3 \bmod 6 & -1 & 1 & 1 \\ \equiv 5 \bmod 6 & 1 & -1 & 1 \end{array} \quad . \tag{33}$$

Then

$$x_{-\sigma+2}=\begin{cases} 1 & \text{if } \sigma\equiv 1 \bmod 6\ , \\ 2 & \text{if } \sigma\equiv 3 \bmod 6\ , \\ 1+c_2-c_1 & \text{if } \sigma\equiv 5 \bmod 6\ . \end{cases}$$

Proof. Consider the case $\sigma\equiv 1 \bmod 6$, or $\sigma=6j+1$ for $j\geq 1$. Using the x_{-m} given in (33), one may prove by induction the continuant values in Table 1. Substitute $z_0=x_0+c_2$ in the second of the formulas (29):

$$\begin{aligned} x_{-\sigma+1} &= -K_{\sigma-5}(\mathsf{x})_{-\sigma+3}^{-2}-(x_0+c_2)K_{\sigma-4}(\mathsf{x})_{-\sigma+3}^{-1} \\ &= \left(-x_0K_{\sigma-4}(\mathsf{x})_{-\sigma+3}^{-1}-K_{\sigma-5}(\mathsf{x})_{-\sigma+3}^{-2}\right)-c_2K_{\sigma-4}(\mathsf{x})_{-\sigma+3}^{-1}\ . \end{aligned}$$

Using Definition 3's RA recurrence, the difference within parentheses becomes

$$x_{-\sigma+1}=-K_{\sigma-3}(\mathsf{x})_{-\sigma+3}^{0}-c_2K_{\sigma-4}(\mathsf{x})_{-\sigma+3}^{-1}\ .$$

Reading from Table 1, and recalling that $c_1\neq c_2$, this simplifies to

$$x_{-\sigma+1}=c_1-c_2\neq 0\ .$$

Turning now to $x_{-\sigma+2}$, we have by a similar calculation

$$\begin{aligned} x_{-\sigma+2} &= \left(-1+K_{\sigma-6}(\mathsf{x})_{-\sigma+4}^{-2}+(x_0+c_2)K_{\sigma-5}(\mathsf{x})_{-\sigma+4}^{-1}\right)/(c_1-c_2) \\ &= \left(-1+\left(x_0K_{\sigma-5}(\mathsf{x})_{-\sigma+4}^{-1}+K_{\sigma-6}(\mathsf{x})_{-\sigma+4}^{-2}\right)+c_2K_{\sigma-5}(\mathsf{x})_{-\sigma+4}^{-1}\right)/(c_1-c_2) \\ &= \left(-1+K_{\sigma-4}(\mathsf{x})_{-\sigma+4}^{0}+c_2K_{\sigma-5}(\mathsf{x})_{-\sigma+4}^{-1}\right)/(c_1-c_2) \\ &= \left(-1+(1+c_1)+c_2(-1)\right)/(c_1-c_2)=1\ . \end{aligned}$$

The proofs for $\sigma\equiv 3 \bmod 6$ and $\sigma\equiv 5 \bmod 6$ are similar. □

n			$K_n(\mathsf{x})^0_{-n}$	$K_n(\mathsf{x})^{-1}_{-n-1}$
$\sigma-5$	$=$	$6j-4$	1	-1
$\sigma-4$	$=$	$6j-3$	$1+c_1$	1
$\sigma-3$	$=$	$6j-2$	$-c_1$	0
$\sigma-2$	$=$	$6j-1$	1	1
$\sigma-1$	$=$	$6j$	$1-c_1$	-1
σ	$=$	$6j+1$	c_1	0

Table 1: Values of the continuants $K_n(\mathsf{x})^0_{-n}$ and $K_n(\mathsf{x})^{-1}_{-n-1}$, where n is expressed in terms of σ and in terms of $j \geq 1$. The continuant input sequences $(\mathsf{x})^0_{-n}$ and $(\mathsf{x})^{-1}_{-n-1}$ are from the first row of (33).

