



SIGNED INVERSION SEQUENCES AVOIDING A SET OF LENGTH-2 SIGNED PATTERNS

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Received: 9/25/25, Accepted: 5/18/26, Published: 6/8/26

Abstract

A signed inversion sequence of length n is a sequence $e_1 \cdots e_n$ of integers, possibly barred, such that $e_{i+1} \in \{0, \bar{0}, 1, \bar{1}, \dots, i, \bar{i}\}$, for all $i = 0, 1, \dots, n-1$. For a set of signed patterns B , let $\tilde{\mathcal{I}}_n(B)$ be the set of signed inversion sequences of length n that avoid all the signed patterns in B . We say that two sets of signed patterns B and C are Wilf-equivalent if $|\tilde{\mathcal{I}}_n(B)| = |\tilde{\mathcal{I}}_n(C)|$ for all $n \geq 0$. Let sw_k be the number of Wilf-equivalences among sets of k length-2 signed patterns. In this paper, we show that $sw_3 = 93$, $sw_4 = 195$, $239 \leq sw_5 \leq 240$, $sw_6 = 190$, $sw_7 = 107$, $sw_8 = 49$, $sw_9 = 22$, $sw_{10} = 8$, $sw_{11} = 3$, and $sw_{12} = 1$.

1. Introduction

An *inversion sequence* [5, 10] $\pi = \pi_1\pi_2 \cdots \pi_n$ of length n is a sequence of integers such that $0 \leq \pi_i < i$ for all $i = 1, 2, \dots, n$. Pattern-avoiding inversion sequences have received a lot of attention in recent years (see, for instance, [1–4, 6–8] and references therein).

Recently, the authors extended the study of inversion sequences to signed inversion sequences [9], where they determined the number of Wilf-equivalences among sets of k length-2 signed patterns for $k = 1, 2$. The aim of this paper is to continue this study by determining the number of Wilf-equivalences among sets of k length-2 signed patterns for $k = 3, 4, \dots, 12$. A *signed inversion sequence* of length n is a sequence $e_1 \cdots e_n$ of integers possibly barred such that $e_{i+1} \in \{0, \bar{0}, 1, \bar{1}, \dots, i, \bar{i}\}$ for all $i = 0, 1, \dots, n-1$. Note that signed inversion sequences are the Lehmer code of signed permutations (a signed permutation is a word in which each of

the symbols $1, 2, \dots, n$ appears exactly once and may be barred). We denote the set of all signed inversion sequences of length n by $\bar{\mathcal{I}}_n$. For example, $\bar{\mathcal{I}}_2 = \{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}1, 0\bar{1}, 0\bar{0}, 00, 01\}$. For any integer a , we define the *barring operation* as the operation that changes the symbol a to \bar{a} and the symbol \bar{a} to a . Thus, it is an involution; that is, $\bar{\bar{a}} = a$. Also, we define the absolute value by $|a| = |\bar{a}| = a$ for all $a \geq 0$.

Let $\theta = \theta_1\theta_2 \cdots \theta_n \in \bar{\mathcal{I}}_n$ and τ be any word containing all letters $0, 1, \dots, k$ possibly barred. We say that θ *contains* τ , if there exists a sequence of k indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $|\theta_{i_1}||\theta_{i_2}| \cdots |\theta_{i_k}|$ is order-isomorphic to $|\tau_1||\tau_2| \cdots |\tau_k|$ and that θ_{i_j} is *barred* if and only if τ_j is barred for all $1 \leq j \leq k$. Otherwise, we say that θ *avoids* τ . In this context, τ is usually called a *signed pattern*. For example, the signed inversion sequence $0\bar{0}\bar{2}0\bar{2}$ avoids 001 but contains $0\bar{0}\bar{1}$, and the signed inversion sequence $0\bar{0}0\bar{2}\bar{1}$ avoids 000 but contains $\bar{0}\bar{2}\bar{1}$. We denote the set of all signed inversion sequences in $\bar{\mathcal{I}}_n$ that avoid τ by $\bar{\mathcal{I}}_n(\tau)$. For an arbitrary finite collection of signed patterns B , we say that π *avoids* B if π avoids every $\tau \in B$; we denote the corresponding subset of $\bar{\mathcal{I}}_n$ by $\bar{\mathcal{I}}_n(B)$. We say that two sets of signed patterns B and C are *Wilf-equivalent* if $|\bar{\mathcal{I}}_n(B)| = |\bar{\mathcal{I}}_n(C)|$ for all $n \geq 0$.

The authors [9] showed that the number of Wilf-equivalences among length-2 signed patterns is 3, and the number of Wilf-equivalences among pairs of length-2 signed patterns is 30. In this paper, we extend these results by determining the number of Wilf-equivalences among sets of k length-2 signed patterns for $k \geq 3$. In particular, we show the following result.

Theorem 1. *Let sw_k be the number of Wilf-equivalences among sets of k length-2 signed patterns. Then $sw_3 = 93$, $sw_4 = 195$, $239 \leq sw_5 \leq 240$, $sw_6 = 190$, $sw_7 = 107$, $sw_8 = 49$, $sw_9 = 22$, $sw_{10} = 8$, $sw_{11} = 3$, and $sw_{12} = 1$.*

To prove this theorem, in the next section, we describe a general procedure, as introduced in [9], using generating trees to find an explicit formula for the generating function for the number of signed inversion sequences of length n that avoid a set of signed patterns. In the subsequent sections, we apply this procedure to show that $sw_3 = 93$ and $sw_4 = 195$. We also briefly describe how sw_k is determined for $k = 5, 6, \dots, 12$. Further, to complete the classifications of Wilf-equivalences among sets of five length-2 signed patterns, it remains to show that $\{\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, \bar{0}0\} \sim \{\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{1}, 0\bar{1}, 00\}$, see Class 141 in Table 6.

2. Main Procedure

For each class \mathcal{C} of signed inversion sequences avoiding a fixed set of signed patterns, there is a corresponding generating tree whose nodes encode the elements of \mathcal{C} , as

described in [9]. More precisely, for given a set B of nonempty signed patterns, we define $\bar{\mathcal{I}}(B) = \cup_{n \geq 0} \bar{\mathcal{I}}_n(B)$. We then construct a pattern-avoidance tree $\mathcal{T}(B)$ for the set $\bar{\mathcal{I}}(B)$. Let the empty word ϵ be the root of $\mathcal{T}(B)$. Starting from this root at level 0, the nodes at level $n+1$ in $\mathcal{T}(B)$ are generated from the nodes at level n as follows. The descendants of $\pi = \pi_1\pi_2 \cdots \pi_n \in \bar{\mathcal{I}}_n(B)$ are $\pi' = \pi_1\pi_2 \cdots \pi_n j \in \bar{\mathcal{I}}_{n+1}(B)$, where j belongs to the ordered set $\{\bar{n}, \overline{n-1}, \dots, \bar{0}, 0, 1, \dots, n\}$. To study the tree $\mathcal{T}(B)$, we introduce a new labeling for its nodes. Let $\mathcal{T}(B; \pi)$ be the subtree rooted at the signed inversion sequence π , including all its descendants in $\mathcal{T}(B)$. For any two nodes π, π' in $\mathcal{T}(B)$, we say that π is *equivalent* to π' , denoted $\pi \sim \pi'$, if and only if $\mathcal{T}(B; \pi) \cong \mathcal{T}(B; \pi')$ as (ordered) plane tree. We define the set of all equivalent classes under \sim by $\mathcal{E}(B)$. Each equivalence class in $\mathcal{E}(B)$ is represented by the label of the unique node in $\mathcal{T}(B)$ that is the left-most node at the lowest level among all nodes in the same class. Finally, let $\mathcal{T}[B]$ be the tree obtained from $\mathcal{T}(B)$ by replacing each node with the label of its corresponding equivalence class.

Example 1. Let $B = \{\bar{0}\bar{0}, \bar{1}0, 0\bar{1}\}$. The generating tree $\mathcal{T}[B]$ is given by

$$\begin{aligned} \epsilon &\rightsquigarrow \bar{0}, b_1, & \bar{0} &\rightsquigarrow \bar{0}, c_0, a_1, \\ a_m &\rightsquigarrow c_{m-1}, c_m, (a_{m+1})^{m+1}, \quad m \geq 1, & b_m &\rightsquigarrow c_{m-1}, (b_{m+1})^{m+1}, \quad m \geq 1, \\ c_m &\rightsquigarrow (c_{m+1})^{m+3}, \quad m \geq 0, \end{aligned}$$

where $a_m = \bar{0}1^m$, $b_m = 0^m$, and $c_m = 0\bar{0}0^m$. To prove this, we show that each of these rules holds:

- The children of ϵ are $\bar{0}, b_1 = 0$. Thus, $\epsilon \rightsquigarrow \bar{0}, b_1$.
- The children of $\bar{0}$ are $\bar{0}\bar{1}, \bar{0}0, \bar{0}1$. Clearly, $\bar{0}\bar{1} \sim \bar{0}$, $\bar{0}0 \sim c_0$ and $\bar{0}1 = a_1$. Thus, $\bar{0} \rightsquigarrow \bar{0}, c_0, a_1$.
- The children of a_m are $\bar{0}1^m\bar{1} \sim c_{m-1}$, $\bar{0}1^m0 \sim c_m$, and $\bar{0}1^mj \sim a_{m+1}$ with $j = 1, 2, \dots, m+1$, which leads to $a_m \rightsquigarrow c_{m-1}, c_m, (a_{m+1})^{m+1}$. Note that $\bar{0}1^m\bar{1} \sim c_{m-1}$ holds because any signed inversion sequence $\pi = \bar{0}1^m\bar{1}\pi'$ that avoids B can be mapped to $\alpha = 0\bar{0}0^{m-1}\pi''$, where π'' is obtained from π' by mapping j to $j-1$ and \bar{j} to $\overline{j-1}$. Thus, π avoids B if and only if α avoids B , which gives the required result. Similarly, the other equivalences can be shown.
- As before, we show that all the other succession rules hold.

For several detailed examples, we refer the reader to [9].

Now, we are ready to describe our general procedure, which explains how to obtain an explicit formula for the generating function $F_B(x) = \sum_{n \geq 0} |\bar{\mathcal{I}}_n(B)|x^n$ for the number of signed inversion sequences of length n that avoid a set of signed patterns B . The main procedure is as follows:

- Given a set of signed patterns B and a natural number N ;
- Built the first N levels of $\mathcal{T}[B]$;
- Attempt to guess the succession rules of $\mathcal{T}[B]$ for all levels. If this fails, either increase N and return to the previous step, or stop and declare that the procedure fails;
- Translate the succession rules of $\mathcal{T}[B]$ into a system of equations and solve for $F_B(x) = A_\epsilon(x)$. Note that the rule $e \rightsquigarrow v^{(1)}, \dots, v^{(s)}$ can be translated to $A_e(x) = 1 + x \sum_{j=1}^s A_{v^{(j)}}(x)$, where

$$A_w(x) = \sum_{n \geq 0} (\# \text{the nodes at level } n \text{ in } \mathcal{T}(B; w)) x^n$$

is the generating function for the number of the nodes at level $n \geq 0$ in the subtree of $\mathcal{T}(B; w)$, where its root stays at level 0;

- Using any method, solve the resulting system of equations to obtain $F_B(x) = A_\epsilon(x)$.

Example 2. By translating the succession rules in Example 1 to equations, we obtain

$$\begin{aligned} A_\epsilon(x) &= 1 + xA_{\bar{0}}(x) + xA_{b_1}(x), \\ A_{\bar{0}}(x) &= 1 + xA_{\bar{0}}(x) + xA_{c_0}(x) + xA_{a_1}(x), \\ A_{a_m}(x) &= 1 + xA_{c_{m-1}}(x) + xA_{c_m}(x) + (m + 1)xA_{a_{m+1}}(x), \quad m \geq 1, \\ A_{b_m}(x) &= 1 + xA_{c_{m-1}}(x) + (m + 1)xA_{b_{m+1}}(x), \quad m \geq 1, \\ A_{c_m}(x) &= 1 + (m + 3)xA_{c_{m+1}}(x), \quad m \geq 0. \end{aligned}$$

By iterating, we obtain

$$\begin{aligned} A_{c_m}(x) &= \sum_{j \geq 0} \frac{(m + 2 + j)! x^j}{(m + 2)!}, \\ A_{b_m}(x) &= \sum_{j \geq 0} \frac{(m + j)! x^j}{m!} + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(m + 1 + j + i)! x^{i+j+1}}{m!(m + 1 + j)}, \\ A_{a_m}(x) &= \sum_{j \geq 0} \frac{(m + j)! x^j}{m!} + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(m + 1 + j + i)! x^{i+j+1}}{m!(m + 1 + j)} \\ &\quad + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(m + 2 + j + i)! x^{i+j+1}}{m!(m + 2 + j)(m + 1 + j)}. \end{aligned}$$

Thus,

$$\begin{aligned}
 A_{\bar{0}}(x) &= \frac{1}{1-x} \left(1 + \sum_{j \geq 1} \frac{(j+1)!x^j}{2} \right. \\
 &\quad \left. + \sum_{j \geq 1} j!x^j + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(j+i+2)!x^{i+j+2}}{j+2} + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(j+i+3)!x^{i+j+2}}{(j+3)(j+2)} \right), \\
 A_0(x) = A_{b_1}(x) &= \sum_{j \geq 0} (j+1)!x^j + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(j+i+2)!x^{i+j+1}}{j+2}.
 \end{aligned}$$

Hence, by $A_\epsilon(x) = 1 + xA_{\bar{0}}(x) + xA_{b_1}(x)$, we obtain

$$\begin{aligned}
 A_\epsilon(x) &= \frac{1}{1-x} + \frac{x}{1-x} \left(\sum_{j \geq 1} \frac{(j+1)!x^j}{2} \right. \\
 &\quad \left. + \sum_{j \geq 1} j!x^j + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(j+i+2)!x^{i+j+2}}{j+2} + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(j+i+3)!x^{i+j+2}}{(j+3)(j+2)} \right), \\
 &\quad + x \sum_{j \geq 0} (j+1)!x^j + x \sum_{j \geq 0} \sum_{i \geq 0} \frac{(j+i+2)!x^{i+j+1}}{j+2}.
 \end{aligned}$$

By comparing the coefficients of x^n in both sides, we obtain that the number of signed inversion sequences of length n that avoid $\{\bar{0}\bar{0}, \bar{1}0, 0\bar{1}\}$ is given by

$$1 + \sum_{i=1}^{n-1} \left(\frac{1}{2}(i+1)! + i! + \sum_{j=0}^{i-2} \left(\frac{i!}{j+2} + \frac{(i+1)!}{(j+2)(j+3)} \right) \right) + \sum_{i=1}^n \frac{n!}{i},$$

which simplifies to

$$n! + \sum_{j=1}^n j!H_j,$$

where $H_n = \sum_{j=1}^n \frac{1}{j}$ is the n th Harmonic number.

As an application, in the next sections, we apply the main procedure for any set B of length-2 signed patterns with $|B| \geq 3$.

3. Avoiding a Triple of Length-2 Signed Patterns

There are 12 length-2 signed patterns, and $\binom{12}{3} = 220$ triples of length-2 signed patterns. Calculating the first 7 terms of the counting sequence for the avoiders of each triple gives 93 different sequences, and divides the 220 triples into 93 classes – the *candidate classes*. A candidate class is *trivial* if it contains exactly one triple

of signed patterns; otherwise, it is *nontrivial*. Table 1 shows that there are exactly 83 trivial and 10 nontrivial candidate classes. A trivial class certainly constitutes a (singleton) Wilf-equivalence class, and we do not attempt to enumerate the avoiders in such a class, since such enumeration is not needed for our purpose of identifying Wilf-equivalence classes. The aim of this section is to show that the 10 nontrivial candidate classes of triples of length-2 signed patterns are indeed 10 Wilf-equivalences, that is, in each class, all the counting sequences agree forever, not just in the first 7 terms. Note that by the barring operation, we see that $|\bar{I}_n(B)| = |\bar{I}_n(\bar{B})|$ for all $n \geq 0$. So, in Table 1, we do not include the barring of a set B if B is included.

No.	B	$\{ \bar{I}_n(B) \}_{n=0}^7$
1	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{1}\}$	1, 2, 5, 10, 19, 36, 69, 134
2	$\{\bar{0}\bar{1}, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 5, 11, 23, 47, 95, 191
3	$\{\bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{1}\}$	1, 2, 5, 12, 37, 158, 879, 5920
4	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{1}\}, \{\bar{0}\bar{1}, \bar{0}\bar{1}, 0\bar{1}\}$	1, 2, 5, 13, 34, 89, 233, 610
5	$\{\bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 5, 13, 35, 100, 309, 1041
6	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{0}\}$	1, 2, 5, 13, 35, 99, 296, 935
7	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{0}\}$	1, 2, 5, 14, 43, 146, 551, 2304
8	$\{\bar{0}\bar{1}, \bar{0}\bar{1}, 0\bar{1}\}$	1, 2, 5, 14, 46, 188, 976, 6272
9	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{0}\}$	1, 2, 5, 14, 51, 192, 793, 3568
10	$\{\bar{0}\bar{1}, 0\bar{1}, 0\bar{0}\}$	1, 2, 5, 15, 52, 203, 877, 4140
11	$\{\bar{0}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\}, \{\bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{1}\}$	1, 2, 5, 15, 57, 273, 1593, 10953
12	$\{\bar{0}\bar{1}, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 5, 15, 60, 308, 1910, 13794
13	$\{\bar{0}\bar{0}, \bar{0}\bar{0}, 0\bar{0}\}$	1, 2, 5, 16, 64, 308, 1730, 11104
14	$\{\bar{0}\bar{0}, 0\bar{1}, 0\bar{0}\}$	1, 2, 5, 16, 65, 326, 1957, 13700
15	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{1}\}, \{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{1}\}, \{\bar{0}\bar{1}, 0\bar{1}, 0\bar{0}\}$	1, 2, 5, 17, 74, 394, 2484, 18108
16	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{1}\}, \{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{1}\}$	1, 2, 5, 17, 75, 407, 2619, 19487
17	$\{\bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 5, 17, 75, 408, 2643, 19875
18	$\{\bar{0}\bar{0}, \bar{0}\bar{0}, 0\bar{0}\}$	1, 2, 5, 17, 79, 474, 3468, 29799
19	$\{\bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 5, 18, 85, 494, 3400, 27000
20	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{0}\}$	1, 2, 5, 18, 85, 494, 3401, 27027
21	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{0}\}$	1, 2, 5, 18, 85, 495, 3421, 27330
22	$\{\bar{0}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\}$	1, 2, 5, 18, 86, 508, 3556, 28722
23	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{0}\}$	1, 2, 5, 19, 94, 572, 4122, 34278
24	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{1}\}, \{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{1}\}$	1, 2, 6, 17, 45, 115, 290, 730
25	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\}$	1, 2, 6, 19, 59, 168, 616, 2183
26	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{1}\}$	1, 2, 6, 19, 64, 247, 1166, 6890
27	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{1}\}$	1, 2, 6, 19, 69, 307, 1695, 11311
28	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 6, 20, 69, 243, 870, 3158
29	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\}$	1, 2, 6, 20, 70, 254, 948, 3618
30	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 6, 20, 71, 273, 1149, 5288
31	$\{\bar{1}\bar{0}, 0\bar{1}, 0\bar{1}\}$	1, 2, 6, 21, 80, 321, 1333, 5672
32	$\{\bar{1}\bar{0}, 0\bar{1}, 0\bar{0}\}$	1, 2, 6, 21, 81, 341, 1558, 7679
33	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{1}\}$	1, 2, 6, 21, 85, 407, 2325, 15673
34	$\{\bar{1}\bar{0}, 0\bar{1}, 0\bar{1}\}$	1, 2, 6, 21, 86, 418, 2413, 16330
35	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{1}\}$	1, 2, 6, 21, 87, 433, 2571, 17869
36	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 6, 21, 88, 445, 2676, 18739
37	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\}$	1, 2, 6, 21, 89, 459, 2823, 20211
38	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{1}\}$	1, 2, 6, 22, 90, 394, 1806, 8558
39	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\}$	1, 2, 6, 22, 91, 409, 1955, 9800
40	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{1}\}$	1, 2, 6, 22, 96, 496, 3008, 21120
41	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 6, 22, 96, 497, 3018, 21155
42	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\}$	1, 2, 6, 22, 97, 511, 3178, 22954
43	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\}$	1, 2, 6, 22, 98, 524, 3298, 23960
44	$\{\bar{1}\bar{0}, 0\bar{0}, 0\bar{1}\}$	1, 2, 6, 23, 100, 470, 2330, 12009
45	$\{\bar{1}\bar{0}, 0\bar{1}, 0\bar{0}\}$	1, 2, 6, 23, 102, 499, 2635, 14857
46	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, 0\bar{0}\}$	1, 2, 6, 23, 106, 566, 3415, 22872
47	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, 0\bar{0}\}$	1, 2, 6, 23, 106, 568, 3456, 23469
48	$\{\bar{1}\bar{0}, 0\bar{1}, 0\bar{0}\}$	1, 2, 6, 23, 106, 574, 3603, 25912
49	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\}, \{\bar{1}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\}$	1, 2, 6, 23, 106, 575, 3627, 26289
50	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 6, 23, 107, 581, 3614, 25522

51	$\{\bar{1}0, \bar{0}0, 0\bar{1}\}$	1, 2, 6, 23, 107, 584, 3675, 26382
52	$\{\bar{1}0, 0\bar{0}, 01\}$	1, 2, 6, 23, 107, 591, 3806, 28127
53	$\{\bar{1}0, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 6, 23, 108, 603, 3913, 28992
54	$\{\bar{1}0, \bar{0}0, 01\}$	1, 2, 6, 23, 110, 634, 4279, 33103
55	$\{\bar{1}0, 0\bar{0}, 01\}$	1, 2, 6, 24, 112, 574, 3131, 17857
56	$\{\bar{1}0, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 24, 117, 668, 4379, 32556
57	$\{\bar{1}0, \bar{0}0, 0\bar{0}\}$	1, 2, 6, 24, 118, 678, 4408, 31746
58	$\{\bar{1}0, 0\bar{0}, 01\}$	1, 2, 6, 24, 118, 690, 4692, 36444
59	$\{\bar{1}0, \bar{0}\bar{1}, 0\bar{0}\}, \{\bar{1}0, \bar{0}\bar{1}, 0\bar{0}\}$	1, 2, 6, 24, 119, 702, 4798, 37286
60	$\{\bar{1}0, 0\bar{0}, 0\bar{0}\}$	1, 2, 6, 24, 120, 716, 4940, 38588
61	$\{\bar{1}0, \bar{0}\bar{1}, 0\bar{0}\}, \{\bar{1}0, \bar{0}\bar{1}, 0\bar{0}\}, \{\bar{1}0, \bar{0}\bar{1}, 0\bar{1}\}$ $\{\bar{1}0, \bar{0}\bar{1}, 0\bar{0}\}, \{\bar{1}0, \bar{0}\bar{1}, 0\bar{1}\}, \{\bar{1}0, 0\bar{0}, 0\bar{1}\}$ $\{\bar{1}0, 0\bar{0}, 0\bar{1}\}, \{\bar{1}0, 0\bar{1}, 0\bar{0}\}$	1, 2, 6, 24, 120, 720, 5040, 40320
62	$\{\bar{1}0, 0\bar{0}, 0\bar{0}\}$	1, 2, 6, 25, 131, 811, 5723, 45002
63	$\{\bar{1}0, 0\bar{0}, 0\bar{0}\}$	1, 2, 6, 25, 132, 837, 6184, 52177
64	$\{\bar{1}0, 0\bar{0}, 0\bar{0}\}$	1, 2, 6, 25, 134, 875, 6715, 59082
65	$\{\bar{1}0, 0\bar{0}, 0\bar{0}\}$	1, 2, 6, 25, 134, 875, 6717, 59144
66	$\{\bar{1}0, 0\bar{0}, 0\bar{0}\}$	1, 2, 6, 25, 134, 877, 6769, 60120
67	$\{\bar{1}0, 0\bar{0}, 0\bar{0}\}$	1, 2, 6, 25, 135, 896, 7042, 63841
68	$\{\bar{1}0, 0\bar{0}, 0\bar{0}\}$	1, 2, 6, 26, 148, 1038, 8624, 82652
69	$\{\bar{1}0, 0\bar{0}, 0\bar{0}\}$	1, 2, 6, 26, 148, 1040, 8674, 83582
70	$\{\bar{1}0, \bar{1}0, 01\}$	1, 2, 7, 29, 128, 581, 2674, 12399
71	$\{\bar{1}0, \bar{1}0, 0\bar{1}\}$	1, 2, 7, 29, 132, 657, 3632, 22752,
72	$\{\bar{1}0, \bar{0}\bar{1}, 10\}$	1, 2, 7, 30, 140, 680, 3375, 16980
73	$\{\bar{1}0, \bar{0}\bar{1}, 10\}$	1, 2, 7, 30, 141, 695, 3523, 18200
74	$\{\bar{1}0, 0\bar{1}, 10\}$	1, 2, 7, 30, 143, 729, 3897, 21599
75	$\{\bar{1}0, \bar{0}\bar{1}, 10\}$	1, 2, 7, 30, 146, 800, 4949, 34552
76	$\{\bar{1}0, \bar{0}\bar{1}, 10\}$	1, 2, 7, 30, 147, 806, 4920, 33488
77	$\{\bar{1}0, \bar{0}\bar{1}, 10\}$	1, 2, 7, 30, 147, 817, 5156, 36897
78	$\{\bar{1}0, \bar{0}\bar{1}, 10\}$	1, 2, 7, 30, 147, 819, 5184, 37109
79	$\{\bar{1}0, \bar{0}\bar{1}, 10\}$	1, 2, 7, 30, 151, 884, 5959, 45706
80	$\{\bar{1}0, \bar{1}0, 0\bar{1}\}$	1, 2, 7, 31, 162, 978, 6756, 52970
81	$\{\bar{1}0, \bar{1}0, 0\bar{1}\}$	1, 2, 7, 31, 164, 1012, 7164, 57396
82	$\{\bar{1}0, 0\bar{0}, 10\}$	1, 2, 7, 32, 175, 1098, 7710, 59586
83	$\{\bar{1}0, 0\bar{0}, 10\}$	1, 2, 7, 32, 177, 1147, 8539, 71965
84	$\{\bar{1}0, 0\bar{0}, 10\}$	1, 2, 7, 33, 185, 1162, 7912, 57226
85	$\{\bar{1}0, \bar{1}0, 00\}$	1, 2, 7, 33, 185, 1170, 8121, 60846
86	$\{\bar{1}0, 0\bar{0}, 10\}$	1, 2, 7, 33, 187, 1208, 8650, 67534
87	$\{\bar{1}0, 0\bar{0}, 10\}$	1, 2, 7, 33, 188, 1229, 8960, 71616
88	$\{\bar{1}0, \bar{1}0, 0\bar{0}\}$	1, 2, 7, 33, 188, 1234, 9104, 74409
89	$\{\bar{1}0, 0\bar{0}, 10\}$	1, 2, 7, 33, 189, 1240, 8980, 70052
90	$\{\bar{1}0, \bar{1}0, 0\bar{0}\}, \{\bar{1}0, \bar{1}0, 0\bar{0}\}$	1, 2, 7, 33, 191, 1297, 10063, 87669
91	$\{\bar{1}0, 0\bar{0}, 10\}, \{\bar{1}0, 0\bar{0}, 10\}$	1, 2, 7, 34, 209, 1546, 13327, 130922
92	$\{\bar{1}0, \bar{1}0, 10\}$	1, 2, 8, 42, 258, 1762, 13006, 101952
93	$\{\bar{1}0, \bar{1}0, 10\}$	1, 2, 8, 42, 258, 1768, 13196, 105898

Table 1: Number signed inversion sequences in $\bar{\mathcal{I}}_n(B)$, where $n = 0, 1, \dots, 7$ and B is any triple of length-2 signed patterns.

As discussed above, to study the Wilf-equivalences, we focus on Cases

4, 11, 15, 16, 24, 49, 59, 61, 90, and 91

in Table 1. Table 2 presents the succession rules of each triple in each of these cases whenever we successfully guessed them using our main procedure.

No	B	Succession rules of $\mathcal{T}[B]$
4	$\{\bar{0}\bar{1}, \bar{0}0, 01\}$	$\epsilon \rightsquigarrow \bar{0}, 0; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow a_m, a_{m-1}, \dots, a_1, \bar{0}\bar{0}, a_m; b_m \rightsquigarrow a_m, a_{m-1}, \dots, a_1, \bar{0}\bar{0}, b_{m+1}; c_m \rightsquigarrow c_{m+1}, d_1, d_2, \dots, d_m; d_m \rightsquigarrow d_m, d_1, d_2, \dots, d_m,$ where $a_m = 0^m \bar{m}, b_m = 0^m, c_m = \bar{0}^m,$ and $d_m = \bar{0}^m \bar{m}$
	$\{\bar{0}\bar{1}, 01, 01\}$	$\epsilon \rightsquigarrow \bar{0}, 0; a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0, a_{m+1}; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0, b_m,$ where $a_m = 0^m$ and $b_m = 0^m \bar{m}$ (Lemma 1)
11	$\{\bar{0}\bar{0}, \bar{0}0, 0\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow b_{m-1}, (a_{m+1})^{m+1}; b_m \rightsquigarrow (b_{m+1})^{m+2},$ where $a_m = 0^m$ and $b_m = 0\bar{0}1^m$

	$\{\bar{0}0, \bar{0}1, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow b_0, a_1; \bar{0}0 \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow \bar{0}\bar{0}, (a_{m+1})^{m+1}; b_m \rightsquigarrow (b_{m+1})^{m+2}$, where $a_m = 0^m$ and $b_m = \bar{0}\bar{1} \cdots \bar{m}$ (Lemma 2)
15	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow b_0, b_0; a_m \rightsquigarrow b_{m-1}, (a_{m+1})^{m+1}; b_m \rightsquigarrow (b_{m+1})^{m+3}$, where $a_m = 0^m$ and $b_m = \bar{0}\bar{0}0^m$
	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow b_1, b_1; a_m \rightsquigarrow b_m, (a_{m+1})^{m+1}; b_m \rightsquigarrow (b_{m+1})^{m+2}$, where $a_m = 0^m$ and $b_m = \bar{0}\bar{0}^m$
	$\{\bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow b_1, a_1; b_m \rightsquigarrow b_{m+1}, a_{m+1}, (b_{m+1})^m; a_m \rightsquigarrow (a_{m+1})^{m+1}$, where $a_m = 0^m$ and $b_m = \bar{0}^m$ (Lemma 3)
16	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow b_0, a_1; a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0, (a_{m+1})^{m+1}; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_0, (b_m)^{m+1}$, where $a_m = 0^m$ and $b_m = 0^m \bar{m}$ (Lemma 4)
24	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow b_1, a_1; b_m \rightsquigarrow b_{m+1}, \bar{0}\bar{1}, c_1, c_2, \dots, c_m; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}, \bar{0}\bar{1}; a_m \rightsquigarrow (\bar{0}\bar{1})^{m+1}, a_{m+1}; c_m \rightsquigarrow c_m, \bar{0}\bar{1}, c_1, c_2, \dots, c_m$, where $a_m = 0^m, b_m = \bar{0}^m$, and $c_m = \bar{0}^m m$
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow b_1, a_1; b_m \rightsquigarrow b_{m+1}, \bar{0}\bar{0}, c_1, c_2, \dots, c_m; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}\bar{0}; \bar{0}\bar{1}\bar{0} \rightsquigarrow \bar{0}\bar{1}\bar{0}; a_m \rightsquigarrow d_m, d_{m-1}, \dots, d_1, \bar{0}\bar{0}, a_{m+1}; c_m \rightsquigarrow c_m, \bar{0}\bar{0}, c_1, c_2, \dots, c_m; d_m \rightsquigarrow d_m, d_{m-1}, \dots, d_1, \bar{0}\bar{1}\bar{0}$, where $a_m = 0^m, b_m = \bar{0}^m, c_m = \bar{0}^m m$, and $d_m = 0^m \bar{m}$ (Lemma 5)
49	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, b_0; a_m \rightsquigarrow c_{m,0}, c_{m-1,1}, \dots, c_{1,m-1}, b_{m-1}, (a_{m+1})^{m+1}; b_m \rightsquigarrow \bar{0}, b_0, b_1, \dots, b_{m+1}; c_{i,j} \rightsquigarrow c_{i,0}, c_{i,1}, \dots, c_{i,j}, (c_{i,j+1})^{i+1}$, where $a_m = 0^m, b_m = \bar{0}\bar{0}0^m$, and $c_{i,j} = 0^{i\bar{i}0^j}$
	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, b_1; a_m \rightsquigarrow c_{m,1}, c_{m-1,2}, \dots, c_{1,m}, b_m, (a_{m+1})^{m+1}; b_m \rightsquigarrow \bar{0}, b_1, b_2, \dots, b_{m+1}; c_{i,j} \rightsquigarrow c_{i,1}, c_{i,2}, \dots, c_{i,j}, (c_{i,j+1})^{i+1}$, where $a_m = 0^m, b_m = \bar{0}\bar{0}^m$, and $c_{i,j} = 0^{i\bar{i}j}$ (Lemma 6)
59	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow b_0, b_0; a_m \rightsquigarrow c_{m,0}, c_{m-1,1}, \dots, c_{1,m-1}, b_{m-1}, (a_{m+1})^{m+1}; b_m \rightsquigarrow (b_{m+1})^{m+3}; c_{i,j} \rightsquigarrow c_{i-1,j+1}, c_{i-2,j+1}, \dots, c_{1,j+1}, b_j, (c_{i,j+1})^{j+2}$, where $a_m = 0^m, b_m = \bar{0}\bar{0}0^m$, and $c_{i,j} = 0^{i\bar{i}j}$
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow b_1, b_1; a_m \rightsquigarrow c_{m,1}, c_{m-1,2}, \dots, c_{1,m}, b_m, (a_{m+1})^{m+1}; b_m \rightsquigarrow (b_{m+1})^{m+2}; c_{i,j} \rightsquigarrow c_{i,j+1}, c_{i-1,j+1}, \dots, c_{1,j+1}, b_j, (c_{i,j+1})^j$, where $a_m = 0^m, b_m = \bar{0}\bar{0}^m$, and $c_{i,j} = 0^{i\bar{i}j}$ (Lemma 7)
61	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_1)^2; a_m \rightsquigarrow (b_m)^{m+1}, (a_{m+1})^{m+1}; b_m \rightsquigarrow (b_{m+1})^{m+2}$, where $a_m = 0^m$ and $b_m = \bar{0}\bar{1}\bar{0}^m$
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_1)^2; a_m \rightsquigarrow (b_m)^{m+1}, (a_{m+1})^{m+1}; b_m \rightsquigarrow (b_{m+1})^{m+2}$, where $a_m = 0^m$ and $b_m = \bar{0}\bar{1}^m$
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, \bar{0}; \bar{0} \rightsquigarrow (a_1)^3; a_m \rightsquigarrow (a_{m+1})^{m+3}$, where $a_m = \bar{0}\bar{0}^m$
	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_1)^2; a_m \rightsquigarrow (b_m)^{m+1}, (a_{m+1})^{m+1}; b_m \rightsquigarrow (b_{m+1})^{m+2}$, where $a_m = 0^m$ and $b_m = \bar{0}\bar{1}\bar{2} \cdots \bar{m}$
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	(Lemma 8)
90	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, (b_0)^2; a_m \rightsquigarrow \bar{0}, b_0, b_1, \dots, b_{m-1}, (a_{m+1})^{m+1}; b_m \rightsquigarrow \bar{0}, b_0, b_1, \dots, b_m, (b_{m+1})^{m+3}$, where $a_m = 0^m$ and $b_m = \bar{0}\bar{0}0^m$
	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, (b_1)^2; a_m \rightsquigarrow \bar{0}, b_1, b_2, \dots, b_m, (a_{m+1})^{m+1}; b_m \rightsquigarrow \bar{0}, b_1, b_2, \dots, b_m, (b_{m+1})^{m+2}$, where $a_m = 0^m$ and $b_m = \bar{0}\bar{0}^m$ (Lemma 9)
91	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$	(Lemma 10)

Table 2: Succession rules of $\mathcal{T}[B]$, where B is a triple in Cases 4, 11, 15, 16, 24, 49, 59, 61, 90, and 91 in Table 1.

Now, we show that each class in Table 2 is a Wilf-equivalence class.

Lemma 1. *We have that $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\} \sim \{\bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$. Moreover, the generating function for the number of signed inversion sequences of length n that avoid $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ is given by $\frac{1-x}{1-3x+x^2}$.*

Proof. By translating the succession rules in Table 2 for the generating tree $\mathcal{T}[\bar{0}\bar{1}, \bar{0}\bar{0}]$,

01] to equations, we obtain

$$\begin{aligned}
 A_\epsilon(x) &= 1 + xA_{\bar{0}}(x) + xA_0(x), \\
 A_{0\bar{0}}(x) &= 1 + xA_{0\bar{0}}(x), \\
 A_{a_m}(x) &= 1 + x \sum_{i=1}^m A_{a_i}(x) + xA_{0\bar{0}}(x) + xA_{a_m}(x), \\
 A_{b_m}(x) &= 1 + x \sum_{i=1}^m A_{a_i}(x) + xA_{0\bar{0}}(x) + xA_{b_{m+1}}(x), \\
 A_{c_m}(x) &= 1 + xA_{c_{m+1}}(x) + x \sum_{i=1}^m A_{d_i}(x), \\
 A_{d_m}(x) &= 1 + xA_{d_m}(x) + x \sum_{i=1}^m A_{d_i}(x).
 \end{aligned}$$

Define $A(x, v) = \sum_{m \geq 1} A_{a_m}(x)v^{m-1}$, $B(x, v) = \sum_{m \geq 1} A_{b_m}(x)v^{m-1}$, $C(x, v) = \sum_{m \geq 1} A_{c_m}(x)v^{m-1}$, $D(x, v) = \sum_{m \geq 1} A_{d_m}(x)v^{m-1}$. Then, the above system of recurrences can be written as

$$\begin{aligned}
 A_\epsilon(x) &= 1 + xA_{\bar{0}}(x) + xA_0(x), \\
 A_{0\bar{0}}(x) &= 1 + xA_{0\bar{0}}(x), \\
 A(x, v) &= \frac{1}{1-v} + \frac{x}{1-v}A(x, v) + \frac{x}{1-v}A_{0\bar{0}}(x) + xA(x, v), \\
 B(x, v) &= \frac{1}{1-v} + \frac{x}{1-v}A(x, v) + \frac{x}{1-v}A_{0\bar{0}}(x) + \frac{x}{v}(B(x, v) - A_0(x)), \\
 C(x, v) &= \frac{1}{1-v} + \frac{x}{v}(C(x, v) - A_{\bar{0}}(x)) + \frac{x}{1-v}D(x, v), \\
 D(x, v) &= \frac{1}{1-v} + xD(x, v) + \frac{x}{1-v}D(x, v).
 \end{aligned}$$

Clearly, $D(x, v) = \frac{1}{1-2x-v(1-x)}$ and $A(x, v) = \frac{1}{(1-x)(1-2x-v(1-x))}$. By taking $v = x$ into the equations of $B(x, v)$ and $C(x, v)$, we obtain

$$\begin{aligned}
 A_0(x) &= \frac{1}{1-x} + \frac{x}{1-x}A(x, x) + \frac{x}{1-x}A_{0\bar{0}}(x) = \frac{1}{1-3x+x^2}, \\
 A_{\bar{0}}(x) &= \frac{1}{1-x} + \frac{x}{1-x}D(x, x) = \frac{1-x}{1-3x+x^2},
 \end{aligned}$$

Hence, $A_\epsilon(x) = \frac{1-x}{1-3x+x^2}$.

By translating the succession rules in Table 2 for the generating tree $\mathcal{T}[\bar{0}\bar{1}, \bar{0}\bar{1}, 01]$

to equations, we obtain

$$\begin{aligned}
 A_\epsilon(x) &= 1 + xA_{\bar{0}}(x) + xA_0(x), \\
 A_{a_m}(x) &= 1 + x \sum_{j=0}^m A_{b_j}(x) + xA_{a_{m+1}}(x), \quad m \geq 1, \\
 A_{b_m}(x) &= 1 + x \sum_{j=0}^m A_{b_j}(x) + xA_{b_m}(x), \quad m \geq 0.
 \end{aligned}$$

Define $A(x, v) = \sum_{m \geq 1} A_{a_m}(x)v^{m-1}$ and $B(x, v) = \sum_{m \geq 0} A_{b_m}(x)v^m$. Then, the above system of recurrences can be written as

$$\begin{aligned}
 A_\epsilon(x) &= 1 + xA_{\bar{0}}(x) + xA_0(x), \\
 A(x, v) &= \frac{1}{1-v} + \frac{x}{1-v}(B(x, 0) + \frac{1}{v}(B(x, v) - B(x, 0))) + \frac{x}{v}(A(x, v) - A_0(x)), \\
 B(x, v) &= \frac{1 + xB(x, v)}{1-v} + xB(x, v).
 \end{aligned}$$

Clearly, $B(x, v) = \frac{1}{1-2x-v(1-x)}$. Then by taking $x = v$, we obtain

$$A_0(x) = \frac{(1-x)^2}{(1-2x)(1-3x+x^2)}.$$

Thus, by the fact that $A_{\bar{0}}(x) = B(x, 0)$, we have that $A_\epsilon(x) = \frac{1-x}{1-3x+x^2}$. □

Lemma 2. *We have that $\{\bar{0}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\} \sim \{\bar{0}\bar{0}, \bar{0}1, 0\bar{1}\}$. Moreover, the generating function for the number of signed inversion sequences of length n that avoid $\{\bar{0}\bar{0}, \bar{0}\bar{0}, 0\bar{1}\}$ is given by $1 + \sum_{j=0}^{n-1} (2j+1)j!$.*

Proof. By translating the succession rules in Table 2 for the generating tree $\mathcal{T}[\bar{0}\bar{0}, \bar{0}\bar{0}, 0\bar{1}]$ to equations, we obtain

$$\begin{aligned}
 A_\epsilon(x) &= 1 + xA_{\bar{0}}(x) + xA_{a_1}(x), \\
 A_{\bar{0}}(x) &= 1 + xA_{\bar{0}}(x) + xA_{a_1}(x), \\
 A_{a_m}(x) &= 1 + xA_{b_{m-1}}(x) + (m+1)xA_{a_{m+1}}(x), \quad m \geq 1, \\
 A_{b_m}(x) &= 1 + (m+2)xA_{b_{m+1}}(x), \quad m \geq 0.
 \end{aligned}$$

By iterating, we obtain

$$A_{b_m}(x) = \sum_{j \geq 0} \frac{(m+1+j)!x^j}{(m+1)!},$$

which implies

$$A_{a_m}(x) = \sum_{j \geq 0} \frac{(m+j)!x^j}{m!} + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(m+j+i)!x^{j+i+1}}{m!}.$$

In particular,

$$A_{a_1}(x) = \sum_{j \geq 0} (j + 1)!x^j + \sum_{j \geq 0} \sum_{i \geq 0} (j + i + 1)!x^{j+i+1},$$

which implies

$$\begin{aligned} A_\epsilon(x) &= \frac{1}{1-x} + \frac{x}{1-x} A_{a_1}(x) \\ &= \frac{1}{1-x} + \frac{x}{1-x} \sum_{j \geq 0} (j + 1)!x^j + \frac{x}{1-x} \sum_{j \geq 0} \sum_{i \geq 0} (j + i + 1)!x^{j+i+1}. \end{aligned}$$

By comparing the coefficients of x^n in both sides, we obtain that the number of signed inversion sequences of length n that avoid $\{\bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ is given by $1 + \sum_{j=0}^{n-1} (2j + 1)j!$.

As before, by translating the succession rules in Table 2 for the generating tree $\mathcal{T}[\bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}]$ to equations, we obtain that the number of signed inversion sequences of length n that avoid $\{\bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ is given by $1 + \sum_{j=0}^{n-1} (2j + 1)j!$. \square

Lemma 3. *We have that $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\} \sim \{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\} \sim \{\bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$. Moreover, the number of signed inversion sequences of length n that avoid $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ is given by $n!(1 + H_n)$, where H_n is the n th Harmonic number.*

Proof. By translating the succession rules in Table 2 for the generating tree $\mathcal{T}[\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}]$ to equations, we obtain

$$\begin{aligned} A_\epsilon(x) &= 1 + xA_{\bar{0}}(x) + xA_{a_1}(x), \\ A_{\bar{0}}(x) &= 1 + 2xA_{b_0}(x), \\ A_{a_m}(x) &= 1 + xA_{b_{m-1}}(x) + (m + 1)xA_{a_{m+1}}(x), \quad m \geq 1, \\ A_{b_m}(x) &= 1 + (m + 3)xA_{b_{m+1}}(x), \quad m \geq 0. \end{aligned}$$

As in the proof of Lemma 2, we obtain

$$A_{b_m}(x) = \sum_{j \geq 0} \frac{(m + 2 + j)!x^j}{(m + 2)!}$$

and

$$A_{a_m}(x) = \sum_{j \geq 0} \frac{(m + j)!x^j}{m!} + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(m + 1 + j + i)!x^{j+i+1}}{m!(m + 1 + j)}.$$

Thus, by $A_\epsilon(x) = 1 + x(1 + 2xA_{b_0}(x)) + xA_{a_1}(x)$, we obtain

$$A_\epsilon(x) = 1 + x + x^2 \sum_{j \geq 0} (j + 2)!x^j + x \sum_{j \geq 0} (j + 1)!x^j + x \sum_{i, j \geq 0} \frac{(j + i + 2)!x^{j+i+1}}{j + 2}.$$

By comparing the coefficients of x^n in both sides, we obtain that the number of signed inversion sequences of length n that avoid $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ is given by $n!(1 + H_n)$.

Note that the succession rules of the generating tree $\mathcal{T}[\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}]$ are given by

$$\epsilon \rightsquigarrow \bar{0}, a_1, \quad \bar{0} \rightsquigarrow b_0, b_0, \quad a_m \rightsquigarrow b_{m-1}, (a_{m+1})^{m+1}, \quad b_m \rightsquigarrow (b_{m+1})^{m+3}$$

and the succession rules of the generating tree $\mathcal{T}[\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}]$ are given by

$$\epsilon \rightsquigarrow \bar{0}, a_1, \quad \bar{0} \rightsquigarrow c_1, c_1, \quad a_m \rightsquigarrow c_m, (a_{m+1})^{m+1}, \quad c_m \rightsquigarrow (c_{m+1})^{m+2},$$

where $a_m = 0^m$, $b_m = 0\bar{0}0^m$, and $c_m = 0\bar{0}^m$. By mapping each node b_m in $\mathcal{T}[\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}]$ to c_{m+1} in $\mathcal{T}[\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}]$, we obtain that $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\} \sim \{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$.