Examples. Proposition 3's sequences for $\sigma = 7, 9, \ldots, 17$ (which include $x_{-\sigma+2}$ in bold) are

$$\begin{aligned}
\sigma = 7 &: (x_{-5}, \ldots, x_{-1}) = (\mathbf{1}, -1, z_0 + c_2, 1, -1) \\
\sigma = 9 &: (x_{-7}, \ldots, x_{-1}) = (\mathbf{2}, -1, 1, 1, z_0 + c_2, 1, -1) \\
\sigma = 11 &: (x_{-9}, \ldots, x_{-1}) = (\boldsymbol{1 + c_2 - c_1}, -1, 1, 1, -1, 1, z_0 + c_2, 1, -1) \\
\sigma = 13 &: (x_{-11}, \ldots, x_{-1}) = (\mathbf{1}, -1, 1, 1, -1, 1, 1, -1, z_0 + c_2, 1, -1) \\
\sigma = 15 &: (x_{-13}, \ldots, x_{-1}) = (\mathbf{2}, -1, 1, 1, -1, 1, 1, -1, 1, 1, z_0 + c_2, 1, -1) \\
\sigma = 17 &: (x_{-15}, \ldots, x_{-1}) = (\boldsymbol{1 + c_2 - c_1}, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, z_0 + c_2, 1, -1)\,.
\end{aligned}$$

To summarize: When $\sigma \geq 5$ is odd, Propositions 2 and 3 show that the number of σ-tuples of possible silent terms is (at least) countably infinite; the silent terms are not unique.

6.4. Is There a Canonical Form for BpCFs?

For even $\sigma \geq 0$, no σ-silent BpCF exists. For odd $\sigma \geq 5$, the silent terms need not be unique. One could argue for the 3-silent case as best for a canonical BpCF; however, its expansions for $0 < \overline{\alpha} < 1$ are not unique, since the 1-silent version covers that range. It appears, then, that a canonical BpCF form is not possible.

Nonetheless, the 3-silent case with $x_{-1} = 0$ might serve as a general-purpose solution, applicable to all $\overline{\alpha} \in \mathbb{R}$. For a closer look at the 3-silent form, we turn to Herzog's work.

7. Herzog's Taxonomy and Its Application to BpCFs

Herzog calculated, for any quadratic irrational α, the exact form of its conjugate's (eastbound) SCF [8]. A notational convention will facilitate our review of those results. Given $s \geq 1$ and an s-tuple $(b_0, b_1, \ldots, b_{s-1})$, the bi-infinite sequence

$$(x_i) = (b_{i \bmod s})_{i \in \mathbb{Z}} \ ,$$

is s-periodic. Henceforth, input sequences of the b_i are indexed in the usual way, but we understand the individual terms b_i to be indexed modulo s. Thus,

$$(\mathsf{b})_h^n = (b_{h \bmod s}, b_{(h+1) \bmod s}, b_{(h+2) \bmod s}, \ldots, b_{(n-1) \bmod s}, b_{n \bmod s}) \ . \tag{34}$$

This applies only to input terms b_i and sequences (b).

7.1. A Condensed Version of Herzog's Theorem 1

The main result in [8] is a list of 28 cases, organized by α's number of nonperiodics. Selections from that list are reproduced in Table 2, where we have synchronized the original notation to ours. Herzog's list shows that the reversed sequence of α's periodics is cyclically permuted in one of just three ways to produce $\overline{\alpha}$'s periodics:

$$\begin{aligned} \overrightarrow{P}_1 &= (\overrightarrow{b_{s-2}, b_{s-3}, \ldots, b_0, b_{-1}}) \\ \overrightarrow{P}_2 &= (\overrightarrow{b_{s-3}, b_{s-4}, \ldots, b_0, b_{-1}, b_{-2}}) \\ \overrightarrow{P}_3 &= (\overrightarrow{b_{s-4}, b_{s-5}, \ldots, b_0, b_{-1}, b_{-2}, b_{-3}}) \ , \end{aligned} \tag{35}$$

where, per (34), all subscripts on the right are taken modulo s.