Now, by translating the succession rules in Table 2 for the generating tree $\mathcal{T}[\bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}]$ to equations, we obtain

$$\begin{aligned} A_\epsilon(x) &= 1 + xA_{b_1}(x) + xA_{a_1}(x), \\ A_{a_m}(x) &= 1 + (m + 1)xA_{a_{m+1}}(x), \quad m \geq 1, \\ A_{b_m}(x) &= 1 + xA_{a_{m+1}}(x) + (m + 1)xA_{b_{m+1}}(x), \quad m \geq 1. \end{aligned}$$

As in the proof of Lemma 2, we obtain

$$A_{a_m}(x) = \sum_{j \geq 0} \frac{(m + j)!x^j}{m!}$$

and

$$A_{b_m}(x) = \sum_{j \geq 0} \frac{(m + j)!x^j}{m!} + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(m + 1 + j + i)!x^{j+i+1}}{m!(m + 1 + j)}.$$

Thus, by $A_\epsilon(x) = 1 + xA_{b_1}(x) + xA_{a_1}(x)$, we obtain

$$A_\epsilon(x) = 1 + 2 \sum_{j \geq 0} (j + 1)!x^{j+1} + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(j + i + 2)!x^{j+i+2}}{(j + 2)}.$$

By comparing the coefficients of x^n in both sides, we obtain that the number of signed inversion sequences of length n that avoid $\{\bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ is given by $n!(1 + H_n)$. □

Lemma 4. *We have that $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\} \sim \{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$. Moreover, the number of signed inversion sequences of length n that avoid $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ is given by $\sum_{j=0}^n j!(j + 1)^{n-j}$.*

Proof. By Table 2 we see that the generating trees $\mathcal{T}[\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}]$ and $\mathcal{T}[\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}]$ are equal. So, $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\} \sim \{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$. By translating the succession rules in

Table 2 for the generating tree $\mathcal{T}[\overline{01}, \overline{00}, \overline{01}]$ to equations, we obtain

$$\begin{aligned} A_\epsilon(x) &= 1 + xA_{b_0}(x) + xA_{a_1}(x), \\ A_{a_m}(x) &= 1 + x \sum_{j=0}^m A_{b_j}(x) + (m+1)x A_{a_{m+1}}(x), \quad m \geq 1, \\ A_{b_m}(x) &= 1 + x \sum_{j=0}^m A_{b_j}(x) + mx A_{b_m}(x), \quad m \geq 0. \end{aligned}$$

By induction on m , we have that $A_{b_m}(x) = \frac{1}{(1-mx)(1-(m+1)x)}$. Thus,

$$1 + x \sum_{j=0}^m A_{b_j}(x) = \frac{1}{1 - (m+1)x},$$

which implies

$$A_{a_m}(x) = \frac{1}{1 - (m+1)x} + (m+1)x A_{a_{m+1}}(x),$$

for all $m \geq 1$. By iterating, we obtain

$$A_{a_m}(x) = \sum_{j \geq 0} \frac{(m+j)!x^j}{m!(1 - (m+1+j)x)}.$$

Hence,

$$A_\epsilon(x) = \sum_{j \geq 0} \frac{j!x^j}{(1 - (j+1)x)},$$

as claimed. □

Lemma 5. *We have that $\{\overline{10}, \overline{01}, 01\} \sim \{\overline{10}, \overline{01}, 01\}$. Moreover, the generating function for the number of signed inversion sequences of length n that avoid $\{\overline{10}, \overline{01}, 01\}$ is given by*

$$1 + \frac{x(1-x)^2}{(1-2x)(1-3x+x^2)} + \frac{x(1-x+x^2)}{(1-2x)(1-x)^2}.$$

Proof. By translating the succession rules in Table 2 for the generating tree $\mathcal{T}[\overline{10}, \overline{01}, 01]$ to equations, we obtain

$$\begin{aligned} A_\epsilon(x) &= 1 + xA_{b_1}(x) + xA_{a_1}(x), \\ A_{b_m}(x) &= 1 + xA_{b_{m+1}}(x) + xA_{0\overline{1}}(x) + x \sum_{j=1}^m A_{c_j}(x), \\ A_{0\overline{1}}(x) &= 1 + 2xA_{0\overline{1}}(x), \\ A_{a_m}(x) &= 1 + (m+1)x A_{0\overline{1}}(x) + xA_{a_{m+1}}(x), \\ A_{c_m}(x) &= 1 + xA_{c_m}(x) + xA_{0\overline{1}}(x) + x \sum_{j=1}^m A_{c_j}(x). \end{aligned}$$

Clearly, $A_{0\bar{1}}(x) = \frac{1}{1-2x}$. By iterating, we obtain that $A_{a_m}(x) = \frac{1+(m-2)x-(m-2)x^2}{(1-2x)(1-x)^2}$. By induction on m , we have that $A_{c_m}(x) = \frac{(1-x)^m}{(1-2x)^{m+1}}$. Thus,

$$A_{b_m}(x) = xA_{b_{m+1}}(x) + \frac{1-x}{1-2x} + x \sum_{j=1}^m \frac{(1-x)^j}{(1-2x)^{j+1}},$$

with $m \geq 1$, which, by iterating, implies

$$A_{b_m}(x) = \frac{(1-x)^{m+1}}{(1-2x)^m(1-3x+x^2)}.$$

Hence,

$$A_\epsilon(x) = 1 + \frac{x(1-x)^2}{(1-2x)(1-3x+x^2)} + \frac{x(1-x+x^2)}{(1-2x)(1-x)^2}.$$

Now, by translating the succession rules in Table 2 for the generating tree $\mathcal{T}[\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}]$ to equations, we obtain

$$\begin{aligned} A_\epsilon(x) &= 1 + xA_{b_1}(x) + xA_{a_1}(x), \\ A_{b_m}(x) &= 1 + xA_{b_{m+1}}(x) + xA_{0\bar{0}\bar{0}}(x) + x \sum_{j=1}^m A_{c_j}(x), \\ A_{0\bar{0}\bar{0}}(x) &= 1 + 2xA_{0\bar{0}\bar{0}}(x), \\ A_{0\bar{0}\bar{1}\bar{0}}(x) &= 1 + xA_{0\bar{0}\bar{1}\bar{0}}(x), \\ A_{a_m}(x) &= 1 + x \sum_{j=1}^m A_{d_j}(x) + xA_{0\bar{0}\bar{0}}(x) + xA_{a_{m+1}}(x), \\ A_{c_m}(x) &= 1 + xA_{c_m}(x) + xA_{0\bar{0}\bar{0}}(x) + x \sum_{j=1}^m A_{c_j}(x), \\ A_{d_m}(x) &= 1 + x \sum_{j=1}^m A_{d_j}(x) + xA_{0\bar{0}\bar{1}\bar{0}}(x). \end{aligned}$$

Clearly, $A_{0\bar{0}\bar{1}\bar{0}}(x) = \frac{1}{1-x}$ and $A_{0\bar{0}\bar{0}}(x) = \frac{1}{1-2x}$. By induction on m , we have that $A_{d_m}(x) = \frac{1}{(1-x)^{m+1}}$ and $A_{c_m}(x) = \frac{(1-x)^m}{(1-2x)^{m+1}}$. By iterating, we obtain $A_{a_m}(x) = \frac{1}{(1-x)^m(1-2x)} + \frac{x^2}{(1-2x)(1-x)^2}$ and $A_{b_m}(x) = \frac{(1-x)^{m+1}}{(1-2x)^m(1-3x+x^2)}$. Hence,

$$A_\epsilon(x) = 1 + \frac{x(1-x)^2}{(1-2x)(1-3x+x^2)} + \frac{x(1-x+x^2)}{(1-2x)(1-x)^2},$$

as claimed. □

Lemma 6. *We have that $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\} \sim \{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$.*

Proof. The succession rules of $\mathcal{T}_1 = \mathcal{T}[\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}]$ are given by

$$\begin{aligned} \epsilon &\rightsquigarrow \bar{0}, a_1, & \bar{0} &\rightsquigarrow \bar{0}, b_0, \\ a_m &\rightsquigarrow c_{m,0}, c_{m-1,1}, \dots, c_{1,m-1}, b_{m-1}, (a_{m+1})^{m+1}, & b_m &\rightsquigarrow \bar{0}, b_0, b_1, \dots, b_{m+1}, \\ c_{i,j} &\rightsquigarrow c_{i,0}, c_{i,1}, \dots, c_{i,j}, (c_{i,j+1})^{i+1} \end{aligned}$$

and the succession rules of $\mathcal{T}_2 = \mathcal{T}[\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}]$ are given by

$$\begin{aligned} \epsilon &\rightsquigarrow \bar{0}, a'_1, & \bar{0} &\rightsquigarrow \bar{0}, b'_1, \\ a'_m &\rightsquigarrow c'_{m,1}, c'_{m-1,2}, \dots, c'_{1,m}, b'_m, (a'_{m+1})^{m+1}, & b'_m &\rightsquigarrow \bar{0}, b'_1, b'_2, \dots, b'_{m+1}, \\ c'_{i,j} &\rightsquigarrow c'_{i,1}, c'_{i,2}, \dots, c'_{i,j}, (c'_{i,j+1})^{i+1}, \end{aligned}$$

where $a_m = 0^m$, $b_m = 0\bar{0}0^m$, $c_{i,j} = 0^i\bar{i}0^j$, $a'_m = 0^m$, $b'_m = 0\bar{0}^m$, and $c'_{i,j} = 0^i\bar{i}^j$. By mapping the nodes of \mathcal{T}_1 as $a_m \mapsto a'_m$, $b_m \mapsto b'_{m+1}$, and $c_{i,j} \mapsto c'_{i,j+1}$, we obtain that the succession rules of \mathcal{T}_1 map to the succession rules of \mathcal{T}_2 . Hence, \mathcal{T}_1 is isomorphic, as a plane tree, to \mathcal{T}_2 . \square

Lemma 7. *We have that $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\} \sim \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$.*

Proof. The succession rules of $\mathcal{T}_1 = \mathcal{T}[\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}]$ are given by

$$\begin{aligned} \epsilon &\rightsquigarrow \bar{0}, a_1, & \bar{0} &\rightsquigarrow b_0, b_0, \\ a_m &\rightsquigarrow c_{m,0}, c_{m-1,1}, \dots, c_{1,m-1}, b_{m-1}, (a_{m+1})^{m+1}, & b_m &\rightsquigarrow (b_{m+1})^{m+3}, \\ c_{i,j} &\rightsquigarrow c_{i-1,j+1}, c_{i-2,j+1}, \dots, c_{1,j+1}, b_j, (c_{i,j+1})^{j+2} \end{aligned}$$

and the succession rules of $\mathcal{T}_2 = \mathcal{T}[\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}]$ are given by

$$\begin{aligned} \epsilon &\rightsquigarrow \bar{0}, a'_1, & \bar{0} &\rightsquigarrow b'_1, b'_1, \\ a'_m &\rightsquigarrow c'_{m,1}, c'_{m-1,2}, \dots, c'_{1,m}, b'_m, (a'_{m+1})^{m+1}, & b'_m &\rightsquigarrow (b'_{m+1})^{m+2}, \\ c'_{i,j} &\rightsquigarrow c'_{i,j+1}, c'_{i-1,j+1}, \dots, c'_{1,j+1}, b'_j, (c'_{i,j+1})^j, \end{aligned}$$

where $a_m = 0^m$, $b_m = 0\bar{0}0^m$, $c_{i,j} = 0^i\bar{i}^j$, $a'_m = 0^m$, $b'_m = 0\bar{0}^m$, and $c'_{i,j} = 0^i\bar{i}^j$. By mapping the nodes of \mathcal{T}_1 as $a_m \mapsto a'_m$, $b_m \mapsto b'_{m+1}$, and $c_{i,j} \mapsto c'_{i,j+1}$, we obtain that the succession rules of \mathcal{T}_1 map to the succession rules of \mathcal{T}_2 . Hence, \mathcal{T}_1 is isomorphic as plane trees to \mathcal{T}_2 . \square

Lemma 8. *For all $n \geq 0$,*

- (i) $|\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0})| = (n + 1)!$;
- (ii) $|\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0})| = (n + 1)!$;
- (iii) $|\mathcal{I}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{1})| = |\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, 0\bar{1})| = (n + 1)!$;
- (iv) $|\mathcal{I}_n(\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1})| = (n + 1)!$;

(v) $|\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0})| = (n + 1)!;$

(vi) $|\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1})| = (n + 1)!;$

(vii) $|\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1})| = (n + 1)!.$

Proof. (i) By translating the succession rules of the generating tree $\mathcal{T}[\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}]$ in Table 2 to equations, we obtain $A_\epsilon(x) = 1 + xA_{\bar{0}}(x) + xA_{a_1}(x)$, $A_{\bar{0}}(x) = 1 + 2xA_{b_1}(x)$, $A_{a_m}(x) = 1 + (m + 1)xA_{b_m}(x) + (m + 1)xA_{a_{m+1}}(x)$, and $A_{b_m}(x) = 1 + (m + 2)xA_{b_{m+1}}(x)$. By iterating, we obtain $A_{b_m}(x) = \sum_{j \geq 0} \frac{(m+1+j)!x^j}{(m+1)!}$, and then

$$A_{a_m}(x) = \sum_{j \geq 0} \frac{(m + j)!x^j}{m!} + \sum_{j \geq 0} \sum_{i \geq 0} \frac{(m + j + 1 + i)!x^{j+i+1}}{m!}.$$

Hence,

$$A_\epsilon(x) = \sum_{j \geq 0} j!x^j + \sum_{j \geq 1} j!x^j + \sum_{j \geq 0} \sum_{i \geq 0} (j + i + 2)!x^{j+i+2},$$

which, by comparing the coefficient of x^n in both sides, implies that

$$|\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0})| = (n + 1)!.$$

Note that any signed inversion sequence in $\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0})$ is given by a signed inversion sequence where at most one of its letters is barred. Since there are $n!$ inversion sequences of length n , we have $(n + 1)!$ signed inversion sequences in $\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0})$.

(ii) This case follows immediately from the fact that the generating tree of this case is isomorphic to the generating tree in Case (i).

(iii) By translating the succession rules of the generating tree $\mathcal{T}[\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}]$ in Table 2 to equations, we obtain $A_\epsilon(x) = 1 + 2xA_{\bar{0}}(x)$, $A_{\bar{0}}(x) = 1 + 3xA_{a_1}(x)$, and $A_{a_m}(x) = 1 + (m + 3)xA_{a_{m+1}}(x)$. By iterating, we obtain that $A_{a_m}(x) = \sum_{j \geq 0} \frac{(m+2+j)!x^j}{(m+2)!}$. Thus, $A_{a_1}(x) = \sum_{j \geq 0} \frac{(j+3)!x^j}{6}$, which gives that $A_{\bar{0}}(x) = 1 + \sum_{j \geq 1} \frac{(j+2)!x^j}{2}$ and then $A_\epsilon(x) = \sum_{j \geq 0} (j + 1)!x^j$. Hence,

$$|\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1})| = |\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1})| = (n + 1)!$$

(both cases have the same generating tree).

(iv) The proof is very similar to Cases (i) and (iii).

(v) Note that any signed inversion sequence π in $\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0})$ satisfies either (1) π has no barring letter, or (2) there exists a letter k such that π contains the subsequence $\bar{k}^a k^b$ with $a \geq 1$ and $b \geq 0$ and no other barring letters in π . Also, any signed inversion sequence π in $\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0})$ satisfies either (1) π has no barring letter, or (2) there exists a letter k such that π contains the subsequence $k^b \bar{k}^a$ with $a \geq 1$ and $b \geq 0$ and no other barring letters in π . By this easy to see that $|\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0})| = |\bar{\mathcal{I}}_n(\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0})| = (n + 1)!.$

(vi) Note that any signed inversion sequence $\pi = \pi_1\pi_2 \cdots \pi_n$ in $\bar{\mathcal{I}}_n(\bar{10}, \bar{00}, 0\bar{1})$ satisfies either (1) π has no barring letter, or (2) there exist i, k such that $\pi_i = \bar{k}$, where π_j is barring if and only if $|\pi_j| < k$ or $\pi_j = \bar{k}$ with $j \geq i$. Since the number of inversion sequences of length n is $n!$ and the letter k has $n + 1$ possibilities (any letter from 0 to $n - 1$, or else k does not exist), we obtain that there are exactly $(n + 1)!$ signed inversion sequences in $\bar{\mathcal{I}}_n(\bar{10}, \bar{00}, 0\bar{1})$.

(vii) Note that any signed inversion sequence $\pi = \pi_1\pi_2 \cdots \pi_n$ in $\bar{\mathcal{I}}_n(\bar{10}, 0\bar{0}, 0\bar{1})$ satisfies either (1) π has no barring letter, or (2) there exist i, k such that $\pi_i = \bar{k}$, where π_j is barring if and only if $|\pi_j| < k$ or $\pi_j = \bar{k}$ with $j \leq i$. So, there are exactly $(n + 1)!$ signed inversion sequences in $\bar{\mathcal{I}}_n(\bar{10}, 0\bar{0}, 0\bar{1})$. \square

Lemma 9. *We have that $\{\bar{10}, \bar{10}, \bar{00}\} \sim \{\bar{10}, \bar{10}, 0\bar{0}\}$.*

Proof. The succession rules of $\mathcal{T}_1 = \mathcal{T}[\bar{10}, \bar{10}, \bar{00}]$ are given by

$$\begin{aligned} \epsilon &\rightsquigarrow \bar{0}, 0, & \bar{0} &\rightsquigarrow \bar{0}, (b_0)^2, \\ a_m &\rightsquigarrow \bar{0}, b_0, b_1, \dots, b_{m-1}, (a_{m+1})^{m+1}, & b_m &\rightsquigarrow \bar{0}, b_0, b_1, \dots, b_m, (b_{m+1})^{m+3} \end{aligned}$$

and the succession rules of $\mathcal{T}_2 = \mathcal{T}[\bar{10}, \bar{10}, 0\bar{0}]$ are given by

$$\begin{aligned} \epsilon &\rightsquigarrow \bar{0}, 0, & \bar{0} &\rightsquigarrow \bar{0}, (b'_1)^2, \\ a'_m &\rightsquigarrow \bar{0}, b'_1, b'_2, \dots, b'_m, (a'_{m+1})^{m+1}, & b'_m &\rightsquigarrow \bar{0}, b'_1, b'_2, \dots, b'_m, (b'_{m+1})^{m+2}, \end{aligned}$$

where $a_m = 0^m$, $b_m = 0\bar{0}0^m$, $a'_m = 0^m$, and $b'_m = 0\bar{0}^m$. By mapping the nodes of \mathcal{T}_1 as $a_m \mapsto a'_m$ and $b_m \mapsto b'_{m+1}$, we obtain that the succession rules of \mathcal{T}_1 map to the succession rules of \mathcal{T}_2 . Hence, \mathcal{T}_1 is isomorphic as plane trees to \mathcal{T}_2 . \square

Lemma 10. *We have that $\{\bar{10}, \bar{00}, 1\bar{0}\} \sim \{\bar{10}, 0\bar{0}, 1\bar{0}\}$.*

Proof. Let us describe a bijection f between

$$A_n = \bar{\mathcal{I}}_n(\bar{10}, \bar{00}, 1\bar{0}) \text{ and } B_n = \bar{\mathcal{I}}_n(\bar{10}, 0\bar{0}, 1\bar{0}).$$

Let $\pi = \pi_1\pi_2 \cdots \pi_n \in A_n$. Since π avoids $\{\bar{10}, \bar{00}\}$, all the barred letters of π forms an increasing subsequence $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_m}$ (here $|\pi_{i_1}| < \cdots < |\pi_{i_m}|$ and $m \geq 0$). Since $\pi \in A_n$, then there are no letters on left side of π_{i_j} in π with absolute value bigger than $|\pi_{i_j}|$. Moreover, each letter on the left side of π_{i_j} whose absolute value is equal to $|\pi_{i_j}|$ is not barred.

Now, we define $f(\pi)$ to be the same as π , where we barring all the letters π_a of π such that there exists i_j with $|\pi_a| = |\pi_{i_j}|$ and $a < i_j$. Clearly, $f(\pi)$ avoids $\{\bar{10}, 1\bar{0}\}$. Also, by definitions, $f(\pi)$ avoids $0\bar{0}$ if and only if π avoids $\bar{00}$. Hence, $f(\pi) \in B_n$. Since $f^2 = id$, we obtain that f is a bijection. \square

By Lemmas 1-10 and Table 1, we have the following result.

Theorem 2. *The number of Wilf-equivalences among sets of triples of length-2 signed patterns is 93.*

4. Set of Four of Length-2 Signed Patterns

There are $\binom{12}{4} = 495$ sets of four of length-2 signed patterns. Calculating the first 7 terms of the counting sequence for the avoiders for each of these sets gives 195 different sequences, and divides the 495 sets of four of length-2 signed patterns into 195 candidate classes. Table 3 shows that there are exactly 158 trivial and 37 nontrivial candidate classes. The aim of this section is to show that the 37 nontrivial candidate classes of sets of four of length-2 signed patterns are indeed 37 Wilf-equivalences, that is, in each class, all the counting sequences agree forever, not just in the first 7 terms. Note that by the barring operation, we see that $|\bar{I}_n(B)| = |\bar{I}_n(\bar{B})|$, for all $n \geq 0$. So, in Table 3, we do not include barring of a set B if B is included.

No.	B	$\{\bar{I}_n(B)\}_{n=0}^7$
1	{01, 00, 00, 01}	1,2,4,10,26,68,178,466
2	{01, 00, 01, 00}, {01, 01, 00, 00}	1,2,4,10,30,104,406,1754
3	{01, 00, 01, 01}, {01, 00, 01, 01}	
	{00, 00, 01, 00}	1,2,4,10,34,154,874,5914
4	{01, 00, 00, 01}, {01, 00, 01, 00}	
	{01, 00, 01, 00}, {00, 01, 01, 00}	1,2,4,12,48,240,1440,10080
5	{01, 00, 01, 00}, {01, 00, 01, 00}	1,2,4,12,50,262,1640,11920
6	{00, 00, 01, 00}	1,2,4,12,52,287,1888,14356
7	{01, 00, 00, 00}	1,2,4,13,58,324,2166,16827
8	{01, 00, 00, 01}	12,4,14,62,338,2186,16346
9	{01, 00, 00, 01}	1,2,4,2,0,0,0,0
10	{00, 01, 01, 00}	1,2,4,4,4,4,4,4
11	{01, 00, 01, 00}	1,2,4,6,10,16,26,42
12	{01, 00, 01, 01}, {01, 01, 01, 00}	
	{01, 01, 00, 00}	1,2,4,6,8,10,12,14
13	{01, 00, 00, 01}	1,2,4,7,12,21,38,71
14	{01, 00, 00, 01}	1,2,4,7,13,25,49,97
15	{01, 00, 01, 01}, {01, 00, 01, 00}	
	{01, 00, 00, 00}, {01, 00, 01, 00}	
	{01, 01, 01, 01}, {00, 00, 01, 00}	
	{00, 00, 00, 00}	1,2,4,8,16,32,64,128
16	{01, 00, 01, 01}, {01, 01, 00, 01}	1,2,4,8,17,39,95,240
17	{01, 00, 01, 01}	1,2,4,9,22,56,145,378
18	{01, 00, 00, 00}	1,2,4,9,22,57,156,452
19	{00, 00, 01, 00}	1,2,4,9,23,66,210,733
20	{01, 00, 00, 00}	1,2,4,9,24,76,279,1156
21	{01, 01, 01, 00}, {00, 01, 01, 00}	1,2,4,9,28,125,726,5047
22	{00, 00, 01, 01}	1,2,4,9,31,145,841,5761
23	{10, 01, 01, 00}, {10, 00, 01, 01}	1,2,5,10,17,26,37,50
24	{10, 01, 00, 01}	1,2,5,10,19,36,69,134
25	{10, 00, 01, 01}, {10, 01, 01, 00}	1,2,5,11,23,47,95,191
26	{10, 01, 01, 00}	1,2,5,11,25,64,184,573
27	{10, 01, 01, 00}	1,2,5,11,31,129,731,5053
28	{10, 00, 01, 01}	1,2,5,11,35,155,875,5915
29	{10, 00, 01, 00}	1,2,5,12,26,52,99,184
30	{10, 01, 00, 00}, {10, 01, 01, 01}	
	{10, 01, 01, 01}, {10, 00, 01, 00}	
	{00, 01, 00, 10}	1,2,5,12,27,58,121,248
31	{10, 00, 00, 01}	1,2,5,12,27,60,135,312
32	{10, 01, 00, 01}, {10, 01, 00, 01}	1,2,5,12,28,65,152,360
33	{10, 01, 00, 01}	1,2,5,12,29,71,176,441
34	{10, 00, 00, 01}	1,2,5,12,29,72,182,466
35	{10, 00, 01, 00}	1,2,5,12,31,88,275,942
36	{10, 00, 01, 01}	1,2,5,12,37,158,879,5920
37	{10, 01, 00, 01}	1,2,5,13,33,82,202,498
38	{10, 01, 01, 01}, {10, 01, 01, 01}	1,2,5,13,34,89,233,610
39	{10, 00, 00, 01}, {10, 01, 01, 01}	1,2,5,13,34,91,254,746