r	Case	Conditions	Eastbound SCF for $\overline{\alpha}$
0	$0A$	$b_{s-1} = 1$	$[-1, b_{s-2}+1, \overrightarrow{P_2})$
0	$0B$	$b_{s-1} \geq 2$	$[-1, 1, b_{s-1}-1, \overrightarrow{P_1})$
1	$1A$	$b_{s-2} = 1$	$[a_0 - b_{s-1} - 1, b_{s-3}+1, \overrightarrow{P_3})$
1	$1B$	$b_{s-2} \geq 2$	$[a_0 - b_{s-1} - 1, 1, b_{s-2}-1, \overrightarrow{P_2})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
≥ 4	rG	$a_{r-1} \geq b_{s-1}+2,\ b_{s-2} = 1$	$[a_0, \ldots, a_{r-2}, a_{r-1} - b_{s-1} - 1, b_{s-3}+1, \overrightarrow{P_3})$
≥ 4	rH	$a_{r-1} \geq b_{s-1}+2,\ b_{s-2} \geq 2$	$[a_0, \ldots, a_{r-2}, a_{r-1} - b_{s-1} - 1, 1, b_{s-2}-1, \overrightarrow{P_2})$

Table 2: Six of the 28 cases from [8, Table I], with adjusted notation. The $\overrightarrow{P_i}$ are the conjugates' periodics defined in (35). Given an s-periodic α having r nonperiodics, and which satisfies the conditions, the eastbound SCF of $\overline{\alpha}$ is shown.

Table 3 is an edited version of [8, Table I], constructed using [8, Table II], which shows which of the 28 cases are conjugate pairs, along with [8, Theorem 2], which compares the initial terms of the SCFs of a conjugate pair. (The example α_0 in (1) is of type $0A_*$.) We have assumed $\overline{\alpha} < \alpha$; by removing from [8, Table I] the lesser conjugate of each pair, and re-expressing the conditions, we obtain Table 3. Its last column lists the following parameter.

Case (rX_*)	a_{r-1}	Condition	Eastbound SCF for $\overline{\alpha}$	$\dot{r}$	η
$0A_*$		$b_{s-1}=1$	$[-1, b_{s-2}+1, \overrightarrow{P_2})$	2	2
$0B_*$		$b_{s-1}\geq 2$	$[-1, 1, b_{s-1}-1, \overrightarrow{P_1})$	3	1
$1A_*$		$b_{s-2}=1$	$[a_0-b_{s-1}-1, b_{s-3}+1, \overrightarrow{P_3})$	2	3
$1B_*$		$b_{s-2}\geq 2$	$[a_0-b_{s-1}-1, 1, b_{s-2}-1, \overrightarrow{P_2})$	3	2
$2A_*$	$\leq b_{s-1}-2$		$[a_0-1, 1, -a_1+b_{s-1}-1, \overrightarrow{P_1})$	3	1
$2B_*$	$=b_{s-1}-1$		$[a_0-1, b_{s-2}+1, \overrightarrow{P_2})$	2	2
$(2k+1)E_*$	$=b_{s-1}+1$	$b_{s-2}=1$	$[a_0,\ldots,a_{r-3}, a_{r-2}+b_{s-3}+1, \overrightarrow{P_3})$	$r-1$	3
$(2k+1)F_*$	$=b_{s-1}+1$	$b_{s-2}\geq 2$	$[a_0,\ldots,a_{r-3}, a_{r-2}+1, b_{s-2}-1, \overrightarrow{P_2})$	r	2
$(2k+1)G_*$	$\geq b_{s-1}+2$	$b_{s-2}=1$	$[a_0,\ldots,a_{r-2}, a_{r-1}-b_{s-1}-1, b_{s-3}+1, \overrightarrow{P_3})$	$r+1$	3
$(2k+1)H_*$	$\geq b_{s-1}+2$	$b_{s-2}\geq 2$	$[a_0,\ldots,a_{r-2}, a_{r-1}-b_{s-1}-1, 1, b_{s-2}-1, \overrightarrow{P_2})$	$r+2$	2
$(2k+2)A_*$	$\leq b_{s-1}-2$	$a_{r-2}=1$	$[a_0,\ldots,a_{r-4}, a_{r-3}+1, -a_{r-1}+b_{s-1}-1, \overrightarrow{P_1})$	$r-1$	1
$(2k+2)B_*$	$\leq b_{s-1}-2$	$a_{r-2}\geq 2$	$[a_0,\ldots,a_{r-3}, a_{r-2}-1, 1, -a_{r-1}+b_{s-1}-1, \overrightarrow{P_1})$	$r+1$	1
$(2k+2)C_*$	$=b_{s-1}-1$	$a_{r-2}=1$	$[a_0,\ldots,a_{r-4}, a_{r-3}+b_{s-2}+1, \overrightarrow{P_2})$	$r-2$	2
$(2k+2)D_*$	$=b_{s-1}-1$	$a_{r-2}\geq 2$	$[a_0,\ldots,a_{r-3}, a_{r-2}-1, b_{s-2}+1, \overrightarrow{P_2})$	r	2