40	{ $\bar{1}0, \bar{0}0, \bar{0}0, 01$ }	1,2,5,13,34,92,265,827
41	{ $\bar{1}0, \bar{0}0, 0\bar{1}, 00$ }, { $\bar{1}0, \bar{0}0, \bar{0}0, 01$ }	1,2,5,13,35,100,309,1041
42	{ $\bar{1}0, 0\bar{1}, \bar{0}0, 00$ }	1,2,5,13,35,99,296,935
43	{ $\bar{1}0, \bar{0}0, \bar{0}1, 01$ }	1,2,5,13,36,108,353,1255
44	{ $\bar{1}0, 0\bar{1}, 0\bar{1}, 01$ }	1,2,5,13,39,151,783,5167
45	{ $\bar{1}0, \bar{0}0, \bar{0}1, 01$ }, { $\bar{1}0, \bar{0}0, \bar{0}1, 00$ }	1,2,5,14,42,132,429,1430
46	{ $\bar{1}0, 0\bar{1}, \bar{0}0, 00$ }, { $\bar{1}0, 0\bar{1}, 0\bar{0}, 00$ }	
	{ $\bar{1}0, 0\bar{1}, 0\bar{0}, 00$ }	1,2,5,14,43,146,551,2304
47	{ $\bar{1}0, \bar{0}1, \bar{0}0, 00$ }	1,2,5,14,44,150,537,1978
48	{ $\bar{1}0, \bar{0}0, 0\bar{1}, 01$ }	1,2,5,14,46,168,647,2559
49	{ $\bar{1}0, 0\bar{1}, 0\bar{1}, 00$ }	1,2,5,14,46,184,916,5665
50	{ $\bar{1}0, 0\bar{1}, 0\bar{1}, 01$ }, { $\bar{1}0, 0\bar{1}, 0\bar{1}, 01$ }	1,2,5,14,46,188,976,6272
51	{ $\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{1}$ }	1,2,5,14,47,195,1005,6342
52	{ $\bar{1}0, \bar{0}1, \bar{0}0, 00$ }	1,2,5,14,48,209,1135,7412
53	{ $\bar{1}0, 0\bar{1}, 0\bar{1}, 0\bar{0}$ }	1,2,5,14,50,232,1337,9147
54	{ $\bar{1}0, 0\bar{1}, 0\bar{1}, 0\bar{0}$ }	1,2,5,14,51,244,1445,10086
55	{ $\bar{1}0, \bar{0}1, 00, 01$ }	1,2,5,15,49,168,594,2145
56	{ $\bar{1}0, \bar{0}0, \bar{0}0, 01$ }	1,2,5,15,49,169,607,2249
57	{ $\bar{1}0, \bar{0}0, \bar{0}1, 01$ }	1,2,5,15,50,180,691,2807
58	{ $\bar{1}0, 0\bar{1}, \bar{0}0, 00$ }	1,2,5,15,51,192,793,3568
59	{ $\bar{1}0, \bar{0}0, \bar{0}0, 00$ }	1,2,5,15,52,203,876,4119
60	{ $\bar{1}0, 0\bar{1}, 0\bar{1}, 00$ }, { $\bar{1}0, 0\bar{1}, 0\bar{1}, 00$ }	1,2,5,15,52,203,877,4140
61	{ $\bar{1}0, \bar{0}0, \bar{0}0, 00$ }	1,2,5,15,53,216,993,5056
62	{ $\bar{1}0, 0\bar{1}, 00, 01$ }	1,2,5,15,55,249,1371,8953
63	{ $\bar{1}0, \bar{0}0, \bar{0}1, 00$ }	1,2,5,15,57,270,1552,10551
64	{ $\bar{1}0, \bar{0}0, \bar{0}0, 0\bar{1}$ }, { $\bar{1}0, \bar{0}0, \bar{0}0, 0\bar{1}$ }	
	{ $\bar{1}0, \bar{0}0, 0\bar{1}, 0\bar{0}$ }, { $\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{1}$ }	1,2,5,15,57,273,1593,10953
65	{ $\bar{1}0, \bar{0}0, 0\bar{1}, 01$ }	1,2,5,15,59,294,1764,12344
66	{ $\bar{1}0, 0\bar{1}, \bar{0}1, 0\bar{0}$ }	1,2,5,15,59,295,1782,12573
67	{ $\bar{1}0, 0\bar{1}, 00, 01$ }	1,2,5,16,57,215,843,3398
68	{ $\bar{1}0, 0\bar{1}, 0\bar{0}, 01$ }	1,2,5,16,59,234,967,4104
69	{ $\bar{1}0, \bar{0}0, 00, 01$ }	1,2,5,16,59,238,1020,4566
70	{ $\bar{1}0, \bar{0}1, \bar{0}0, 01$ }	1,2,5,16,60,248,1092,5024
71	{ $\bar{1}0, \bar{0}0, 0\bar{1}, 00$ }	1,2,5,16,61,261,1215,6060
72	{ $\bar{1}0, \bar{0}0, \bar{0}0, 00$ }	1,2,5,16,63,292,1546,9164
73	{ $\bar{1}0, \bar{0}0, \bar{0}0, 00$ }	1,2,5,16,63,293,1566,9437
74	{ $\bar{1}0, 00, 00, 00$ }	1,2,5,16,64,305,1660,10033
75	{ $\bar{1}0, 0\bar{1}, 00, 01$ }	1,2,5,16,64,316,1876,13032
76	{ $\bar{1}0, 0\bar{1}, \bar{0}0, \bar{0}1$ }	1,2,5,16,64,318,1902,13308
77	{ $\bar{1}0, 0\bar{1}, \bar{0}0, 01$ }	1,2,5,16,65,324,1928,13387
78	{ $\bar{1}0, \bar{0}0, 0\bar{1}, 0\bar{0}$ }, { $\bar{1}0, 0\bar{1}, \bar{0}0, \bar{0}1$ }	
	{ $\bar{1}0, \bar{0}0, \bar{0}1, 00$ }, { $\bar{1}0, \bar{0}0, \bar{0}1, 01$ }	
	{ $\bar{1}0, \bar{0}1, 00, 01$ }	1,2,5,16,65,326,1957,13700
79	{ $\bar{1}0, 0\bar{1}, \bar{0}0, \bar{0}1$ }, { $\bar{1}0, 0\bar{1}, \bar{0}0, \bar{0}1$ }	1,2,5,16,66,338,2070,14750
80	{ $\bar{1}0, \bar{0}0, \bar{0}0, 0\bar{0}$ }	1,2,5,16,66,341,2119,15329
81	{ $\bar{1}0, \bar{0}0, \bar{0}0, 01$ }	1,2,5,17,69,311,1500,7583
82	{ $\bar{1}0, 0\bar{1}, 0\bar{0}, 00$ }	1,2,5,17,70,325,1648,8994
83	{ $\bar{1}0, \bar{0}0, 0\bar{1}, 0\bar{0}$ }	1,2,5,17,72,362,2117,14251
84	{ $\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{0}$ }, { $\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{0}$ }	1,2,5,17,73,381,2358,16944
85	{ $\bar{1}0, 00, 01, 00$ }	1,2,5,17,74,392,2446,17574
86	{ $\bar{1}0, 0\bar{1}, 0\bar{0}, 0\bar{0}$ }, { $\bar{1}0, 0\bar{1}, 0\bar{0}, 0\bar{0}$ }	1,2,5,17,74,392,2446,17577
87	{ $\bar{1}0, \bar{0}0, 0\bar{1}, 00$ }, { $\bar{1}0, 0\bar{1}, 0\bar{0}, 00$ }	1,2,5,17,74,393,2469,17946
88	{ $\bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{1}$ }, { $\bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{1}$ }	
	{ $\bar{1}0, 0\bar{1}, 0\bar{1}, 0\bar{0}$ }, { $\bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{1}$ }	
	{ $\bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{1}$ }, { $\bar{1}0, 0\bar{1}, 0\bar{1}, 0\bar{0}$ }	1,2,5,17,74,394,2484,18108
89	{ $\bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1$ }	1,2,5,17,75,407,2614,19367
90	{ $\bar{1}0, \bar{0}0, \bar{0}0, 01$ }	1,2,5,17,75,407,2618,19461
91	{ $\bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1$ }	1,2,5,17,75,407,2619,19487
92	{ $\bar{1}0, \bar{0}0, \bar{0}0, 00$ }	1,2,5,17,78,454,3175,25783
93	{ $\bar{1}0, \bar{0}0, \bar{0}0, 0\bar{0}$ }	1,2,5,17,78,456,3224,26623
94	{ $\bar{1}0, \bar{0}0, 00, 01$ }	1,2,5,18,75,342,1654,8339
95	{ $\bar{1}0, 00, 00, 01$ }	1,2,5,18,81,441,2830,20960
96	{ $\bar{1}0, 0\bar{1}, 00, 00$ }	1,2,5,18,83,465,3062,23160
97	{ $\bar{1}0, \bar{0}0, 00, 01$ }	1,2,5,18,84,478,3204,24704
98	{ $\bar{1}0, 0\bar{1}, \bar{0}0, \bar{0}0$ }, { $\bar{1}0, 0\bar{1}, \bar{0}0, \bar{0}0$ }	
	{ $\bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{0}$ }, { $\bar{1}0, \bar{0}0, 0\bar{1}, 0\bar{0}$ }	1,2,5,18,84,480,3240,25200
99	{ $\bar{1}0, 0\bar{1}, \bar{0}0, 01$ }	1,2,5,9,14,20,27,35

100	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 01$ }	1,2,5,9,14,21,31,46
101	{ $\bar{1}0, \bar{0}\bar{1}, 00, 01$ }	1,2,5,9,17,33,65,129
102	{ $\bar{1}0, \bar{0}\bar{1}, 01, 1\bar{0}$ }, { $\bar{1}0, \bar{0}\bar{1}, 01, 10$ }	
	{ $\bar{1}0, \bar{0}\bar{1}, 01, 1\bar{0}$ }	1,2,6,16,38,84,178,368
103	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, 01$ }	1,2,6,16,40,99,248,631
104	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{0}, 01$ }	1,2,6,18,50,130,322,770
105	{ $\bar{1}0, \bar{0}1, \bar{0}\bar{1}, 10$ }	1,2,6,18,52,150,444,1364
106	{ $\bar{1}0, \bar{0}\bar{1}, 00, 1\bar{0}$ }	1,2,6,18,53,160,512,1766
107	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 1\bar{0}$ }	1,2,6,18,57,210,973,5785
108	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}$ }	1,2,6,18,59,229,1107,6700
109	{ $10, 0\bar{1}, \bar{0}\bar{1}, 10$ }	1,2,6,18,62,270,1502,10206
110	{ $10, \bar{0}\bar{1}, 00, 1\bar{0}$ }	1,2,6,19,59,186,616,2183
111	{ $\bar{1}0, \bar{0}\bar{1}, 00, 10$ }	1,2,6,19,60,191,619,2048
112	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, 00$ }	1,2,6,19,61,199,661,2234
113	{ $\bar{1}0, \bar{0}\bar{1}, 00, 1\bar{0}$ }	1,2,6,19,63,217,769,2782
114	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 10$ }	1,2,6,20,69,242,859,3081
115	{ $\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 10$ }	1,2,6,20,69,243,870,3158
116	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, 01$ }, { $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, 01$ }	1,2,6,20,70,252,924,3432
117	{ $\bar{1}0, \bar{0}\bar{0}, 01, 1\bar{0}$ }	1,2,6,20,70,254,948,3618
118	{ $\bar{1}0, \bar{0}\bar{1}, 01, 1\bar{0}$ }	1,2,6,20,70,256,976,3860
119	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, 00$ }	1,2,6,20,71,273,1149,5288
120	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 1\bar{0}$ }	1,2,6,20,73,310,1597,9984
121	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}$ }	1,2,6,20,76,344,1888,12416
122	{ $10, \bar{0}\bar{1}, \bar{0}\bar{1}, 10$ }	1,2,6,20,78,368,2094,14072
123	{ $10, \bar{0}\bar{0}, \bar{0}\bar{1}, 1\bar{0}$ }	1,2,6,20,79,381,2218,15198
124	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{0}, 01$ }	1,2,6,21,81,333,1428,6298
125	{ $\bar{1}0, \bar{0}\bar{0}, 01, 1\bar{0}$ }	1,2,6,21,81,336,1480,6862
126	{ $\bar{1}0, \bar{0}\bar{1}, 00, 1\bar{0}$ }	1,2,6,21,82,352,1648,8363
127	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}$ }	1,2,6,21,83,373,1945,11961
128	{ $\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 1\bar{0}$ }	1,2,6,21,83,378,2014,12575
129	{ $\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 1\bar{0}$ }	1,2,6,21,84,385,2035,12460
130	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}$ }	1,2,6,21,85,407,2325,15673
131	{ $\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 1\bar{0}$ }	1,2,6,21,88,445,2676,18739
132	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}$ }	1,2,6,21,89,459,2823,20211
133	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 10$ }	1,2,6,22,87,355,1468,6103
134	{ $\bar{1}0, \bar{1}0, 00, 01$ }	1,2,6,22,87,357,1495,6337
135	{ $10, 00, \bar{0}\bar{1}, 10$ }	1,2,6,22,88,366,1553,6671
136	{ $10, 10, 00, 01$ }	1,2,6,22,88,369,1591,6981
137	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 10$ }	1,2,6,22,90,393,1789,8378
138	{ $\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 10$ }	1,2,6,22,90,394,1805,8543
139	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 10$ }, { $\bar{1}0, \bar{0}\bar{1}, 01, 1\bar{0}$ }	1,2,6,22,90,394,1806,8558
140	{ $\bar{1}0, 00, 01, 1\bar{0}$ }	1,2,6,22,90,396,1838,8891
141	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, 00$ }	1,2,6,22,90,398,1880,9436
142	{ $10, \bar{0}\bar{0}, 00, 1\bar{0}$ }	1,2,6,22,92,420,2050,10550
143	{ $\bar{1}0, \bar{0}\bar{0}, 00, 10$ }	1,2,6,22,92,422,2074,10754
144	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{0}, 00$ }	1,2,6,22,92,426,2146,11624
145	{ $\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 10$ }	1,2,6,22,93,430,2108,10760
146	{ $\bar{1}0, \bar{0}\bar{0}, 00, 1\bar{0}$ }	1,2,6,22,94,454,2426,14130
147	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}$ }	1,2,6,22,94,464,2652,17562
148	{ $10, \bar{0}\bar{1}, 00, 10$ }	1,2,6,22,94,465,2654,17363
149	{ $10, \bar{0}\bar{0}, \bar{0}\bar{1}, 1\bar{0}$ }, { $\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 10$ }	1,2,6,22,95,481,2848,19556
150	{ $\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 1\bar{0}$ }	1,2,6,22,95,482,2854,19480
151	{ $\bar{1}0, \bar{0}\bar{0}, 00, 1\bar{0}$ }	1,2,6,22,96,480,2692,16666
152	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 1\bar{0}$ }, { $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 1\bar{0}$ }	1,2,6,22,96,496,3008,21120
153	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 1\bar{0}$ }	1,2,6,22,97,505,3054,21157
154	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 1\bar{0}$ }	1,2,6,22,98,525,3315,24179
155	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 10$ }	1,2,6,23,100,464,2232,10975
156	{ $\bar{1}0, \bar{0}\bar{0}, 01, 1\bar{0}$ }	1,2,6,23,102,496,2568,13918
157	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 1\bar{0}$ }	1,2,6,23,104,549,3344,23239
158	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, 00$ }	1,2,6,23,105,558,3424,24110
159	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 1\bar{0}$ }	1,2,6,23,106,567,3462,23946
160	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 10$ }	1,2,6,23,106,572,3564,25377
161	{ $10, 10, 00, 01$ }	1,2,6,23,106,572,3569,25515
162	{ $\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 1\bar{0}$ }	1,2,6,23,107,587,3721,26833
163	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}$ }	1,2,6,23,107,589,3765,27518
164	{ $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}$ }, { $\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}$ }	1,2,6,23,108,604,3936,29364
165	{ $\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 1\bar{0}$ }, { $\bar{1}0, \bar{0}\bar{1}, 00, 10$ }	1,2,6,23,109,620,4127,31508

166	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, 00$ }	1,2,6,24,114,606,3504,21690
167	{ $\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}0, 10$ }	1,2,6,24,116,637,3831,24665
168	{ $\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, 10$ }	1,2,6,24,116,638,3850,24908
169	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, 00$ }	1,2,6,24,116,642,3950,26530
170	{ $\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}$ }	1,2,6,24,116,644,4008,27606
171	{ $\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, 10$ }	1,2,6,24,118,674,4298,29762
172	{ $\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}$ }	1,2,6,24,118,676,4380,31576
173	{ $\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, 10$ }	1,2,6,24,118,678,4416,31956
174	{ $\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}$ }	1,2,6,24,118,680,4464,32788
175	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}$ }, { $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}$ }	1,2,6,24,118,682,4518,33766
176	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}$ }, { $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}$ }	1,2,6,24,120,720,5040,40320
177	{ $\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}$ }, { $\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, 10$ }	1,2,6,24,120,722,5090,41194
178	{ $\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}$ }	1,2,6,26,146,1002,8102,75266
179	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{1}, \bar{1}\bar{0}$ }	1,2,7,28,116,484,2017,8376
180	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{1}, 10$ }	1,2,7,28,117,497,2125,9108
181	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{1}, 10$ }	1,2,7,28,117,498,2142,9287
182	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{1}, \bar{1}\bar{0}$ }	1,2,7,28,120,551,2766,15635
183	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{1}, 10$ }	1,2,7,29,130,609,2933,14399
184	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{1}, 10$ }	1,2,7,29,131,625,3099,15818
185	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{1}, \bar{1}\bar{0}$ }	1,2,7,29,134,693,4064,27254
186	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{1}, \bar{1}\bar{0}$ }	1,2,7,29,137,738,4534,31640
187	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, \bar{1}\bar{0}$ }	1,2,7,31,156,851,4915,29639
188	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, 10$ }	1,2,7,31,158,888,5371,34420
189	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, 10$ }	1,2,7,31,158,888,5372,34447
190	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, \bar{1}\bar{0}$ }	1,2,7,31,159,911,5731,39266
191	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, 10$ }	1,2,7,32,170,997,6280,41783
192	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, \bar{1}\bar{0}$ }	1,2,7,32,172,1041,6950,50673
193	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, 10$ }	1,2,7,32,174,1071,7238,52631
194	{ $\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, \bar{1}\bar{0}$ }	1,2,7,32,174,1076,7386,55514
195	{ $\bar{1}\bar{0}, \bar{1}0, \bar{1}\bar{0}, 10$ }	1,2,8,40,224,1344,8448,54912

Table 3: Number of signed inversion sequences in $\bar{\mathcal{I}}_n(B)$, where $n = 0, 1, \dots, 7$ and B is any set of four of length-2 signed patterns.

As discussed above, to study the Wilf-equivalences, we have to consider all the nontrivial candidate classes in Table 3. Table 4 presents the succession rules of each set in each of these cases whenever we succeeded in guessing by our main procedure.

No	B	Succession rules of $\mathcal{T}[B]$
2	{ $\bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, 01$ }	$\epsilon \rightsquigarrow b_0, a_1 b_m \rightsquigarrow (b_m)^{m+1}; a_m \rightsquigarrow (a_m)^m, b_m, a_{m+1}$, where $a_m = 0^m, b_m = 0^m \bar{0}$
	{ $\bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01$ }	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow a_1, a_2; a_m \rightsquigarrow (a_m)^m, a_{m+1}$, where $a_m = 0^m$
3	{ $0\bar{1}, 00, 0\bar{1}, 0\bar{1}$ }	
	{ $\bar{0}\bar{0}, \bar{0}\bar{0}, 0\bar{1}, 0\bar{0}$ }	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow (a_{m+1})^{m+1}$, where $a_m = 0^m$
4	{ $0\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, 0\bar{1}$ }	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow b_0; b_m \rightsquigarrow (b_{m+1})^{m+2}; a_m \rightsquigarrow b_{m-1}, (a_{m+1})^{m+1}$, where $a_m = 0^m, b_m = 0\bar{0}1^m$
	{ $0\bar{1}, \bar{0}\bar{0}, 0\bar{1}, 0\bar{0}$ }	
	{ $\bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{1}, 0\bar{0}$ }	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (a_2)^2; a_m \rightsquigarrow (a_{m+1})^{m+1}$, where $a_m = 0^m$
5	{ $0\bar{1}, 00, 0\bar{1}, 0\bar{0}$ }	
	{ $0\bar{1}, 00, \bar{0}\bar{1}, 0\bar{0}$ }	Theorem 4
12	{ $0\bar{1}, 00, 0\bar{1}, 0\bar{1}$ }	$\epsilon \rightsquigarrow 0, 0; 0 \rightsquigarrow 0\bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0; 0\bar{0} \rightsquigarrow 0\bar{0}$
	{ $0\bar{1}, \bar{0}\bar{1}, 0\bar{1}, 0\bar{0}$ }	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}; 0\bar{0} \rightsquigarrow 0\bar{0}$
	{ $0\bar{1}, \bar{0}\bar{1}, 0\bar{0}, 0\bar{0}$ }	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}$
15	{ $0\bar{1}, 00, 0\bar{1}, 0\bar{1}$ }	$\epsilon \rightsquigarrow b_0, a_1; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_0, b_m; a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0, a_{m+1}$, where $a_m = 0^m, b_m = 0^m \bar{m}$
	{ $0\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{0}$ }	$\epsilon \rightsquigarrow b_0, a_0; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0; a_m \rightsquigarrow b_{m+1}, b_m, \dots, b_0, a_{m+1}$, where $a_m = 0\bar{1} \dots m, b_m = 0\bar{1} \dots (m-1)\bar{m}$
	{ $0\bar{1}, \bar{0}\bar{0}, 0\bar{1}, 0\bar{0}$ }	
	{ $0\bar{1}, \bar{0}\bar{0}, 0\bar{0}, 0\bar{0}$ }	
	{ $0\bar{0}, 0\bar{0}, 0\bar{1}, 0\bar{0}$ }	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0, 0; \bar{0} \rightsquigarrow 0, 0$
16	{ $0\bar{1}, 00, 0\bar{1}, 0\bar{1}$ }	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0\bar{0}, 0; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow a_{m+1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow b_m, b_1, b_2, \dots, b_m$, where $a_m = 0^m, b_m = 0^m m$

	$\{\bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, b_m; a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, a_{m+1}, \text{ where } a_m = 0^m, b_m = 0^m \bar{m}$
21	$\{\bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow (a_{m+1})^{m+1}, \text{ where } a_m = 0^m$
23	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow \bar{0}, (\bar{0}\bar{0})^{m+1}, a_{m+1}, \text{ where } a_m = 0\bar{1} \dots m$ $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; \bar{0}\bar{1} \rightsquigarrow (\bar{0}\bar{0})^2, \bar{0}\bar{1}$
25	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow a_0, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0; a_0 \rightsquigarrow a_1, \bar{0}\bar{0}, \bar{0}\bar{1}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; \bar{0}\bar{1} \rightsquigarrow 0, \bar{0}\bar{0}, \bar{0}\bar{1}; a_m \rightsquigarrow a_{m+1}, \bar{0}\bar{0}, 0, c_2, c_3, \dots, c_m, b_{m+1}; c_m \rightsquigarrow \bar{0}\bar{0}, 0, c_2, c_3, \dots, c_m; b_m \rightsquigarrow c_m, \bar{0}\bar{0}, 0, c_2, c_3, \dots, c_{m-1}, b_m, \text{ where } a_m = \bar{0}\bar{1} \dots \bar{m}, b_m = a_{m-1} \bar{m}, c_m = a_m \bar{m}$ $\epsilon \rightsquigarrow \bar{0}, a_0; a_0 \rightsquigarrow \bar{0}\bar{1}, \bar{0}\bar{0}, a_1; \bar{0} \rightsquigarrow 0, \bar{0}\bar{0}; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}, \bar{0}\bar{0}, 0; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; c_m \rightsquigarrow c_m, c_{m-1}, \dots, c_2, \bar{0}, \bar{0}\bar{0}; b_m \rightsquigarrow b_m, c_{m-1}, c_{m-2}, \dots, c_2, \bar{0}, \bar{0}\bar{0}, c_m; a_m \rightsquigarrow b_{m+1}, c_m, c_{m-1}, \dots, c_2, \bar{0}, \bar{0}\bar{0}, a_{m+1}, \text{ where } a_m = 0\bar{1} \dots m, b_m = a_{m-1} \bar{m}, c_m = b_m \bar{m}$
30	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow (\bar{0})^2; a_m \rightsquigarrow (\bar{0})^{m+2}, a_{m+1}, \text{ where } a_m = 0\bar{1} \dots m$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0})^2; a_m \rightsquigarrow (\bar{0})^{m+1}, a_{m+1}, \text{ where } a_m = 0^m$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0})^2; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}, \bar{0}\bar{1}\bar{0}; \bar{0}\bar{1}\bar{0} \rightsquigarrow \bar{0}\bar{1}\bar{0}; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, \bar{0}\bar{1}\bar{0}; a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0, a_{m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{m}$ $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0, 0; \bar{0} \rightsquigarrow \bar{0}, \bar{0}, \bar{0}\bar{1}; \bar{0}\bar{1} \rightsquigarrow 0, \bar{0}, \bar{0}\bar{1}$ $\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (\bar{0})^2; \bar{0}\bar{1}\bar{0} \rightsquigarrow \bar{0}\bar{1}\bar{0}; a_m \rightsquigarrow a_{m+1}, 0, b_1, b_2, \dots, b_m; b_m \rightsquigarrow \bar{0}\bar{1}\bar{0}, b_1, b_2, \dots, b_m, \text{ where } a_m = 0^m, b_m = a_m \bar{m}$
32	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, b_1; a_m \rightsquigarrow a_{m+1}, c_1, c_2, \dots, c_m; b_m \rightsquigarrow d_m, d_{m-1}, \dots, d_1, \bar{0}\bar{0}, b_{m+1}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; c_m \rightsquigarrow c_m, c_1, c_2, \dots, c_m; d_m \rightsquigarrow d_m, d_{m-1}, \dots, d_1, \bar{0}\bar{0}, \text{ where } a_m = \bar{0}^m, b_m = 0^m, c_m = a_m \bar{m}, d_m = b_m \bar{m}$
38	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0, 0; a_m \rightsquigarrow a_{m+1}, 0, b_1, b_2, \dots, b_m; b_m \rightsquigarrow b_m, 0, b_1, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{m}$
39	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow 0, a_1; a_m \rightsquigarrow c_0, c_1, \dots, c_{m-1}, b_m, a_{m+1}; b_m \rightsquigarrow 0\bar{0}\bar{2}, b_1, b_2, \dots, b_m; c_m \rightsquigarrow c_0, c_1, \dots, c_{m+1}; \bar{0} \rightsquigarrow 0, a_1; 0\bar{0}\bar{2} \rightsquigarrow 0\bar{0}\bar{2}, \text{ where } a_m = 0^m, b_m = a_m \bar{0}, c_m = 0\bar{1} a_m$ $\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (\bar{0})^2; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, 0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}, \bar{0}\bar{1}\bar{0}; \bar{0}\bar{1}\bar{0} \rightsquigarrow \bar{0}\bar{1}\bar{0}, \text{ where } a_m = \bar{0}^m, b_m = 0\bar{1} \bar{m}$
41	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_0, 0; 0 \rightsquigarrow (\bar{0})^2; a_0 \rightsquigarrow a_1, 0, b_1; b_1 \rightsquigarrow c_1, 0, b_1; a_m \rightsquigarrow a_{m+1}, (c_m)^{m+1}, b_{m+1}; b_m \rightsquigarrow c_m, (c_{m-1})^m, b_m; c_m \rightsquigarrow (c_m)^{m+2}, \text{ where } a_m = \bar{0}\bar{1} \dots \bar{m}, b_m = a_{m-1} \bar{m}, c_m = a_m \bar{0}$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow (b_m)^{m+1}, a_{m+1}; b_m \rightsquigarrow (b_m)^{m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{m}$
45	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow \bar{0}, a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = 0^m$ $\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}, a_0; a_m \rightsquigarrow \bar{0}, a_0, a_1, \dots, a_{m+1}, \text{ where } a_m = 0\bar{1} \dots m$
46	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow b_1, \bar{0}; b_m \rightsquigarrow b_{m+1}, (b_m)^{m+1}; a_m \rightsquigarrow \bar{0}, (\bar{0}\bar{0})^{m+1}, a_{m+1}, \text{ where } a_m = 0\bar{1} \dots m, b_m = 0\bar{0} \bar{m}$ $\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_{m+1}, b_m, (a_m)^m; b_m \rightsquigarrow (b_m)^{m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{0}$
50	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0})^2; a_m \rightsquigarrow \bar{0}, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m$
60	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_{m+1}, (a_m)^{m+1}, \text{ where } a_m = \bar{0}^m$
64	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; b_m \rightsquigarrow (b_{m+1})^{m+2}; a_m \rightsquigarrow b_{m-1}, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = 0\bar{0}\bar{1} \bar{m}$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, (a_2)^2; a_m \rightsquigarrow (a_{m+1})^{m+1}, \text{ where } a_m = 0^m$ $\epsilon \rightsquigarrow b_0, a_1; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow 0\bar{0}, (a_{m+1})^{m+1}; b_m \rightsquigarrow (b_{m+1})^{m+2}, \text{ where } a_m = 0^m, b_m = 0\bar{1} \dots \bar{m}$
78	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow b_0, a_1; a_m \rightsquigarrow (a_{m+1})^{m+1}; b_m \rightsquigarrow b_{m+1}, (a_{m+2})^{m+2}, \text{ where } a_m = 0^m, b_m = \bar{0}\bar{1} \dots \bar{m}$ $\epsilon \rightsquigarrow b_0, a_1; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0; a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{m}$ $\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow (b_1)^2; b_m \rightsquigarrow (b_{m+1})^{m+2}; a_m \rightsquigarrow (b_{m+1})^{m+2}, a_{m+1}, a_m = 0\bar{1} \dots m, b_m = 0\bar{1}\bar{2} \dots \bar{m}$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_1)^2; b_m \rightsquigarrow (b_{m+1})^{m+2}; a_m \rightsquigarrow (b_m)^{m+1}, a_{m+1}, \text{ where } a_m = 0^m, b_m = \bar{1}\bar{2} \dots \bar{m}$ $\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (b_1)^2; b_m \rightsquigarrow (b_{m+1})^{m+2}; a_m \rightsquigarrow (b_m)^m, a_{m+1}, b_m, \text{ where } a_m = \bar{0}^m, b_m = 0\bar{1}\bar{2} \dots \bar{m}$
79	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow (b_m)^{m+1}; a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{m}$