Table 3: Edited version of [8, Table I], showing only the cases for the larger elements α of the conjugate pairs α and $\overline{\alpha}$, and where $k \geq 1$. The subscript $*$ distinguishes these from Herzog's originals. The P_i are defined in (35). On the right, $\dot{r}$ is the number of nonperiodics in $\overline{\alpha}$'s (eastbound) SCF, and η is the rotation selector (Definition 7).

Definition 7. For a given $\overline{\alpha}$, the *rotation selector* $\eta = \eta(\overline{\alpha})$ is the index $i \in \{1, 2, 3\}$ of the $\overrightarrow{P}_i$ associated with $\overline{\alpha}$ in Table 3.

The following lemmas are proved by inspection of the periodics identities (35) and Table 3, respectively.

Lemma 2. *The left-most term in* $\overrightarrow{P}_\eta$, $\eta \in \{1, 2, 3\}$, *is* $b_{(s-\eta-1) \bmod s}$.

Lemma 3. *Given* $\overline{\alpha} < \alpha$, *the eastbound SCF for* $\overline{\alpha}$ *has at least one nonperiodic; that is,* $\dot{r} \geq 1$.

7.2. Details of the 3-Silent Form

Evidently, all aspects of $\overline{\alpha}$'s SCF and of the 3-silent $\overleftrightarrow{B}$ are determined by α's SCF. Our final result parses the westbound side of $\overleftrightarrow{B}$ so as to better display α's periodics.

Proposition 4. *Let* α *and* $\overline{\alpha}$ *have period* s, *rotation selector* $\eta \in \{1, 2, 3\}$, *and nonperiodics* r *and* $\dot{r}$, *respectively. Then in* $\overleftrightarrow{B}(\overline{\alpha}, \alpha) = (x_i)_{i\in\mathbb{Z}}$, *we have*

$$x_{-\dot{r}-1} = -\dot{a}_{\dot{r}-1} \tag{36}$$

$$x_{-\dot{r}-2} = -b_{s-(\eta+2-1)} \tag{37}$$

$$x_{-\dot{r}-j} = -b_{s-(\eta+j-1)}\,,\ j \geq 2 \tag{38}$$

In particular,

$$x_{-\dot{r}-(s-\eta+1)} = -b_0\,, \tag{39}$$

$$x_{-\dot{r}-(s-\eta+2)} = -b_{s-1}\,, \tag{40}$$

where all b_i are indexed modulo s, per (34).

Proof. By Theorem 6, $x_{-n} = -z_{n-2}$ for $n \geq 2$. Table 4 shows positive westbound values of n, from $n = 4$ on the right to $n = \dot{r} + (s - \eta + 2)$ on the left, along with the corresponding x_{-n} and $-z_{n-2}$.

n	$\dot{r}+(s-\eta+2)$	$\dot{r}+(s-\eta+1)$	$\ldots$	$\dot{r}+j$	$\ldots$	$\dot{r}+2$	$\dot{r}+1$	$\ldots$	5	4
$-z_{n-2}$	$-z_{\dot{r}+s-\eta}$	$-z_{\dot{r}+s-\eta-1}$	$\ldots$	$-z_{\dot{r}+j-2}$	$\ldots$	$-z_{\dot{r}}$	$-z_{\dot{r}-1}$	$\ldots$	$-z_3$	$-z_2$
x_{-n}	$-b_{s-1}$	$-b_0$	$\ldots$	$-b_{s-(\eta+j-1)}$	$\ldots$	$-b_{s-(\eta+2-1)}$	$-\dot{a}_{\dot{r}-1}$	$\ldots$	$-\dot{a}_3$	$-\dot{a}_2$

Table 4: Values of n, $-z_{n-2}$, and x_{-n} for westbound $n \in \big(\dot{r} + (s - \eta + 2), \ldots, 5, 4\big)$, as described in Proposition 4.