84	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	Theorem 5
86	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	Theorem 5
87	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	Theorem 6
88	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_0)^2; b_m \rightsquigarrow (b_{m+1})^{m+3}; a_m \rightsquigarrow b_{m-1}, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = \bar{0}\bar{0}^m$ ----- $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_1)^2; b_m \rightsquigarrow (b_{m+1})^{m+2}; a_m \rightsquigarrow b_m, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = \bar{0}\bar{0}^m$ ----- $\epsilon \rightsquigarrow b_1, a_1; a_m \rightsquigarrow (a_{m+1})^{m+1}; b_m \rightsquigarrow b_{m+1}, a_{m+1}, (b_{m+1})^m, \text{ where } a_m = 0^m, b_m = \bar{0}^m$
98	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	Theorem 7
102	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow a_1, c_1; \bar{0}\bar{1} \rightsquigarrow (\bar{0}\bar{1})^2; \bar{0}\bar{1}\bar{0} \rightsquigarrow \bar{0}\bar{1}\bar{0}; c_m \rightsquigarrow (\bar{0}\bar{1})^{m+1}, c_{m+1}; a_m \rightsquigarrow a_{m+1}, \bar{0}\bar{1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow \bar{0}\bar{1}\bar{0}, b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}^m m, c_m = 0^m$ ----- $\epsilon \rightsquigarrow a_1, c_1; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}, \bar{0}\bar{1}\bar{0}; \bar{0}\bar{0} \rightsquigarrow (\bar{0}\bar{0})^2; \bar{0}\bar{1}\bar{0} \rightsquigarrow \bar{0}\bar{1}\bar{0}; d_m \rightsquigarrow d_m, d_{m-1}, \dots, d_1, \bar{0}\bar{1}\bar{0}; c_m \rightsquigarrow d_m, d_{m-1}, \dots, d_1, \bar{0}\bar{0}, c_{m+1}; a_m \rightsquigarrow a_{m+1}, \bar{0}\bar{0}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow \bar{0}\bar{1}\bar{0}, b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}^m m, c_m = 0^m, d_m = 0^m \bar{m}$ ----- $\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{1} \rightsquigarrow (\bar{0}\bar{1})^2; b_m \rightsquigarrow (\bar{0}\bar{1})^{m+1}, b_{m+1}; a_m \rightsquigarrow a_{m+1}, (\bar{0}\bar{1})^{m+1}, \text{ where } a_m = \bar{0}^m, b_m = 0^m$
116	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow b_1, (a_1)^2; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}, (a_{m+1})^2, \text{ where } a_m = \bar{0}\bar{0}^m, b_m = \bar{0}\bar{1}^m$ ----- $\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, b_0, b_1, \dots, b_m; b_m \rightsquigarrow b_0, b_1, \dots, b_m, b_0, b_1, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m$
139	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow (a_1)^2, a_0; a_m \rightsquigarrow a_{m+1}, a_{m+1}, a_m, \dots, a_0, \text{ where } a_m = \bar{0}\bar{0}^m$ ----- $\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow a_0, a_1, a_1; a_m \rightsquigarrow a_0, a_1, \dots, a_{m+1}, a_{m+1}, \text{ where } a_m = \bar{0}\bar{0}^m$
149	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	Theorem 5
152	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow (a_1)^2, b_0; a_m \rightsquigarrow (a_{m+1})^2, (b_m)^{m+1}; b_m \rightsquigarrow (b_{m+1})^{m+3}, \text{ where } a_m = \bar{0}\bar{0}^m, b_m = a_m 1$
164	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_0)^2; b_m \rightsquigarrow (b_{m+1})^{m+3}; a_m \rightsquigarrow \bar{0}, b_0, b_1, \dots, b_{m-1}, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = \bar{0}\bar{0}^m$ ----- $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_1)^2; b_m \rightsquigarrow (b_{m+1})^{m+2}; a_m \rightsquigarrow \bar{0}, b_1, b_2, \dots, b_m, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = \bar{0}\bar{0}^m$
165	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$	Theorem 8
175	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	Theorem 9
176	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow (a_1)^3; a_m \rightsquigarrow (a_{m+1})^{m+3}, \text{ where } a_m = \bar{0}\bar{0}^m$ ----- $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow \bar{0}, a_1, a_2, \dots, a_m, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m$
177	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$	Theorem 8

Table 4: Succession rules of $\mathcal{T}[B]$ for some sets B of four of length-2 signed patterns in Table 3.

By using the main procedure (as in Examples 1 and 2), we obtain the following result (here Cx means Class x in Table 4).

Theorem 3. *We have*

$$C2: F_{\{\bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}}(x) = F_{\{\bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}}(x) = 1 + 2 \sum_{k \geq 1} \frac{x^k}{\prod_{i=1}^k (1-ix)};$$

$$C3: F_{\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}}(x) = F_{\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}}(x) = F_{\{\bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}}(x) = \sum_{k \geq 0} \sum_{i=0}^k i! x^k;$$

$$\begin{aligned}
 C4: & F_{\{\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = F_{\{\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = \\
 & 1 + 2 \sum_{k \geq 1} k!x^k; \\
 C12: & F_{\{\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = F_{\{\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{0}\}}(x) = \frac{1+x^2}{(1-x)^2}; \\
 C15: & F_{\{\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = F_{\{\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{0},\bar{0}\bar{0}\}}(x) = \\
 & F_{\{\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = F_{\{\bar{0}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{0}\bar{0},\bar{0}\bar{0},\bar{0}\bar{0},\bar{0}\bar{0}\}}(x) = \frac{1}{1-2x}; \\
 C16: & F_{\{\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = F_{\{\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = \frac{1-3x+2x^2-x^3}{(1-x)^2(1-3x+x^2)}; \\
 C21: & F_{\{\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = \frac{1}{(1-x)^2} + \sum_{j \geq 0} j!x^j; \\
 C23: & F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = \frac{1-x+2x^2}{(1-x)^3}; \\
 C25: & F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = \frac{1-x+x^2}{(1-x)(1-2x)}; \\
 C30: & F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = \\
 & F_{\{\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{1}\bar{0}\}}(x) = \frac{1-2x+2x^2}{(1-x)^2(1-2x)}; \\
 C32: & F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = \frac{1-4x+5x^2-3x^3}{(1-x)(1-2x)(1-3x+x^2)}; \\
 C38: & F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = \frac{1-x}{1-3x+x^2}; \\
 C39: & F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = \frac{1-3x+2x^2+2x^3-(1-3x+2x^2)\sqrt{1-4x}}{2x(1-x)^2(1-2x)}; \\
 C41: & F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = \frac{1}{1-x} + \frac{1}{1-x} \sum_{j \geq 1} \frac{x^j}{1-(j+1)x}; \\
 C45: & F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = C(x); \\
 C46: & F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{0}\}}(x) = \\
 & \frac{1}{1-x} + \frac{1}{1-x} \sum_{j \geq 1} \frac{x^j(1-jx)}{\prod_{i=1}^{j+1}(1-ix)}; \\
 C50: & F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = \frac{1-x}{1-2x} \sum_{j \geq 0} j!x^j; \\
 C60: & F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = \sum_{j \geq 0} \frac{x^j}{\prod_{i=1}^{j+1}(1-ix)}; \\
 C64: & F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = \\
 & 1 + \frac{2-x}{1-x} \sum_{j \geq 1} j!x^j; \\
 C78: & F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1}\}}(x) = \\
 & F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = \sum_{n \geq 0} \sum_{j=0}^n \frac{n!}{j!} x^n; \\
 C79: & F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = \frac{1}{1-x} + \sum_{k \geq 1} k!x^k + x \sum_{k \geq 1} \sum_{i=1}^{k+1} \frac{k!x^k}{1-ix}; \\
 C88: & F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0},\bar{0}\bar{1}\}}(x) = \\
 & F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = \sum_{k \geq 0} k!(1 + \sum_{j=1}^k \frac{1}{j})x^k;
 \end{aligned}$$

$$C102: F_{\{\bar{1}\bar{0},\bar{0}\bar{1},0\bar{1},\bar{1}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},0\bar{1},\bar{1}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},0\bar{1},\bar{1}\bar{0}\}}(x) = \frac{1-2x+3x^2}{(1-x)^2(1-2x)};$$

$$C116: F_{\{\bar{1}\bar{0},\bar{1}\bar{0},\bar{0}\bar{1},0\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{1}\bar{0},\bar{0}\bar{1},0\bar{1}\}}(x) = \frac{1}{\sqrt{1-4x}};$$

$$C139: F_{\{\bar{1}\bar{0},\bar{0}\bar{1},0\bar{1},\bar{1}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},0\bar{1},\bar{1}\bar{0}\}} = \frac{1-x-\sqrt{1-6x+x^2}}{2x};$$

$$C152: F_{\{\bar{1}\bar{0},\bar{0}\bar{1},0\bar{1},\bar{1}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{0}\bar{1},0\bar{1},\bar{1}\bar{0}\}}(x) = \frac{1}{1-2x} + \sum_{j \geq 2} \sum_{i=0}^{j-2} \frac{j!2^{i+1}x^j}{(i+2)!};$$

$$C164: F_{\{\bar{1}\bar{0},\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = F_{\{\bar{1}\bar{0},\bar{1}\bar{0},\bar{0}\bar{1},\bar{0}\bar{0}\}}(x) = \left(\sum_{j \geq 0} j!x^j\right)^2 + \sum_{i \geq 1} \sum_{k=1}^i \sum_{j \geq 0} \frac{i!(k+j+1)!}{(k+1)!} x^{i+j+1};$$

$$C176: F_{\{\bar{1}\bar{0},\bar{1}\bar{0},\bar{0}\bar{1},0\bar{1}\}}(x) = F_{\{\bar{1}\bar{0},\bar{1}\bar{0},\bar{0}\bar{0},\bar{0}\bar{0}\}}(x) = \sum_{j \geq 0} (j+1)!x^j.$$

Note that Theorem 3 does not cover all the classes in Table 4 (the nontrivial candidate classes of four of length-2 signed patterns), namely, it does not cover Classes 5, 84, 86, 87, 98, 149, 165, 175, and 177. For any set B in these classes, we failed to obtain the generating tree $\mathcal{T}[B]$. In the next theorems, we consider the Wilf-equivalences for these classes.

Theorem 4 (Class 5). *We have that $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{0}\} \sim \{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{0}\}$.*

Proof. Let $A_n = \bar{I}_n(\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{0})$ and $B_n = \bar{I}_n(\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{0})$. Note that any signed inversion sequence in A_n can be written as

$$\begin{aligned} \pi = & \pi^{(0)} a_{0,0} \pi^{(0,0)} a_{0,1} \pi^{(0,1)} \dots a_{0,i_0} \pi^{(0,i_0)} \\ & a_{1,0} \pi^{(1,0)} a_{1,1} \pi^{(1,1)} \dots a_{1,i_1} \pi^{(1,i_1)} \\ & \dots a_{m,0} \pi^{(m,0)} a_{m,1} \pi^{(m,1)} \dots a_{m,i_m} \pi^{(m,i_m)}, \end{aligned}$$

where only the barred letters in π are $a_{0,0}, \dots, a_{m,0}$ ($m \geq -1$) such that $|a_{0,0}| > \dots > |a_{m,0}|$, $|a_{j,0}| = a_{j,1} = \dots = a_{j,i_j}$, $\pi^{(j,i)}$ is a subword of π , and any letter in $\pi^{(j,0)} \dots \pi^{(j,i_j)}$ is smaller than $|a_{j,0}|$, for all $j = 0, 1, \dots, m$.

On the other hand, any signed inversion sequence in B_n can be written as

$$\begin{aligned} \pi = & \pi^{(0)} a_{0,0} \pi^{(0,0)} a_{0,1} \pi^{(0,1)} \dots a_{0,i_0} \pi^{(0,i_0)} \\ & a_{1,0} \pi^{(1,0)} a_{1,1} \pi^{(1,1)} \dots a_{1,i_1} \pi^{(1,i_1)} \\ & \dots a_{m,0} \pi^{(m,0)} a_{m,1} \pi^{(m,1)} \dots a_{m,i_m} \pi^{(m,i_m)}, \end{aligned}$$

where only the barred letters in π are $a_{0,0}, a_{0,1}, \dots, a_{0,i_0}, \dots, a_{m,0}, a_{m,1}, \dots, a_{m,i_m}$ ($m \geq -1$) such that $|a_{0,0}| = \dots = |a_{0,i_0}| > |a_{1,0}| = \dots = |a_{1,i_1}| > \dots > |a_{m,0}| = \dots = |a_{m,i_m}|$, $\pi^{(j,i)}$ is a subword of π , and any letter in $\pi^{(j,0)} \dots \pi^{(j,i_j)}$ is smaller than $|a_{j,0}|$, for all $j = 0, 1, \dots, m$.

For any $\pi \in A_n$, we define π' to be the same as π where we barred the elements $a_{j,1}, \dots, a_{j,i_j}$, for all $j = 1, 2, \dots, m$. Clearly, $\pi \in A_n$ if and only if $\pi' \in B_n$. Hence, the map $\pi \mapsto \pi'$ is a bijection. \square

Theorem 5 (Class 84). *We have that*

- (i) $\{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{0}\} \sim \{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{0}\}$.
- (ii) $\{\bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{0}\} \sim \{\bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{0}\}$.
- (iii) $\{\bar{1}0, \bar{0}0, \bar{0}1, 1\bar{0}\} \sim \{\bar{1}0, \bar{0}0, \bar{0}1, 1\bar{0}\}$.

Proof. (i) Let $A = \{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{0}\}$ and $B = \{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{0}\}$. Any signed inversion sequence that avoids A with at least one barred letter can be presented as

$$\pi^{(0)}\pi_{i_1}\pi^{(1)}\pi_{i_2}\pi^{(2)} \dots \pi_{i_m}\pi^{(m)},$$

where the subword $\pi^{(j)}$ is given by $\pi^{(j)} = \pi^{(j,0)}|\pi_{i_1}|\pi^{(j,1)} \dots |\pi_{i_1}|\pi^{(j,n_j)}$, $1 \leq j \leq m$, such that

- any letter of $\pi^{(0)}\pi^{(1)} \dots \pi^{(m)}$ is unbarred;
- any letter of $\pi_{i_1}\pi_{i_2} \dots \pi_{i_m}$ is barred such that $|\pi_{i_1}| < |\pi_{i_2}| < \dots < |\pi_{i_m}|$;
- any letter of $\pi^{(1,0)}\pi^{(1,1)} \dots \pi^{(m,n_m)}$ is smaller than $|\pi_{i_1}|$.

On the other hand, any signed inversion sequence that avoids B with at least one barred letter can be presented as

$$\theta^{(0)}\theta_{i_1}\theta^{(1)}\theta_{i_2}\theta^{(2)} \dots \theta_{i_d}\theta^{(d)}$$

such that

- any letter of $\theta^{(0)}\theta^{(1)} \dots \theta^{(d)}$ is unbarred;
- any letter of $\theta_{i_1}\theta_{i_2} \dots \theta_{i_d}$ is barred such that $|\theta_{i_1}| \leq |\theta_{i_2}| < \dots < |\theta_{i_d}|$;
- any letter of $\theta^{(1)}\theta^{(2)} \dots \theta^{(d)}$ is smaller than $|\theta_{i_1}|$.

We are now ready to present a bijection α from $\bar{\mathcal{I}}_n(A)$ to $\bar{\mathcal{I}}_n(B)$. Let $\pi \in \bar{\mathcal{I}}_n(A)$. If π has no barred letters, then we define $\alpha(\pi) = \pi$. Otherwise, by the above, we can define $\alpha(\pi) = \theta^{(0)}\theta_{i_1}\theta^{(1)}\theta_{i_2}\theta^{(2)} \dots \theta_{i_m}\theta^{(m)}$, where $\theta^{(0)} = \pi^{(0)}$ and $\theta^{(j)} = \pi^{(j,0)}|\pi_{i_j}|\pi^{(j,1)} \dots |\pi_{i_j}|\pi^{(j,n_j)}$, for all $j = 1, 2, \dots, m$. Hence, $\pi \in \bar{\mathcal{I}}_n(A)$ if and only if $\alpha(\pi) \in \bar{\mathcal{I}}_n(B)$, which shows that α is a bijection.

(ii) Let $A' = \{\bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{0}\}$ and $B' = \{\bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{0}\}$. The proof of (2) is similar to the proof of (1), where we only look at the decreasing/non-increasing subsequence of barred letters in $\bar{\mathcal{I}}_n(A')/\bar{\mathcal{I}}_n(B')$ instead increasing/non-decreasing subsequence of barred letters in $\bar{\mathcal{I}}_n(A)/\bar{\mathcal{I}}_n(B)$, respectively.

(iii) It follows from the proof of Case (i). □

Theorem 6 (Class 87). *We have that $\{\bar{1}0, \bar{0}0, 0\bar{1}, 0\bar{0}\} \sim \{\bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{0}\}$.*

Proof. Let us construct a bijection α between $A_n = \bar{\mathcal{I}}_n(\bar{1}0, \bar{0}0, 0\bar{1}, 00)$ and $B_n = \bar{\mathcal{I}}_n(\bar{1}0, 0\bar{1}, 0\bar{0}, 00)$. Let $\pi = \pi_1\pi_2 \cdots \pi_n \in A_n$. If each letter of π is barred, we define $\alpha(\pi) = \pi \in B_n$. Otherwise, let π_i be a minimal unbarred letter in π ; in this case we assume that i is minimal. Thus, $\pi = \pi'\pi_i\pi''$ where (1) each letter s of π' is unbarred if $|s| > \pi_i$, and barred if $|s| < \pi_i$ (because of the minimality of i and avoiding $\bar{0}0$, we see that $|s| \neq \pi_i$), and (2) each letter t of π'' is unbarred if $|t| > \pi_i$, and barred if $|t| \leq \pi_i$. Define $\alpha(\pi)$ to be the same as π where we change any letter t of π'' such that $t = \bar{\pi}_i$ to π_i . Clearly, we see that $\alpha(\pi) \in B_n$. The inverse of α can be defined exactly as α where we change any letter t of π'' such that $t = \pi_i$ to $\bar{\pi}_i$. Hence, α is a bijection. \square

Theorem 7 (Class 98). *Let $A = \{\bar{1}0, \bar{0}1, \bar{0}\bar{0}, \bar{0}0\}$, $B = \{\bar{1}0, \bar{0}1, \bar{0}\bar{0}, 0\bar{0}\}$, $C = \{\bar{1}0, \bar{0}1, 00, 0\bar{0}\}$, and $D = \{\bar{1}0, \bar{0}0, 0\bar{1}, 0\bar{0}\}$. Then $A \sim B \sim C \sim D$.*

Proof. We first show that $A \sim B$. Here we construct a bijection α between $\bar{\mathcal{I}}_n(A)$ and $\bar{\mathcal{I}}_n(B)$. Note that for any $\pi = \pi_1\pi_2 \cdots \pi_n \in \bar{\mathcal{I}}_n(A)$, we have that either (1) π has no barred letters, or (2) there exists i such that π_i is barred; in the latter case, every π_j with $j \neq i$ is an unbarred, and there is no $k > i$ for which $|\pi_k| = |\pi_i|$.

Also, for any $\pi = \pi_1\pi_2 \cdots \pi_n \in \bar{\mathcal{I}}_n(B)$, we have that either (1') π has no barred letters, or (2') there exists i such that π_i is barred; in the latter case, every π_j with $j \neq i$ is an unbarred, and there is no $k < i$ for which $|\pi_k| = |\pi_i|$.

In Case (1), we define $\alpha(\pi) = \pi$. In Case (2), let $\pi_{i_1} = \pi_i, \pi_{i_2} = \pi_i, \dots, \pi_{i_{k-1}} = \pi_i, \pi_{i_k} = \bar{\pi}_i$ with k maximal. Then we define $\alpha(\pi)$ to be the same as π , where we exchange π_{i_1} with π_{i_k} . Hence, by (1)-(2) and (1')-(2'), we have that $\pi \in \bar{\mathcal{I}}_n(A)$ if and only if $\alpha(\pi) \in \bar{\mathcal{I}}_n(B)$, which implies that α is a bijection.

Next, we show that $B \sim C$. Here, we construct a bijection β between $\bar{\mathcal{I}}_n(B)$ and $\bar{\mathcal{I}}_n(C)$. Note that for any $\pi = \pi_1\pi_2 \cdots \pi_n \in \bar{\mathcal{I}}_n(C)$, we have that either (1'') π has no barred letters, or (2'') there exists i minimal such that π_i is barred; in this case, every π_j with $j \neq i$ and $\pi_j \neq \pi_i$ is unbarred and π_j is barred whenever $|\pi_j| = |\pi_i|$.

In Case (1''), we define $\beta(\pi) = \pi$. In Case (2''), we define $\beta(\pi)$ to be the same as π where we replace π_k by $\bar{\pi}_k$ whenever $k > i$ and $\pi_k = |\pi_i|$. Hence, by (1'')-(2'') and (1')-(2'), we have that $\pi \in \bar{\mathcal{I}}_n(B)$ if and only if $\beta(\pi) \in \bar{\mathcal{I}}_n(C)$, which implies that β is a bijection.

Next, we show that $C \sim D$. Here, we construct a bijection γ between $\bar{\mathcal{I}}_n(C)$ and $\bar{\mathcal{I}}_n(D)$. Note that for any $\pi = \pi_1\pi_2 \cdots \pi_n \in \bar{\mathcal{I}}_n(D)$, we have that either (1''') π has no barred letters, or (2''') there exists i such that π_i is barred ($|\pi_i|$ maximal) and there is no barred letter π_j with $|\pi_j| > |\pi_i|$, so any letter π_j with $|\pi_j| \leq |\pi_i|$ is barred.

In Case (1'''), we define $\gamma(\pi) = \pi$. In Case (2'''), we define $\gamma(\pi)$ to be the same as π where we replace π_k by $\bar{\pi}_k$ whenever $|\pi_k| < |\pi_i|$. Hence, by (1''')-(2''') and (1'')-(2''), we have that $\pi \in \bar{\mathcal{I}}_n(C)$ if and only if $\gamma(\pi) \in \bar{\mathcal{I}}_n(D)$, which implies that γ is a bijection. \square

Theorem 8 (Class 165). *We have*

- (i) $\{\bar{1}0, \bar{0}1, \bar{0}0, 1\bar{0}\} \sim \{\bar{1}0, \bar{0}1, 0\bar{0}, 1\bar{0}\}$.
- (ii) $\{\bar{1}0, \bar{0}0, \bar{0}0, 1\bar{0}\} \sim \{\bar{1}0, \bar{0}0, 0\bar{0}, 1\bar{0}\}$.

Proof. (i) Let $A = \{\bar{1}0, \bar{0}1, \bar{0}0, 1\bar{0}\}$ and $B = \{\bar{1}0, \bar{0}1, 0\bar{0}, 1\bar{0}\}$. Here we construct a bijection α between $\bar{\mathcal{I}}_n(A)$ and $\bar{\mathcal{I}}_n(B)$. Note that for any $\pi = \pi_1\pi_2 \cdots \pi_n \in \bar{\mathcal{I}}_n(A)$, we have that either (1) π has no barred letters, or (2) there exists i such that π_i is barred, so π_j with $j \neq i$ is unbarred letter and there is no $k < i$ such that $\pi_k > |\pi_i|$. Also, for any $\pi = \pi_1\pi_2 \cdots \pi_n \in \bar{\mathcal{I}}_n(B)$, we have that either (1') π has no barred letters, or (2') there exists i such that π_i is barred, so π_j with $j \neq i$ and $|\pi_j| \neq |\pi_i|$ is unbarred letter and if $|\pi_j| = |\pi_i|$ then either $j < i$ and $\pi_j = \pi_i$ or $j > i$ and $\pi_j = |\pi_i|$. In Case (1) we define $\alpha(\pi) = \pi$ and in Case (2) we define $\alpha(\pi)$ to be the same as π where we replace π_k by $\bar{\pi}_k$ whenever $k < i$ and $\pi_k = |\pi_i|$. Hence, by (1)-(2) and (1')-(2'), we have that $\pi \in \bar{\mathcal{I}}_n(A)$ if and only if $\alpha(\pi) \in \bar{\mathcal{I}}_n(B)$, which implies that α is a bijection.

(ii) Define $A_n = \bar{\mathcal{I}}_n(\bar{1}0, \bar{0}0, \bar{0}0, 1\bar{0})$ and $B_n = \bar{\mathcal{I}}_n(\bar{1}0, \bar{0}0, 0\bar{0}, 1\bar{0})$. Let $\pi \in A_n$ and let $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_k}$ be a subsequence of π of all barred letters. So, $|\pi_{i_1}| < \dots < |\pi_{i_k}|$. Clearly, there is no letter π_j in π such that $|\pi_j| > |\pi_{i_s}|$ and $j < i_s$, or $|\pi_j| = |\pi_{i_s}|$ and $j > i_s$. Define $\beta(\pi)$ to be the same as π where we replace π_j by $\bar{\pi}_j$ whenever $|\pi_j| = |\pi_{i_s}|$ and $j < i_s$ for some $s = 1, 2, \dots, k$. Clearly, $\beta(\pi) \in B_n$. Since $\beta^2 = id$, we obtain that $\beta : A_n \rightarrow B_n$ is a bijection. \square

Theorem 9 (Class 175). *Let $A = \{\bar{1}0, \bar{1}0, \bar{0}0, 0\bar{0}\}$ and $B = \{\bar{1}0, \bar{1}0, \bar{0}0, 0\bar{0}\}$. Then $A \sim B$.*

Proof. Here we construct a bijection α between $\bar{\mathcal{I}}_n(A)$ and $\bar{\mathcal{I}}_n(B)$. Let $\pi = \pi_1 \cdots \pi_n$ in $\bar{\mathcal{I}}_n(A)$, so all barred letters of π stay at positions i_1, i_2, \dots, i_m , $m \geq 0$, such that $i_1 < i_2 < \dots < i_m$ and $|\pi_{i_1}| < |\pi_{i_2}| < \dots < |\pi_{i_m}|$. We define $\alpha(\pi)$ to be the same π where we change π_k (respectively, $\bar{\pi}_k$) to $\bar{\pi}_k$ (respectively, π_k) whenever $i_j < k < i_{j+1}$ and $\pi_k = |\pi_{i_j}|$, where $1 \leq j \leq m$ (here $i_{m+1} = n + 1$). Clearly, π avoids $\{\bar{1}0, \bar{1}0, 0\bar{0}\}$ if and only if $\alpha(\pi)$ avoids $\{\bar{1}0, \bar{1}0, 0\bar{0}\}$. Note that π contain $0\bar{0}$ if and only if $\alpha(\pi)$ avoids $0\bar{0}$. Hence, α is a bijection. \square

As a sequence of Table 4 and Theorems 3-9, we obtain the main result of this section.

Theorem 10. *The number of Wilf-equivalences among sets of four of length-2 patterns is 195.*

5. Sets of k Length-2 Signed Patterns with $k \geq 5$

There are $\binom{12}{k}$ sets of k length-2 signed patterns. Calculating the first 7 terms of the counting sequence for the avoiders for each of these sets gives the number of different sequences, as shown in Table 5.

k	5	6	7	8	9	10	11	12
$\binom{12}{k}$	792	924	792	495	220	66	12	1
Number of Candidate Classes	239	190	107	49	22	8	3	1
Number of nontrivial Candidate Classes	62	69	48	29	12	6	3	0

Table 5: Number of candidate classes and nontrivial candidate classes.

The aim of this section is to show that the number of nontrivial candidate classes of sets of k length-2 signed patterns is indeed the number of Wilf-equivalences (see Table 5), that is, in each class, all the counting sequences agree forever, not just in the first 7 terms (here we do not present the tables of the first 7 terms for the counting sequences). Note that by the barring operation, we see that $|\bar{I}_n(B)| = |\bar{I}_n(\bar{B})|$, for all $n \geq 0$. So, in the next tables, Tables 6-13, we do not include barring of a set B if B is included. In particular, we show the following result.

Theorem 11. *Let sw_k be the number of Wilf-equivalences among sets of k length-2 signed patterns. Then $239 \leq sw_5 \leq 240$, $sw_6 = 190$, $sw_7 = 107$, $sw_8 = 49$, $sw_9 = 22$, $sw_{10} = 8$, $sw_{11} = 3$, and $sw_{12} = 1$.*

As in the previous two sections, we prove this theorem using our main procedure; when we fail to obtain the desired result (the succession rules and/or the generating function), we present an alternative proof (see the following theorems). We mark a trivial candidate class by a star whenever we fail to obtain the succession rules of its generating tree by our main procedure.

Case $k = 5$

No	B	Succession rules of $\mathcal{T}[B]$	Generating function $F_B(x)$
1	{01, 00, 00, 00, 01} {01, 00, 01, 00, 01}	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 0\bar{0}; \bar{0} \rightsquigarrow 0\bar{0}$	$1 + 2x + 3x^2 + x^3$
2	{01, 00, 01, 01, 00}	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0; \bar{0} \rightsquigarrow 0\bar{0}$	$1 + x^2 + \frac{2x}{1-x}$
3	{01, 00, 01, 00, 00} {01, 01, 01, 00, 00} {01, 00, 00, 01, 01} {00, 00, 01, 01, 00}	$\epsilon \rightsquigarrow 0, 0; 0 \rightsquigarrow 0, 0; 0 \rightsquigarrow 0\bar{0}$ $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0, 0; \bar{0} \rightsquigarrow 0, 0\bar{0}$ ----- $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0; \bar{0} \rightsquigarrow 0$	----- ----- ----- $1 + 2x + \frac{3x^2}{1-x}$
4	{01, 00, 01, 00, 01} {01, 00, 01, 01, 00} {01, 00, 01, 01, 01} {01, 00, 01, 00, 00} {01, 00, 00, 00, 00} {01, 00, 01, 01, 01}	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0, 0; 0 \rightsquigarrow 0, 0$ ----- ----- ----- ----- ----- -----	----- ----- ----- ----- ----- -----

	$\{\bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}$	$\frac{1}{(1-x)^2}$
5	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow b_0, a_0; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_0; a_m \rightsquigarrow b_{m+1}, b_m, \dots, b_0, a_{m+1}$, where $a_m = 01 \dots m$, $b_m = a_{m-1} \bar{m}$	$\frac{1+x}{1-x-x^2}$
6	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_2, \bar{0}, b_m;$ $a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_2, \bar{0}, a_{m+1}$, where $a_m = 0^m$, $b_m = a_m \bar{m}$	-----
	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow (0)^2; \bar{0} \rightsquigarrow \bar{0}$	$\frac{1-x-x^2}{(1-x)(1-2x)}$
7	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_1, \bar{0}, b_m;$ $a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, \bar{0}, a_{m+1}$, where $a_m = 0^m$, $b_m = a_m \bar{m}$	$\frac{1-x^2}{1-2x}$
8	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, b_m;$ $a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, a_{m+1}$, where $a_m = 0^m$, $b_m = a_m \bar{m}$	$\frac{x^3-x^2-2x+1}{(1-x)(1-3x+x^2)}$
9	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_{m+1}, (a_m)^m$	$\frac{1}{1-x} + \sum_{k \geq 1} \frac{x^k}{i^k (1-ix)}$
10	$\{\bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow (a_{m+1})^{m+1}$, where $a_m = 0^m$	$\frac{1}{1-x} + \sum_{k \geq 1} k! x^k$
11	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow a_1; a_m \rightsquigarrow (a_{m+1})^{m+1}$, where $a_m = 0^m$	-----
	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow \bar{0}, (a_{m+1})^{m+1}$, where $a_m = 0^m$	$1 + \sum_{k \geq 1} (k! + (k-1)!) x^k$
12*	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	-----	-----
13	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{1}, 0\bar{1}; \bar{0} \rightsquigarrow 0\bar{1}, 0\bar{1}; \bar{0}\bar{1} \rightsquigarrow 0\bar{1}$	$1 + 2x + 4x^2 + x^3$
14	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{1}, 0\bar{0}; \bar{0} \rightsquigarrow 0\bar{0}, 0\bar{1}; 0\bar{1} \rightsquigarrow 0\bar{0}$	$1 + 2x + 4x^2 + 2x^3$
15	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0; \bar{0} \rightsquigarrow 0\bar{1}, 0\bar{0}; \bar{0}\bar{1} \rightsquigarrow 0\bar{1}$	$1 + 2x + 4x^2 + \frac{3x^3}{1-x}$
16	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0; \bar{0} \rightsquigarrow 0, 0\bar{0}$	$1 + 2x + \frac{4x^2}{1-x}$
17	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{0}; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}, \bar{0}\bar{1}; \bar{0}\bar{1} \rightsquigarrow 0\bar{0}, 0\bar{0}$	$1 + 2x + 4x^2 + \frac{5x^3}{1-x}$
18	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0\bar{0}; 0\bar{0} \rightsquigarrow 0\bar{0}$ $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{1}, 0\bar{1}; \bar{0} \rightsquigarrow \bar{0}, 0\bar{1}; 0\bar{1} \rightsquigarrow 0\bar{1}$	-----
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow \bar{0}, (0\bar{0})^{m+1}, a_{m+1}$, where $a_m = 01 \dots m$	$\frac{1+x^2-x^3}{(1-x)^2}$
19	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, 0$ ----- $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}; 0\bar{0} \rightsquigarrow 0\bar{0}$ ----- $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0; \bar{0} \rightsquigarrow 0\bar{0}, 0; 0\bar{0} \rightsquigarrow 0\bar{0}$ ----- $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0, 0\bar{0}; \bar{0} \rightsquigarrow 0, 0\bar{0}; 0\bar{0} \rightsquigarrow 0\bar{0}$ -----	-----
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}$	$\frac{1+x^2}{(1-x)^2}$
20	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow (\bar{0})^m, 0\bar{0}, a_{m+1}$, where $a_m = 0^m$	$\frac{1-x+x^2-x^3+x^4}{(1-x)^3}$
21	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow 0\bar{0}, 0; 0\bar{0} \rightsquigarrow 0\bar{0};$ $b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_1; a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, a_{m+1}$, where $a_m = 0^m$, $b_m = a_m \bar{m}$	$\frac{1-x-x^3}{(1-x)^2(1-x-x^2)}$
22	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_1, 0\bar{0};$ $a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, 0\bar{0}, a_{m+1}$, where $a_m = 0^m$, $b_m = a_m \bar{m}$	$\frac{1-x^3-x^4}{(1-x)(1-x-x^2)}$
23	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow a_0, 0; 0 \rightsquigarrow b_0; a_0 \rightsquigarrow a_1, b_0, c_1; c_1 \rightsquigarrow 0, b_0;$ $a_m \rightsquigarrow a_{m+1}, b_0, 0, b_2, b_3, \dots, b_m, c_{m+1};$ $b_m \rightsquigarrow b_0, 0, b_2, b_3, \dots, b_{m-1}; c_m \rightsquigarrow b_m, b_0, 0, b_2, b_3, \dots, b_{m-1}$, where $a_m = 0\bar{1} \dots \bar{m}$, $b_m = a_m m$, $c_m = a_{m-1} \bar{m}$	-----
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; a_0 \rightsquigarrow c_1, b_0, a_1; \bar{0} \rightsquigarrow b_0;$ $c_1 \rightsquigarrow b_0, 0; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_2, \bar{0}, b_0, b_m;$ $c_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_2, \bar{0}, b_0, b_m;$ $a_m \rightsquigarrow c_{m+1}, b_m, b_{m-1}, \dots, b_2, 0, 0\bar{0}, a_{m+1}$, where $a_m = 01 \dots m$, $b_m = a_m \bar{m}$, $c_m = a_{m-1} \bar{m}$	$\frac{1+x+x^2}{1-x-x^2}$

24	{10, 01, 00, 00, 01}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (0\bar{1})^2; 0\bar{1} \rightsquigarrow 0\bar{1};$ $a_m \rightsquigarrow a_{m+1}, 0\bar{1}, b_2, b_3, \dots, b_m; b_m \rightsquigarrow$ $b_m, 0\bar{1}, b_2, b_3, \dots, b_{m-1},$ where $a_m = 0^m,$ $b_m = a_m m$	$\frac{1-x-2x^3}{(1-x)(1-2x)}$
25	{10, 00, 01, 00, 01}	$\epsilon \rightsquigarrow 0, 0; 0 \rightsquigarrow 0, 0, 0\bar{1}; 0\bar{1} \rightsquigarrow 0, 0\bar{1}$	
	{10, 01, 00, 01, 00}	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow (\bar{0})^{m+2}, a_{m+1},$ where $a_m =$ $\frac{0\bar{1} \dots \bar{m}}$	
	{10, 01, 00, 00, 01}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow 0\bar{1}, \bar{0}; 0\bar{1} \rightsquigarrow 0\bar{1}; a_m \rightsquigarrow (0\bar{1})^m, a_{m+1},$ where $a_m = 0^m$	
	{10, 00, 01, 00, 01}		
	{10, 00, 00, 00, 01}	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{1}, 0; \bar{0} \rightsquigarrow \bar{0}, 0; 0\bar{1} \rightsquigarrow 0\bar{1}$	
	{10, 01, 00, 01, 01}	$\epsilon \rightsquigarrow a_1, 0; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow (\bar{0})^{m+1}, a_{m+1},$ where $a_m =$ $\frac{1-x+x^2}{(1-x)^3}$	
26	{10, 01, 00, 01, 01}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; 0\bar{1} \rightsquigarrow 0\bar{1}\bar{0};$ $b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_1, 0\bar{1}\bar{0}; a_m \rightsquigarrow$ $b_m, b_{m-1}, \dots, b_1, 0, a_{m+1},$ where $a_m = 0^m,$ $b_m = a_m \bar{m}$	$\frac{1-x-x^4}{(1-x)^2(1-x-x^2)}$
27	{10, 01, 00, 00, 01}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0, 00; a_1 \rightsquigarrow a_2, 0\bar{0}; 0\bar{0} \rightsquigarrow$ $0\bar{0}; a_m \rightsquigarrow a_{m+1}, 0\bar{0}, 0, b_3, b_4, \dots, b_m; b_m \rightsquigarrow$ $b_m, 0\bar{0}, 0, b_3, b_4, \dots, b_{m-1},$ where $a_m = 0^m,$ $b_m = a_m m$	
	{10, 01, 01, 00, 01}	$\epsilon \rightsquigarrow \bar{0}, a_1; a_1 \rightsquigarrow 0\bar{1}, a_2; 0 \rightsquigarrow 0, 0\bar{1}; 0\bar{1} \rightsquigarrow$ $0\bar{1}; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_3, 0, 0\bar{1}; a_m \rightsquigarrow$ $b_m, b_{m-1}, \dots, b_3, 0, 0\bar{1}, a_{m+1},$ where $a_m = 0^m,$ $b_m = a_m \bar{m}$	
	{10, 00, 01, 00, 00}	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow 0\bar{1}, 0\bar{0}; 0\bar{1} \rightsquigarrow 0\bar{1}; 0\bar{0} \rightsquigarrow 0\bar{0}, 0\bar{0}$	$\frac{1-2x+x^2-x^3}{(1-x)^2(1-2x)}$
28	{10, 01, 00, 00, 01}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_{m+1}, \bar{0}\bar{0}, b_1, b_2, \dots, b_m;$ $b_m \rightsquigarrow b_m, \bar{0}\bar{0}, b_1, b_2, \dots, b_{m-1},$ where $a_m = 0^m,$ $b_m = a_m m$	
	{10, 00, 01, 00, 01}	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow (0)^2; \bar{0} \rightsquigarrow 0\bar{1}, 0; 0\bar{1} \rightsquigarrow 0\bar{1}$	$\frac{1-x-x^3}{(1-x)(1-2x)}$
29	{10, 00, 01, 00, 00}	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}; 0\bar{0} \rightsquigarrow 0\bar{0}\bar{2}, 0\bar{0}; 0\bar{0}\bar{2} \rightsquigarrow 0\bar{0}\bar{2}$	$\frac{1-2x+2x^2}{(1-x)^4}$
30	{10, 00, 00, 00, 01}	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}, 0\bar{1}; 0\bar{0} \rightsquigarrow 0\bar{0}, 0\bar{0}; 0\bar{1} \rightsquigarrow 0, 0\bar{0}$	$\frac{1-2x+x^2-x^4}{(1-x)^2(1-2x)}$
31	{10, 00, 00, 01, 00}		
	{10, 00, 00, 01, 00}		
	{10, 00, 00, 00, 00}		
	{10, 00, 00, 00, 01}		
	{10, 01, 00, 01, 00}		
	{10, 00, 00, 00, 00}		
	{10, 01, 00, 00, 00}		
	{10, 01, 01, 01, 01}		
	{10, 01, 00, 00, 00}		
	{10, 01, 01, 01, 01}		
	{10, 01, 00, 01, 00}	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow (0)^2; \bar{0} \rightsquigarrow (0)^2$	
	{10, 01, 01, 00, 01}		
	{10, 01, 01, 00, 01}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_{m+1}, 0, b_1, b_2, \dots, b_m;$ $b_m \rightsquigarrow b_m, 0, b_1, b_2, \dots, b_{m-1},$ where $a_m = 0^m,$ $b_m = a_m m$	
	{10, 00, 00, 01, 00}	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow 0; b_m \rightsquigarrow \bar{0}, b_0, b_1, 0, \dots, b_m; a_m \rightsquigarrow$ $\bar{0}, b_0, b_1, 0, \dots, b_m, a_{m+1},$ where $a_m = 0\bar{1} \dots m,$ $b_m = a_m \bar{0}$	
	{10, 01, 00, 01, 00}	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, \bar{0}; a_m \rightsquigarrow$ $b_m, b_{m-1}, \dots, b_1, \bar{0}, a_{m+1},$ where $a_m = 0\bar{1} \dots m,$ $b_m = a_{m-1} \bar{m}$	
	{10, 01, 00, 00, 00}	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}; 0\bar{0} \rightsquigarrow (0\bar{0})^2; a_m \rightsquigarrow$ $\bar{0}, (0\bar{0})^{m+1}, a_{m+1},$ where $a_m = 0\bar{1} \dots m$	
	{10, 00, 01, 00, 00}	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0; \bar{0} \rightsquigarrow \bar{0}, (0\bar{0})^2; 0\bar{0} \rightsquigarrow (0\bar{0})^2$	
	{10, 01, 00, 01, 01}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, \bar{0};$ $a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, \bar{0}, a_{m+1},$ where $a_m =$ $0^m, b_m = a_m \bar{m}$	
	{10, 01, 01, 00, 01}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}; 0\bar{1} \rightsquigarrow (0\bar{1})^2; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow$ $(0\bar{1})^m, a_{m+1},$ where $a_m = 0^m$	
	{10, 01, 00, 01, 01}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; 0\bar{1} \rightsquigarrow (0\bar{1})^2; a_m \rightsquigarrow (0\bar{1},)^m, \bar{0}, a_{m+1},$ where $a_m = 0^m$	