In (36), $x_{-\dot{r}-1} = -z_{\dot{r}-1}$ corresponds to $-\dot{a}_{\dot{r}-1}$, $\overline{\alpha}$'s nonperiodic term of greatest index. This term exists because $\dot{r} \geq 1$, by Lemma 3. We take the b_i indices modulo s in the remaining items. In (37), $x_{-\dot{r}-2} = -z_{\dot{r}} = -b_{s-\eta-1} = -b_{s-(\eta+2-1)}$, since $b_{s-\eta-1}$ is the first element of $\overrightarrow{P}_{\eta}$, per Lemma 2. In (38), the general term $x_{-\dot{r}-j}\,, j \geq 2$, is $-b_{s-(\eta+j-1)}$. The values $j = s-\eta+1$ and $j = s-\eta+2$ in the latter's subscript yield the following two correspondences. In (39), $x_{-\dot{r}-(s-\eta+2)} = -b_0$, because the index $n = \dot{r} + s - \eta - 1$ gives the single appearance of b_0 in $\overrightarrow{P}_{\eta}$. In (40), the term to the right of b_0 in $\overrightarrow{P}_{\eta}$, namely $x_{-\dot{r}-(s-\eta+2)} = b_{s-1}$, is the last of $\overline{\alpha}$'s periodics. □

Example. Let $\alpha_1 = \frac{1}{328}(109 + \sqrt{401}) = [0, 2, 1, 1, \overrightarrow{5, 2, 3})$, which is of type $4A_*$ in Table 3. Here,

$$\begin{aligned} \overline{\alpha}_1 &= [0, 3, 1, \overrightarrow{2, 5, 3}) = (\overleftarrow{3, 5, 2}, 1, 3, 0] \\ \overrightarrow{\mathsf{C}}\,(\alpha_1) &= [\![\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{11}{28}, \tfrac{24}{61}, \tfrac{83}{211}, \ldots)\!) \\ \overrightarrow{\mathsf{C}}\,(\overline{\alpha}_1) &= [\![\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{3}{11}, \tfrac{16}{59}, \tfrac{51}{188}, \ldots)\!)\;. \end{aligned}$$

In tabular form for $-9 \leq n \leq 6$, $\overleftrightarrow{B}(\overline{\alpha}_1, \alpha_1)$ comprises

n	-9	-8	-7	-6	-5	-4	;	-3	-2	-1	|	0	1	2	3	4	5	6
x_n	$\mathbf{-5}$	$\mathbf{-2}$	$\mathbf{-3}$	-5	-2	-1	;	-3	0	0	|	0	2	1	1	**5**	**2**	**3**
k_n	$\frac{51}{188}$	$\frac{16}{59}$	$\frac{3}{11}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{0}{1}$	;	$\frac{1}{0}$	$\frac{0}{1}$	$\frac{1}{0}$	|	$\frac{0}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{11}{28}$	$\frac{24}{61}$	$\frac{83}{211}$

,

where the west- and eastbound periodics are in boldface, and the silent terms are $(x_{-3}, x_{-2}, x_{-1}) = (-3, 0, 0)$. This example has three consecutive 0 terms, the maximum possible. When applied to $\gamma \in \mathbb{R}$, the Euclidean algorithm produces a single, initial 0 only when $\gamma \in (0,1)$. With $y_0 = 0$ for α_1, $z_0 = 0$ for $\overline{\alpha}_1$, and the 3-silent BpCF's $x_{-1} = 0$, all possible 0 terms are accounted for. This example illustrates Herzog's Theorem 2, which states that $\dot{a}_0 = a_0$ if, and only if, $r \geq 4$.

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