	$\{\bar{1}0, \bar{0}0, 0\bar{1}, 0\bar{0}, 01\}$	$\epsilon \rightsquigarrow a_0, b_0; a_m \rightsquigarrow a_{m+1}, b_0, b_1, \dots, b_{m+1};$ $\frac{1}{1-2x}$ $b_m \rightsquigarrow b_0, b_1, b_2, \dots, b_m, \text{ where } a_m \bar{0}\bar{1} \dots \bar{m}, b_m =$ $a_{m-1} \bar{m}$
32	$\{10, 01, 00, 01, 01\}$ $\{10, 01, 00, 01, 01\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0\bar{0}, 0; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow \frac{1-3x+2x^2-x^3}{(1-x)^2(1-3x+x^2)}$ $a_{m+1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow b_m, b_1, b_2, \dots, b_m,$ where $a_m = \bar{0}^m, b_m = a_m \bar{m}$
33	$\{10, 00, 00, 00, 01\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; b_m \rightsquigarrow (b_m)^m; a_m \rightsquigarrow \frac{1}{1-x} \sum_{k \geq 0} \frac{x^k}{1-kx}$ $(b_m)^m, a_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{m}$
34	$\{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{0}, 01\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow b_1, a_2; b_m \rightsquigarrow (b_m)^m; a_m \rightsquigarrow \sum_{k \geq 0} \frac{x^k}{1-kx} + \sum_{k \geq 1} \frac{x^k}{1-kx}$ $(b_m)^m, a_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{m}$
35	$\{10, 00, 01, 01, 01\}$ $\{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{1}, 00\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0\bar{0}, 0; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow \frac{x}{(1-x)^2} + C(x)$ $a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = \bar{0}^m$
36	$\{\bar{1}0, \bar{0}1, 0\bar{1}, 0\bar{0}, 00\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \bar{0}0, \text{ where } \frac{x}{1-x} + (1+x^2 C(x))C(x)$ $a_m = \bar{0}^m$
37	$\{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{1}, 0\bar{0}\}$ $\{\bar{1}0, \bar{0}1, 0\bar{1}, 0\bar{0}, 00\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0}\bar{1})^2; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; a_m \rightsquigarrow (a_{m+1})^{m+1},$ where $a_m = \bar{0}^m$ $\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; b_m \rightsquigarrow (b_{m+1})^{m+2}; a_m \rightsquigarrow x + \frac{2x^2}{1-x} + \sum_{k \geq 0} k!x^k$ $(b_m)^m, a_{m+1}, \bar{0}0, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}\bar{1} \dots \bar{m}$
38	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, 0\bar{0}, 01\}$	$\epsilon \rightsquigarrow a_1, b_1; d_m \rightsquigarrow d_m, d_{m-1}, \dots, d_1;$ $b_m \rightsquigarrow d_m, d_{m-1}, \dots, d_1, b_{m+1}; a_m \rightsquigarrow \frac{2-5x+x^2-x\sqrt{1-2x-3x^2}}{2(x^4-2x^3+7x^2-5x+1)}$ $a_{m+1}, c_1, c_2, \dots, c_m; c_m \rightsquigarrow c_m, c_1, c_2, \dots, c_m,$ where $a_m = \bar{0}^m, b_m = \bar{0}^m, c_m = a_m \bar{m},$ $d_m = b_m \bar{m}$
39*	$\{10, 00, 00, 00, 01\}$	
40	$\{10, 01, 01, 00, 01\}$ $\{\bar{1}0, \bar{0}\bar{1}, 0\bar{1}, 0\bar{0}, 01\}$	$\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow a_{m+1}, b_0, b_1, \dots, b_m; b_m \rightsquigarrow \frac{1-2x}{(1-x)(1-3x+x^2)}$ $b_m, b_0, b_1, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{m}$
41*	$\{10, 01, 00, 00, 00\}$	
42	$\{\bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, 01\}$	$\epsilon \rightsquigarrow b_0, a_1; c_m \rightsquigarrow c_0, c_1, \dots, c_{m+1};$ $\frac{x}{1-2x} + C(x)$ $b_m \rightsquigarrow b_0, b_1, \dots, b_m; a_m \rightsquigarrow$ $c_0, c_1, \dots, c_{m-1}, b_m, a_{m+1}, \text{ where } a_m = \bar{0}^m,$ $b_m = a_m \bar{0}, c_m = \bar{0}\bar{1}a_m$
43	$\{\bar{1}0, \bar{0}1, 0\bar{1}, 00, 01\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}, \bar{0}\bar{1}0; a_m \rightsquigarrow \frac{1}{(1-x)^2} + \frac{(1+x)((1-x)C(x)-1)}{1-x}$ $b_1, b_2, \dots, b_m, a_{m+1}, 0, \text{ where } a_m = \bar{0}^m, b_m =$ $\bar{0}\bar{1}^m$
44	$\{10, 00, 00, 00, 01\}$ $\{\bar{1}0, \bar{0}1, 0\bar{1}, 0\bar{0}, 01\}$ $\{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{0}, 00\}$	$\epsilon \rightsquigarrow 0, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1} \text{ where}$ $\frac{a_m = \bar{0}^m}{\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, 0, \text{ where}}$ $\frac{a_m = \bar{0}^m}{a_m = \bar{0}^m}$ $\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \text{ where } \frac{1}{1-x} C(x)$
45	$\{10, 00, 00, \bar{0}1, 01\}$ $\{\bar{1}0, \bar{0}0, 0\bar{1}, 0\bar{0}, 00\}$ $\{10, 00, 00, \bar{0}1, 00\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow (b_m)^{m+1}, a_{m+1}; b_m \rightsquigarrow$ $\frac{(b_m)^{m+1}, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}^m \bar{m}}{a_m \rightsquigarrow a_0, 0; 0 \rightsquigarrow 0; a_0 \rightsquigarrow a_1, (\bar{0}0)^2; \bar{0}0 \rightsquigarrow (\bar{0}0)^2; a_m \rightsquigarrow}$ $\frac{a_{m+1}, (b_m)^{m+2}; b_m \rightsquigarrow (b_m)^{m+2}, \text{ where } a_m =$ $\frac{\bar{0}\bar{1} \dots \bar{m}, b_m = \bar{0}\bar{1} \dots \bar{m}0}{\epsilon \rightsquigarrow \bar{0}, a_0; a_0 \rightsquigarrow (b_1)^2, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow \sum_{k \geq 0} \frac{x^k}{1-(k+1)x}}$ $\frac{(b_{m+1})^{m+2}, a_{m+1}; b_m \rightsquigarrow (b_m)^{m+1}, \text{ where } a_m =$ $\frac{01 \dots m, b_m = 01 \dots (m-1)\bar{m}}{}$
46	$\{10, 01, 00, 00, 00\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, 0\bar{0}, 00\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_{m+1}, (a_m)^m, \text{ where } \frac{1}{1-x} \sum_{k \geq 0} \frac{x^k}{i!k(1-ix)}$ $a_m = \bar{0}^m$
47	$\{10, 01, 01, 01, 00\}$ $\{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{1}, 0\bar{0}\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}1, 0\bar{1}, 0\bar{0}\}$ $\{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{1}, 00\}$ $\{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{1}, 01\}$ $\{\bar{1}0, \bar{0}1, 0\bar{1}, 0\bar{0}, 01\}$ $\{\bar{1}0, \bar{0}1, 0\bar{1}, 00, 01\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, \bar{0}0; \bar{0}0 \rightsquigarrow \bar{0}0; a_m \rightsquigarrow (a_{m+1})^{m+1},$ where $a_m = \bar{0}^m$ $\epsilon \rightsquigarrow a_0, 0; 0 \rightsquigarrow 0\bar{0}, 0; a_0 \rightsquigarrow (a_1)^2; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow$ $\frac{(a_{m+1})^{m+2}, \text{ where } a_m = \bar{0}\bar{1} \dots \bar{m}}{\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow (b_m)^m, a_{m+1}, 0; b_m \rightsquigarrow \frac{x}{(1-x)^2} + \sum_{k \geq 0} k!x^k}$ $\frac{(b_{m+1})^{m+2}, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}\bar{1} \dots \bar{m}}{}$
48	$\{10, 00, 00, 01, 01\}$ $\{\bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, 0\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow 0\bar{0}, (a_{m+1})^{m+1}, \text{ where } \frac{1}{1-x} + \sum_{k \geq 1} (k! + (k-1)!)x^k$ $a_m = \bar{0}^m$

49	{10, 01, 00, 00, 01}	$\epsilon \sim b_1, a_1; 0\bar{1} \sim (0\bar{1})^2; a_m \sim (0\bar{1})^m, a_{m+1}; b_m \sim \frac{1-5x+8x^2-3x^3-4x^4+2x^5}{(1-x)^2(1-2x)(1-3x+x^2)}$ $b_{m+1}, 0\bar{1}, c_2, \dots, c_m; c_m \sim c_m, 0\bar{1}, c_2, \dots, c_m,$ where $a_m = 0^m, b_m = 0^m, vc_m = b_m m$
50	{10, 00, 01, 00, 01}	$\epsilon \sim 0, a_1; 0 \sim a_1, a_2; a_m \sim a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m$
	{10, 00, 01, 00, 01}	$\epsilon \sim 0, 0; 0 \sim 0, a_1; 0 \sim 0, a_1; a_m \sim 2C(x) - 1$ $0, a_1, a_2, \dots, a_{m+1},$ where $a_m = 00^m$
51	{10, 01, 01, 00, 00}	$\epsilon \sim a_1, 0; 0 \sim 0; a_m \sim a_{m+1}, b_m, (a_m)^m; b_m \sim$ $(b_m)^{m+1},$ where $a_m = 0^m, b_m = a_m 0$
	{10, 01, 01, 00, 00}	
	{10, 01, 00, 01, 00}	$\epsilon \sim 0, 0; 0 \sim a_1, 0; 0 \sim a_1, 0; a_m \sim 1 + 2 \sum_{k \geq 1} \frac{x^k}{1-x}$ $a_{m+1}, (a_m)^{m+1},$ where $a_m = 00^m$
52	{10, 01, 00, 01, 01}	
	{10, 01, 00, 01, 01}	
	{10, 01, 00, 01, 01}	$\epsilon \sim 0, a_1; 0 \sim 0; a_m \sim 0, (a_m+1)^{m+1},$ where $a_m = 0^m$
	{10, 00, 00, 01, 00}	$\epsilon \sim 0, a_1; 0 \sim 0, a_1; a_m \sim (a_m+1)^{m+1},$ where $a_m = 0^m$
	{10, 00, 00, 01, 00}	$\epsilon \sim a_0, 0; 0 \sim a_0, 0; a_m \sim (a_m+1)^{m+2},$ where $\frac{1}{1-x} \sum_{k \geq 0} k! x^k$ $a_m = 0\bar{1} \dots \bar{m}$
53	{10, 01, 00, 00, 01}	$\epsilon \sim a_1, c_0; c_m \sim c_1, c_2, \dots, c_{m+1};$ $b_m \sim b_1, b_2, \dots, b_{m+1}, c_{m+1}; a_m \sim$ $b_1, b_2, \dots, b_m, a_{m+1}, c_m,$ where $a_m = 0^m,$ $b_m = 0\bar{1}^m, c_m = 0\bar{1}^m$
54*	{10, 00, 01, 00, 01}	
55	{10, 00, 01, 00, 01}	$\epsilon \sim a_1, b_0; b_m \sim b_1, b_2, \dots, b_{m+1}, b_{m+1}; a_m \sim C(x) + \frac{1-x-\sqrt{1-6x+x^2}}{2}$ $a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m, b_m = 0\bar{1}^m$
56*	{10, 00, 01, 00, 01}	
57	{10, 00, 01, 01, 00}	$\epsilon \sim a_1, b_1; b_m \sim (b_m+1)^{m+1}; a_m \sim C(x) + \sum_{k \geq 1} k! x^k$ $a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m, b_m = 0^m$
58*	{10, 00, 01, 00, 00}	
59	{10, 00, 01, 00, 01}	$\epsilon \sim a_0, b_1; b_m \sim (a_m-1)^m, b_{m+1}; a_m \sim$ $(a_m+1)^{m+2},$ where $a_m = 0\bar{1} \dots \bar{m}, b_m = 0^m$
	{10, 01, 00, 00, 01}	$\epsilon \sim a_1, 0; 0 \sim 0\bar{1}; a_1 \sim b_1, a_2, 0\bar{1}; 0\bar{1} \sim (b_1)^2; \sum_{k \geq 0} k! x^k + x \sum_{i \geq 0} \sum_{k=0}^i \frac{i!}{k!} x^i$ $b_m \sim (b_m+1)^{m+2}; a_m \sim (b_m)^m, a_{m+1}, b_{m-1},$ where $a_m = 0^m, b_m = 0\bar{1} \dots \bar{m}$
60*	{10, 01, 00, 01, 00}	
61*	{10, 01, 00, 01, 00}	
62*	{10, 00, 00, 01, 00}	
63*	{10, 00, 00, 00, 01}	
64	{10, 01, 00, 00, 01}	
	{10, 01, 00, 00, 01}	$\epsilon \sim 0, a_1; 0 \sim b_0; b_m \sim (b_m+1)^{m+2}; a_m \sim$ $b_{m-1}, (a_m+1)^{m+1},$ where $a_m = 0^m, b_m = 00\bar{1}^m$
	{10, 01, 00, 01, 00}	
	{10, 00, 01, 01, 00}	
	{10, 01, 00, 01, 00}	
	{10, 01, 00, 01, 00}	
	{10, 01, 00, 01, 00}	$\epsilon \sim 0, a_1; 0 \sim (a_2)^2; a_m \sim (a_m+1)^{m+1},$ where $a_m = 0^m$
	{10, 00, 01, 00, 01}	$\epsilon \sim 0, 0; 0 \sim (a_1)^2; 0 \sim (a_1)^2; a_m \sim (a_m+1)^{m+2},$ $1 + 2 \sum_{k \geq 1} k! x^k$ where $a_m = 0\bar{1}2 \dots \bar{m}$
65*	{10, 00, 01, 00, 01}	
66*	{10, 00, 01, 00, 01}	
67*	{10, 01, 00, 00, 00}	
68	{10, 01, 00, 01, 00}	Theorem 12
	{10, 01, 00, 01, 00}	
69*	{10, 00, 00, 01, 00}	
70*	{10, 00, 01, 00, 00}	
71*	{10, 01, 00, 00, 01}	
72*	{10, 01, 00, 00, 01}	
73	{10, 01, 00, 00, 01}	$\epsilon \sim 0, a_1; b_m \sim b_{m-1}, b_{m-2}, \dots, b_1, 0; a_m \sim \sum_{k \geq 0} k! x^k (1+x)^{k+1}$ $b_m, b_{m-1}, \dots, b_1, 0, (a_m+1)^{m+1},$ where $a_m = 0^m, b_m = a_m \bar{m}$

74	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}1\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; b_m \rightsquigarrow (b_m)^m; a_m \rightsquigarrow \sum_{k \geq 0} k!x^k + \sum_{k \geq 1} \sum_{i=0}^k \frac{k!x^{k+1}}{1-ix}$ $b_m, b_{m-1}, \dots, b_1, \bar{0}, (a_{m+1})^{m+1},$ where $a_m = 0^m, b_m = a_m \bar{m}$
75*	$\{10, 00, 00, 00, 01\}$	
76*	$\{10, 01, 00, 00, 00\}$	
77	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 01\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow 0\bar{0}, \bar{0}; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow (0\bar{1})^{m+1}, a_{m+1},$ where $a_m = 0^m$ $\frac{1+2x^2}{(1-x)^2}$
78	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (0\bar{1})^2; 0\bar{1} \rightsquigarrow 0\bar{1}; a_m \rightsquigarrow \frac{1-x+2x^2-2x^3+x^4}{(1-x)^3}$ $(0\bar{1})^{m+1}, a_{m+1},$ where $a_m = 0^m$
79	$\{\bar{1}0, \bar{0}\bar{1}, 00, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (0\bar{1})^2; 0\bar{1} \rightsquigarrow 0\bar{1}; \frac{1-x+x^2-2x^3-x^4+x^5}{(1-x)^2(1-x-x^2)}$ $a_m \rightsquigarrow a_{m+1}, 0\bar{1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow$ $0\bar{1}0, b_1, b_2, \dots, b_{m-1},$ where $a_m = 0^m,$ $b_m = a_m \bar{m}$
80	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow 0\bar{1}, \bar{0}; 0\bar{1} \rightsquigarrow 0\bar{1}; a_m \rightsquigarrow (0\bar{1})^{m+1}, a_{m+1},$ where $a_m = 0^m$
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 0\bar{0}, \bar{0}; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}; 0\bar{0} \rightsquigarrow 0\bar{0}$ $\frac{1-x+2x^2-x^3}{(1-x)^3}$
81	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow 0\bar{0}, \bar{0}; 0\bar{1} \rightsquigarrow 0\bar{1}\bar{0}; 0\bar{0} \rightsquigarrow$ $0\bar{0}; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_1, 0\bar{1}\bar{0}; a_m \rightsquigarrow$ $b_m, b_{m-1}, \dots, b_1, 0\bar{0}, a_{m+1},$ where $a_m = 0^m,$ $b_m = a_m \bar{m}$
82	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, 00, 01\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (0\bar{1})^2; 0\bar{1} \rightsquigarrow 0\bar{1}; \frac{1-x+x^2-2x^3}{(1-x)(1-2x)}$ $a_m \rightsquigarrow a_{m+1}, 0\bar{1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow$ $b_m, 0\bar{1}, b_1, b_2, \dots, b_{m-1},$ where $a_m = 0^m,$ $b_m = a_m \bar{m}$
83	$\{10, 01, 01, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{1}, 0\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}; 0\bar{1} \rightsquigarrow 0\bar{1}, (0\bar{0})^2;$ $0\bar{0} \rightsquigarrow 0\bar{0}$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 00\}$	
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow$ $\bar{0}, (0\bar{0})^{m+1}, a_{m+1},$ where $a_m = 01 \dots m$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, 0\bar{1}, 01\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}, \bar{0}\bar{1}; 0\bar{0} \rightsquigarrow 0\bar{0}; \bar{0}\bar{1} \rightsquigarrow$ $(0\bar{0})^2, \bar{0}\bar{1}$ $\frac{1-x+2x^2}{(1-x)^3}$
84	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, 0\bar{1}, 00\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0\bar{0}, 0; 0\bar{0} \rightsquigarrow 0\bar{0}; b_m \rightsquigarrow$ $b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, 0\bar{0},$ where $a_m = 0^m, b_m = 0\bar{1}^m$ $\frac{x(1+x)}{(1-x)^2} + C(x)$
85	$\{\bar{1}0, \bar{0}\bar{1}, 0\bar{1}, 00, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (0\bar{0})^2; 0\bar{0} \rightsquigarrow 0\bar{0}; \frac{(x^3(x-2)C(x)+x^2-x+1)C(x)}{(1-x)^2}$ $b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}, \bar{0}\bar{1}\bar{0}; a_m \rightsquigarrow$ $b_1, b_2, \dots, b_m, a_{m+1}, 0\bar{0},$ where $a_m = 0^m,$ $b_m = 0\bar{1}^m$
86	$\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 0\bar{1}, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (0\bar{0})^2; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow \frac{x(1+x-x^2)}{(1-x)^2} + \sum_{k \geq 0} k!x^k$ $0\bar{0}, a_{m+1}, (b_m)^m; b_m \rightsquigarrow (b_{m+1})^{m+2},$ where $a_m =$ $0^m, b_m = a_m \bar{1}$
87	$\{10, 10, 00, 00, 01\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{1}, 0; \bar{0} \rightsquigarrow \bar{0}, 0, 0\bar{1}; 0\bar{1} \rightsquigarrow 0\bar{1}; 0\bar{1} \rightsquigarrow 0\bar{1}, 0, 0\bar{1}$ $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 0\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}; 0\bar{0} \rightsquigarrow 0\bar{0}\bar{2}, 0\bar{0}; 0\bar{0}\bar{2} \rightsquigarrow$ $0\bar{0}\bar{2}$
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{1}; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; a_m \rightsquigarrow (0)^{m+2}, a_{m+1},$ where $a_m = 01 \dots m$ $\frac{1-2x+3x^2-x^3}{(1-x)^4}$
88	$\{\bar{1}0, \bar{1}0, 0\bar{1}, \bar{0}\bar{1}, 01\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (0)^2; 0\bar{1} \rightsquigarrow 0\bar{1}; a_m \rightsquigarrow (0\bar{1})^m, \bar{0}, a_{m+1},$ where $a_m = 0^m$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 00\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow 0\bar{1}, \bar{0}; 0\bar{1} \rightsquigarrow 0\bar{1}; \frac{1-3x+4x^2-3x^3}{(1-x)^3(1-2x)}$ $b_m \rightsquigarrow 0\bar{1}, \bar{0}, b_1, b_2, \dots, b_m; a_m \rightsquigarrow$ $(0)^2, b_1, b_2, \dots, b_m, a_{m+1},$ where $a_m = 01 \dots m,$ $b_m = a_m \bar{0}$
89	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow a_1, b_1; 0\bar{1} \rightsquigarrow (0\bar{1})^2; 0\bar{0} \rightsquigarrow 0\bar{0}; b_m \rightsquigarrow$ $(0\bar{1})^m, 0\bar{0}, b_{m+1}; a_m \rightsquigarrow a_{m+1}, 0\bar{0}, c_2, c_3, \dots, c_m;$ $c_m \rightsquigarrow 0\bar{0}, c_2, c_3, \dots, c_m,$ where $a_m = 0^m, b_m =$ $0^m, c_m = a_m \bar{m}$
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow a_1, b_1; 0\bar{0} \rightsquigarrow 0\bar{0}; d_m \rightsquigarrow d_m, d_{m-1}, \dots, d_1, 0\bar{0};$ $b_m \rightsquigarrow d_m, d_{m-1}, \dots, d_1, 0\bar{0}, b_{m+1};$ $a_m \rightsquigarrow a_{m+1}, 0\bar{0}, c_2, c_3, \dots, c_m; c_m \rightsquigarrow$ $0\bar{0}, c_2, c_3, \dots, c_m,$ where $a_m = 0^m, b_m = 0^m,$ $c_m = a_m \bar{m}, d_m = b_m \bar{m}$
	$\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{1}, 0\bar{0}, 0; \bar{0} \rightsquigarrow 0\bar{1}, 0\bar{0}; 0\bar{1} \rightsquigarrow (0\bar{0})^3; 0\bar{0} \rightsquigarrow$ $(0\bar{0})^2, \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}$
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow (0)^2, 0; \bar{0} \rightsquigarrow (0)^2$

	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow a_0, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0; a_0 \rightsquigarrow a_1, \bar{0}\bar{0}, \bar{0}\bar{1}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; \bar{0}\bar{1} \rightsquigarrow 0, \bar{0}\bar{0}, \bar{0}\bar{1}; a_m \rightsquigarrow a_{m+1}, \bar{0}\bar{0}, 0, b_2, b_3, \dots, b_m, c_{m+1}; b_m \rightsquigarrow \bar{0}\bar{0}, 0, b_2, b_3, \dots, b_m; c_m \rightsquigarrow b_m, \bar{0}\bar{0}, 0, b_2, b_3, \dots, b_{m-1}, c_m, \text{ where } a_m = \bar{0}\bar{1} \dots \bar{m}, b_m = a_m m, c_m = a_{m-1} m$	$\frac{1-x+x^2}{(1-x)(1-2x)}$
90	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; b_m \rightsquigarrow (\bar{0}\bar{1})^{m+1}, b_{m+1}; a_m \rightsquigarrow a_{m+1}, c_1, c_2, \dots, c_m; c_m \rightsquigarrow c_m, c_1, c_2, \dots, c_m, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}^m, c_m = a_m m$	$\frac{2x^4-6x^3+6x^2-4x+1}{(1-x)^3(1-3x+x^2)}$
91	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow \bar{0}\bar{0}, a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \rightsquigarrow \bar{0}\bar{1}\bar{0}, b_{m+1}, b_m, \dots, b_1, \text{ where } a_m = \bar{0}^m, b_m = a_m 1$	$\frac{(1-x+x^2+x^3-2x^3C(x))C(x)}{(1-x)^2}$
92	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow \bar{0}\bar{0}, a_{m+1}, (b_m)^m; b_m \rightsquigarrow (b_{m+1})^{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m 1$	$\frac{x(1+x)}{(1-x)^2} + \sum_{k \geq 0} k!x^k$
93	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0}\bar{0})^2; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow \bar{0}\bar{0}, (a_{m+1})^{m+1}, \text{ where } a_m = \bar{0}^m$	$\frac{x^2}{1-x} + \frac{1}{1-x} \sum_{k \geq 0} k!x^k$
94	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}\bar{2}, \bar{0}\bar{0}; \bar{0}\bar{0}\bar{2} \rightsquigarrow \bar{0}\bar{0}\bar{2}; b_m \rightsquigarrow \bar{0}\bar{0}\bar{2}, b_0, b_1, \dots, b_m; a_m \rightsquigarrow \bar{0}, b_0, b_1, \dots, b_m, a_{m+1}, \text{ where } a_m = \bar{0}\bar{1} \dots m, b_m = a_m 0$	$\frac{1-4x+7x^2-6x^3+x^4}{(1-x)^4(1-2x)}$
95	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow (\bar{0})^2; a_m \rightsquigarrow (\bar{0})^{m+2}, a_{m+1}, \text{ where } a_m = \bar{0}\bar{1} \dots m$	
	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; b_m \rightsquigarrow \bar{0}\bar{1}, b_1, b_2, \dots, b_m; a_m \rightsquigarrow \bar{0}\bar{1}, b_1, b_2, \dots, b_m, a_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{0}$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0})^2; \bar{0}\bar{1}\bar{0} \rightsquigarrow \bar{0}\bar{1}\bar{0}; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, \bar{0}, a_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{m}$	
	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow (\bar{0})^2; \bar{0} \rightsquigarrow \bar{0}, 0, \bar{0}\bar{1}; \bar{0}\bar{1} \rightsquigarrow (\bar{0})^2, \bar{0}\bar{1}$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (\bar{0})^2; \bar{0}\bar{1}\bar{0} \rightsquigarrow \bar{0}\bar{1}\bar{0}; a_m \rightsquigarrow a_{m+1}, 0, b_1, b_2, \dots, b_m; b_m \rightsquigarrow \bar{0}\bar{1}\bar{0}, b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{1} \rightsquigarrow (\bar{0}\bar{1})^2; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow a_{m+1}, (\bar{0}\bar{1})^m; b_m \rightsquigarrow (\bar{0}\bar{1})^m, \bar{0}\bar{0}, b_{m+1}; \text{ where } a_m = \bar{0}^m, b_m = \bar{0}^m$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{1} \rightsquigarrow (\bar{0}\bar{1})^2; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; b_m \rightsquigarrow (\bar{0}\bar{1})^m, b_{m+1}; a_m \rightsquigarrow a_{m+1}, \bar{0}\bar{0}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow \bar{0}\bar{0}, b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}^m, c_m = c_m m$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0})^2; a_m \rightsquigarrow (\bar{0})^{m+1}, a_{m+1}, \text{ where } a_m = \bar{0}^m$	$\frac{1-2x+2x^2}{(1-x)^2(1-2x)}$
96	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}\bar{1}, \bar{0}, 0; \bar{0} \rightsquigarrow (\bar{0})^2; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}\bar{0}, (\bar{0})^2; \bar{0}\bar{1}\bar{0} \rightsquigarrow (\bar{0}\bar{1}\bar{0})^3$	$\frac{1-4x+4x^2-2x^3-2x^4}{(1-x)(1-2x)(1-3x)}$
97	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}\bar{1}, \bar{0}\bar{0}; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; b_m \rightsquigarrow \bar{0}\bar{1}, b_0, b_1, \dots, b_m; c_m \rightsquigarrow \bar{0}\bar{1}, b_0, b_1, \dots, b_m, c_{m+1}; a_m \rightsquigarrow \bar{0}\bar{1}, b_0, b_1, \dots, b_{m-2}, c_{m-1}, a_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}\bar{0}a_m 1, c_m = \bar{0}\bar{0}a_m$	$\frac{(1-x)^2}{(1-2x)^2}$
98	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, c_1; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; c_m \rightsquigarrow (\bar{0}\bar{1})^m, c_{m+1}; a_m \rightsquigarrow a_{m+1}, \bar{0}\bar{1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow b_m, \bar{0}\bar{1}, b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m, c_m = \bar{0}^m$	$\frac{1-4x+6x^2-5x^3+x^4}{(1-x)^3(1-3x+x^2)}$
99*	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$		
100	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (\bar{0})^2; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, 0, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}\bar{1}^m$	$\frac{x}{(1-x)(1-2x)} + C(x)$
101	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, a_1, 0; \bar{0} \rightsquigarrow a_1, \bar{0}\bar{1}; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; a_m \rightsquigarrow a_{m+1}, (b_m)^{m+1}; b_m \rightsquigarrow (b_m)^{m+1}, \text{ where } a_m = \bar{0}\bar{0}^m, b_m = a_m 1$	$\frac{1}{(1-x)^2} + \frac{2}{1-x} \sum_{k \geq 2} \frac{x^k}{1-kx}$

102	$\{\bar{1}0, \bar{0}0, 0\bar{1}, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_0, 0; 0 \rightsquigarrow 0\bar{0}, 01; a_0 \rightsquigarrow a_1, 0\bar{0}, c_1;$ $0\bar{0} \rightsquigarrow (0\bar{0})^2; 01 \rightsquigarrow 01; c_1 \rightsquigarrow b_1, (0\bar{0})^2; a_m \rightsquigarrow$ $a_{m+1}, (b_m)^{m+1}, c_{m+1}; b_m \rightsquigarrow (b_m)^{m+2}; c_m \rightsquigarrow$ $b_m, (b_{m-1})^{m+1}, \text{ where } a_m = 0\bar{1} \dots \bar{m}, b_m =$ $a_m 0, c_m = a_{m-1} m$	$\frac{-x}{1-x} + (1+x) \sum_{k \geq 1} \frac{x^{k-1}}{1-kx}$
103	$\{\bar{1}0, \bar{1}0, \bar{0}0, 0\bar{0}, 01\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_2, c_1; b_m \rightsquigarrow b_1, b_2, \dots, b_m;$ $a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}; c_m \rightsquigarrow$ $b_1, b_2, \dots, b_m, a_{m+1}, c_{m+1}, \text{ where } a_m = 0^m,$ $b_m = a_m \bar{1}, c_m = 0\bar{1}^m$	$\frac{1-4x+6x^2-3x^3-x^4}{(1-x)^2(1-2x)^2}$
104	$\{\bar{1}0, \bar{1}0, 0\bar{1}, 0\bar{1}, 01\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (0)^2; a_m \rightsquigarrow a_{m+1}, 0, b_1, b_2, \dots, b_m;$ $b_m \rightsquigarrow b_m, 0, b_1, b_2, \dots, b_m, \text{ where } a_m = 0^m,$ $b_m = a_m m$	$\frac{1-x}{1-3x+x^2}$
105	$\{\bar{1}0, \bar{0}1, 0\bar{1}, 01, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (0)^2; 0\bar{1}0 \rightsquigarrow 0\bar{1}0;$ $b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}, 0\bar{1}0; a_m \rightsquigarrow$ $b_1, b_2, \dots, b_{m+1}, a_{m+1}, 0, \text{ where } a_m = 0^m,$ $b_m = 0\bar{1}^m$	
	$\{\bar{1}0, 0\bar{1}, 0\bar{0}, 00, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; 0\bar{0}2 \rightsquigarrow 0\bar{0}2;$ $a_m \rightsquigarrow a_{m+1}, b_m, c_m, c_{m-1}, \dots, c_1; b_m \rightsquigarrow$ $b_m, b_{m-1}, \dots, b_1, 0\bar{0}2; c_m \rightsquigarrow c_{m+1}, c_m, \dots, c_1,$ $\text{ where } a_m = 0^m, b_m = a_m 0, c_m = a_m \bar{1}$	
	$\{\bar{1}0, \bar{0}0, \bar{0}0, 01, 1\bar{0}\}$	$\epsilon \rightsquigarrow 0, a_1; \bar{0} \rightsquigarrow \bar{0}, 0; 0\bar{0}2 \rightsquigarrow 0\bar{0}2; b_m \rightsquigarrow$ $b_0, b_1, \dots, b_{m+1}; c_m \rightsquigarrow 0\bar{0}2, c_1, c, \dots, c_m; a_m \rightsquigarrow$ $b_0, b_1, \dots, b_{m-1}, c_m, a_{m+1}, \text{ where } a_m = 0^m,$ $b_m = 0\bar{1}a_m, c_m = a_m 0$	
	$\{\bar{1}0, 0\bar{1}, 01, 0\bar{1}, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (0)^2; 01\bar{0} \rightsquigarrow 01\bar{0};$ $a_m \rightsquigarrow 0, a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \rightsquigarrow$ $01\bar{0}, b_{m+1}, b_m, \dots, b_1, \text{ where } a_m = 0^m, b_m =$ $a_m \bar{1}$	$\frac{1}{1-2x} + \frac{x^2}{(1-x)^2} C^3(x)$
106	$\{\bar{1}0, \bar{0}1, \bar{0}0, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow c_1, a_1, 0; \bar{0} \rightsquigarrow a_1, \bar{0}1; c_1 \rightsquigarrow c_2, a_1, \bar{0}1;$ $\bar{0}1 \rightsquigarrow \bar{0}1; c_m \rightsquigarrow c_{m+1}, a_m, (b_{m-1})^m, a_m \rightsquigarrow$ $a_{m+1}, (b_m)^{m+1}; b_m \rightsquigarrow (b_m)^{m+1}, \text{ where } a_m =$ $0\bar{0}^m, b_m = a_m \bar{1}, c_m = 0\bar{1}^m$	$\frac{1+x}{1-x} - \frac{3-x}{1-x} \sum_{k \geq 2} \frac{x^k}{1-kx}$
107	$\{10, 0\bar{1}, 0\bar{0}, 00, 10\}$		
	$\{\bar{1}0, 0\bar{1}, 0\bar{0}, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_{m+1}, (b_m)^{m+1}; b_m \rightsquigarrow$ $(b_m)^{m+1}, \text{ where } a_m = 0^m, b_m = a_m 0$	$\frac{1}{1-x} + \frac{1}{1-x} \sum_{k \geq 2} \frac{x^{k-1}}{1-kx}$
108	$\{10, 10, 0\bar{1}, 00, 00\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow$ $b_1, b_2, \dots, b_m, a_{m+1}, b_m, \text{ where } a_m = 0^m, b_m =$ $0\bar{1}^m$	
	$\{\bar{1}0, \bar{0}0, \bar{0}1, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow 0, 0; 0 \rightsquigarrow \bar{0}, a_1, 0; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow$ $\bar{0}, a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = a\bar{0}^m$	
	$\{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{1}, 10\}$	$\epsilon \rightsquigarrow a_1, b_1; 0\bar{0} \rightsquigarrow 0\bar{0}; b_m \rightsquigarrow 0\bar{0}, b_{m+1}, b_m, \dots, b_1;$ $a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = 0^m, b_m =$ 0^m	$\frac{(2-x)C(x)-1}{1-x}$
109	$\{10, 0\bar{1}, 0\bar{1}, 00, 10\}$		
	$\{\bar{1}0, \bar{0}1, 0\bar{1}, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow c_1, 01; 01 \rightsquigarrow 01; c_m \rightsquigarrow$ $\frac{1}{1-x} + \sum_{k \geq 1} \frac{(2k-1)x^k}{1-kx}$ $c_{m+1}, (b_m)^{m+1}; a_m \rightsquigarrow a_{m+1}, c_m, (b_m)^m; b_m \rightsquigarrow$ $(b_m)^{m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{1}, c_m =$ $0a_m$	
110*	$\{10, 00, 0\bar{1}, 01, 10\}$		
111	$\{10, 0\bar{1}, 0\bar{1}, 0\bar{1}, 1\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0})^2; a_m \rightsquigarrow \bar{0}, a_{m+1}, (b_m)^m; b_m \rightsquigarrow$ $(b_{m+1})^{m+2}, \text{ where } a_m = 0^m, b_m = a_m \bar{1}$	$\frac{x}{(1-x)(1-2x)} + \sum_{k \geq 0} k! x^k$
112	$\{\bar{1}0, \bar{1}0, \bar{0}1, 0\bar{1}, 0\bar{0}\}$	$\epsilon \rightsquigarrow a_1, b_1; 0\bar{0} \rightsquigarrow 0\bar{0}; b_m \rightsquigarrow (b_{m+1})^{m+1},$ $c_m \rightsquigarrow c_1, c_2, \dots, c_{m+1};$ $c_1, c_2, \dots, c_m, a_{m+1}, 0\bar{0}, \text{ where } a_m = 0^m,$ $b_m = 0^m, c_m = 0\bar{1}^m$	
	$\{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{1}, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, b_1; 0\bar{0} \rightsquigarrow 0\bar{0}; b_m \rightsquigarrow 0\bar{0}, b_{m+1}, (c_m)^m;$ $c_m \rightsquigarrow (c_{m+1})^{m+2}; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1},$ $\text{ where } a_m = 0^m, b_m = 0^m, c_m = a_m \bar{1}$	$\frac{x(1-x-x^2C(x))C^2(x)}{(1-x)^2} + \sum_{k \geq 0} k! x^k$
113	$\{\bar{1}0, 0\bar{1}, 00, 00, 10\}$	$\epsilon \rightsquigarrow 0, a_0; \bar{0} \rightsquigarrow b_1, 0; b_m \rightsquigarrow b_{m+1}, b_m, \dots, b_1, \bar{0};$ $a_m \rightsquigarrow 0, (b_1)^{m+1}, a_{m+1}, \text{ where } a_m = 0\bar{1} \dots m,$ $b_m = 0\bar{0}^m$	$\frac{2(1-x)C(x)-1}{(1-x)^2}$
114	$\{10, 10, 0\bar{1}, 00, 01\}$	$\epsilon \rightsquigarrow a_1, b_1; b_m \rightsquigarrow b_1, b_2, \dots, b_m; a_m \rightsquigarrow$ $b_1, b_2, \dots, b_m, a_{m+1}, b_{m+1}, \text{ where } a_m = 0^m,$ $b_m = 0^m$	
	$\{\bar{1}0, \bar{0}0, \bar{0}1, 01, 1\bar{0}\}$		
	$\{10, 10, 00, \bar{0}1, 01\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow \bar{0}, a_1, a_2, \dots, a_{m+1},$ $\text{ where } a_m = 0^m$	

	{10, 10, 01, 00, 01}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0, b_1; b_m \rightsquigarrow 0, b_1, b_2, \dots, b_{m+1};$ $a_m \rightsquigarrow 0, b_1, b_2, \dots, b_{m-1}, a_{m+1}, b_m, \text{ where } a_m =$ $\bar{0}^m, b_m = 0a_m$
	{10, 10, 00, 01, 00}	$\epsilon \rightsquigarrow 0, a_0; \bar{0} \rightsquigarrow 0, a_0; a_m \rightsquigarrow 0, a_0, a_1, \dots, a_{m+1},$ $\text{ where } a_m = 01 \cdot \dots \cdot m$
	{10, 00, 01, 00, 10}	$\epsilon \rightsquigarrow a_0, 0; 0 \rightsquigarrow a_0, 0; a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_0, 0,$ $\text{ where } a_m = \bar{0}1 \cdot \dots \cdot m$
	{10, 01, 01, 00, 10}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_1, 0, C^2(x)$ $\text{ where } a_m = 0^m$
115	{10, 10, 01, 00, 00}	$\epsilon \rightsquigarrow 0, a_0; \bar{0} \rightsquigarrow b_1, \bar{0}; b_m \rightsquigarrow b_{m+1}, (b_m)^{m+1}; a_m \rightsquigarrow$ $\bar{0}, (b_1)^{m+1}, a_{m+1}, \text{ where } a_m = 01 \cdot \dots \cdot m, b_m =$ $0\bar{0}^m$
	{10, 10, 01, 00, 00}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_{m+1}, b_m, (a_m)^m; \text{ Theorem 12}$ $b_m \rightsquigarrow (b_m)^{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m 0$
116	{10, 00, 00, 01, 10}	$\epsilon \rightsquigarrow 0, a_1; 0 \rightsquigarrow 0, a_1; c_m \rightsquigarrow c_{m+1}, c_m, \dots, c_0;$ $a_m \rightsquigarrow c_{m-1}, a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \rightsquigarrow$ $c_m, b_{m+1}, b_m, \dots, b_1, \text{ where } a_m = 0^m, b_m =$ $a_m 1, c_m = 001^m$
	{10, 01, 00, 00, 10}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}, c_m;$ $c_m \rightsquigarrow c_0, c_1, \dots, c_{m+1}; a_m \rightsquigarrow a_{m+1}, c_m;$ $b_1, b_2, \dots, b_m, a_{m+1}, c_{m-1}, \text{ where } a_m = \bar{0}^m,$ $b_m = \bar{0}1^m, c_m = b_m 0$
117*	{10, 00, 00, 00, 10}	
118*	{10, 01, 01, 00, 10}	
119	{10, 10, 00, 01, 01}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0\bar{0}, 0; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow$ $0\bar{0}, b_1, b_2, \dots, b_m, b_1, b_2, \dots, b_m, \text{ where}$ $a_m = \bar{0}^m, b_m = a_m m$
120*	{10, 00, 01, 00, 10}	
121	{10, 01, 01, 00, 10}	$\epsilon \rightsquigarrow a_1, b_1; b_m \rightsquigarrow (b_{m+1})^{m+1}; a_m \rightsquigarrow$ $a_1, a_2, a_{m+1}, \bar{0}\bar{0}, \text{ where } a_m = \bar{0}^m, b_m = 0^m$ $\frac{x\sqrt{1-4x} + (1-x)(1-2x)}{(1-x)^2\sqrt{1-4x}}$
122	{10, 10, 01, 01, 01}	$\epsilon \rightsquigarrow 0, a_1; \bar{0} \rightsquigarrow (\bar{0})^2; a_m \rightsquigarrow \bar{0}, (a_{m+1})^{m+1}, \text{ where}$ $a_m = 0^m$ $\frac{1-x}{1-2x} \sum_{k \geq 0} k!x^k$
123	{10, 10, 00, 01, 01}	$\epsilon \rightsquigarrow a_1, b_1; 0\bar{0} \rightsquigarrow 0\bar{0}; b_m \rightsquigarrow 0\bar{0}, (b_{m+1})^{m+1}; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = 0^m$ $C(x) + \frac{1}{1-x} \sum_{k \geq 1} k!x^k$
124	{10, 00, 00, 01, 10}	
	{10, 00, 00, 01, 10}	$\epsilon \rightsquigarrow 0, a_1; a_1 \rightsquigarrow 0\bar{0}, a_2, b_1; \bar{0} \rightsquigarrow \bar{0}, a_1; 0\bar{0} \rightsquigarrow (b_1)^2;$ $a_m \rightsquigarrow b_{m-1}, a_{m+1}, (b_m)^m; b_m \rightsquigarrow (b_{m+1})^{m+2},$ $\frac{1}{1-x} + \frac{x}{1-x} \sum_{k \geq 0} k!(k + \sum_{j=0}^k \frac{1}{j!})x^k$ $\text{ where } a_m = 0^m, b_m = a_m 1$
125*	{10, 01, 01, 00, 10}	
126*	{10, 10, 01, 01, 00}	
127	{10, 00, 01, 01, 10}	$\epsilon \rightsquigarrow 0, a_1; \bar{0} \rightsquigarrow (b_1)^2; 0\bar{0} \rightsquigarrow 0\bar{0}; b_m \rightsquigarrow (b_{m+1})^{m+1}, \sum_{k \geq 0} (2(k-1)! + k-2)x^k$ $\text{ where } a_m = 0^m, b_m = a_m 1$
128	{10, 10, 01, 00, 01}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1}, 0, b_1, b_2, \dots, b_m; b_m \rightsquigarrow$ $0, b_1, b_2, \dots, b_m, 0, b_1, b_2, \dots, b_{m-1}, \text{ where}$ $a_m = \bar{0}^m, b_m = a_m m$ $\frac{2C(x)-1}{(x-1)C(x)+2}$
129*	{10, 01, 00, 01, 10}	
130	{10, 01, 00, 01, 10}	$\epsilon \rightsquigarrow a_1, b_0; b_m \rightsquigarrow b_0, b_1, \dots, b_{m+1}; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1}, b_m, \text{ where } a_m = \bar{0}^m, b_m =$ $0a_m$
	{10, 01, 00, 01, 10}	$\epsilon \rightsquigarrow 0, a_1; \bar{0} \rightsquigarrow b_0, \bar{0}; b_m \rightsquigarrow b_{m+1}, b_m, \dots, b_1, \bar{0};$ $a_m \rightsquigarrow b_{m-1}, a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = 0^m,$ $b_m = 00a_m$ $\frac{(1-5x)C(x)+2x}{1-4x}$
131*	{10, 00, 00, 01, 10}	
132*	{10, 01, 00, 01, 10}	
133*	{10, 01, 00, 01, 10}	
134	{10, 10, 01, 00, 01}	$\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1}, b_0, b_1, \dots, b_m; b_m \rightsquigarrow$ $b_1, b_2, \dots, b_m, b_0, b_1, \dots, b_m, \text{ where } a_m = 0^m,$ $b_m = a_m m$ $\frac{1}{(1-x)(2-C(x))}$
135	{10, 00, 01, 01, 10}	$\epsilon \rightsquigarrow 0, a_1; \bar{0} \rightsquigarrow 0, b_1; c_m \rightsquigarrow c_1, c_2, \dots, c_{m+1}, c_{m+1};$ $b_m \rightsquigarrow 0, b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow$ $c_1, c_2, \dots, c_m, b_m, a_{m+1}, \text{ where } a_m = 0^m,$ $b_m = 0\bar{0}^m, c_m = 0\bar{1}^m$

$\{\bar{1}0, \bar{0}1, 0\bar{1}, 0\bar{0}, 10\}$	$\epsilon \sim a_1, b_1; b_m \sim b_{m+1}, b_m, \dots, b_1; 1 + xC^3(x) + \frac{1-x-\sqrt{1-6x+x^2}}{2}$ $a_m \sim a_{m+1}, b_{m+1}, c_m, c_{m+1}, \dots, c_1;$ $c_m \sim c_{m+1}, c_{m+1}, c_m, \dots, c_1,$ where $a_m = \bar{0}^m,$ $b_m = 0^m, c_m = a_{m-1}$
136* $\{10, 01, 00, 01, 10\}$	
137* $\{10, 00, 00, 01, 10\}$	
138* $\{10, 00, 00, 00, 10\}$	
139* $\{10, 10, 00, 00, 00\}$	
140* $\{10, 00, 00, 00, 10\}$	
141 $\{10, 10, 00, 00, 00\}$	Open
$\{10, 10, 0\bar{1}, 0\bar{1}, 00\}$	$\epsilon \sim \bar{0}, 0; 0 \sim \bar{0}, 0; a_m \sim a_{m+1}, (a_m)^{m+1},$ where $\sum_{k \geq 0} \frac{x^k}{\prod_{j=1}^{k+1} (1-jx)}$ $a_m = \bar{0}^m$
142* $\{10, 00, 00, 00, 10\}$	
143* $\{10, 01, 00, 01, 10\}$	
144 $\{10, 01, 00, 01, 10\}$	
$\{\bar{1}0, \bar{0}1, \bar{0}0, 0\bar{1}, 1\bar{0}\}$	$\epsilon \sim \bar{0}, b_1; \bar{0} \sim a_1, \bar{0}1; \bar{0}1 \sim (c_1)^2; a_m \sim$ $a_{m+1}, (c_m)^{m+1}; b_m \sim a_m, b_{m+1}, (c_m)^m; c_m \sim$ $(c_{m+1})^{m+2},$ where $a_m = 0\bar{0}^m, b_m = 0^m, c_m =$ a_{m-1}
145* $\{10, 01, 00, 01, 10\}$	
146* $\{10, 10, 00, 01, 00\}$	
147* $\{10, 00, 01, 00, 10\}$	
148 $\{10, 10, 01, 00, 01\}$	
$\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}; a_m \sim (\bar{0})^{m+1}, (a_{m+1})^{m+1},$ where $a_m = 0^m$
$\{\bar{1}0, 10, \bar{0}\bar{0}, \bar{0}0, 0\bar{1}\}$	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}, a_1; b_m \sim (b_{m+1})^{m+2}; a_m \sim$ $b_{m-1}, (a_{m+1})^{m+1},$ where $a_m = 0^m, b_m = 0\bar{0}1^m$
$\{10, 00, 01, 00, 10\}$	
$\{\bar{1}0, 10, \bar{0}\bar{0}, 0\bar{1}, 0\bar{0}\}$	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}, (a_2)^2; a_m \sim (a_{m+1})^{m+1},$ where $\frac{x}{1-x} \sum_{k \geq 0} (k+1)!x^k + \sum_{k \geq 0} k!x^k$ $a_m = 0^m$
149 $\{10, 01, 00, 01, 10\}$	
$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, 1\bar{0}\}$	
$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 0\bar{0}, 1\bar{0}\}$	Theorem 13
150* $\{10, 01, 00, 00, 10\}$	
151* $\{10, 10, 00, 00, 01\}$	
152* $\{10, 00, 00, 01, 10\}$	
153* $\{10, 00, 01, 00, 10\}$	
154* $\{10, 10, 00, 00, 01\}$	
155* $\{10, 10, 01, 00, 00\}$	
156 $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, 0\bar{1}, 10\}$	
$\{10, 01, 00, 01, 10\}$	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim (b_1)^2; b_m \sim$ $b_{m+1}, b_{m+1}, b_m, \dots, b_1; a_m \sim$ $b_m, a_{m+1}, b_m, b_{m-1}, \dots, b_1,$ where $a_m = 0^m,$ $b_m = 0\bar{0}^m$
$\{10, 01, 00, 01, 10\}$	$\epsilon \sim a_1, 0; 0 \sim (b_1)^2; b_m \sim b_1, b_2, \dots, b_{m+1}, b_{m+1};$ $a_m \sim b_1, b_2, \dots, b_m, a_{m+1}, b_m,$ where $a_m = \bar{0}^m,$ $b_m = 0\bar{1}^m$ $\frac{x^2-6x+5+(x-3)\sqrt{1-6x+x^2}}{2}$
157* $\{10, 00, 01, 00, 10\}$	
158* $\{10, 00, 00, 01, 10\}$	
159* $\{10, 10, 00, 01, 00\}$	
160* $\{10, 01, 00, 00, 10\}$	
161* $\{10, 00, 00, 00, 10\}$	
162* $\{10, 00, 01, 00, 10\}$	
163* $\{10, 10, 00, 01, 00\}$	
164* $\{10, 00, 00, 00, 10\}$	
165* $\{10, 10, 00, 00, 00\}$	
$\{10, 00, 00, 01, 10\}$	$\epsilon \sim a_1, 0; 0 \sim a_1, 0; a_m \sim$ $a_1, a_2, \dots, a_{m+1}, (a_m)^m,$ where $a_m = \bar{0}^m$
166* $\{10, 00, 00, 01, 10\}$	
167* $\{10, 01, 00, 00, 10\}$	
168* $\{10, 00, 00, 00, 10\}$	
169* $\{10, 01, 01, 00, 10\}$	
$\{10, 01, 0\bar{1}, 0\bar{0}, 10\}$	$\epsilon \sim a_1, b_1; b_m \sim (b_{m+1})^{m+1}; a_m \sim$ $a_{m+1}, (b_{m+1})^{m+1},$ where $a_m = \bar{0}^m, b_m = 0^m$
$\{\bar{1}0, 10, \bar{0}\bar{0}, \bar{0}0, 0\bar{0}\}$	
$\{\bar{1}0, \bar{0}\bar{0}, 0\bar{1}, 0\bar{0}, 1\bar{0}\}$	$\epsilon \sim a_0, b_1; b_m \sim (b_{m+1})^{m+1}; a_m \sim$ $a_{m+1}, (b_{m+2})^{m+2},$ where $a_m = 0\bar{1} \dots \bar{m}, b_m =$ 0^m
$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 0\bar{1}, 1\bar{0}\}$	

	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_0)^2; b_m \rightsquigarrow (b_{m+1})^{m+3}; a_m \rightsquigarrow b_{m-1}, a_{m+1}, (b_{m-1})^m, \text{ where } a_m = 0^m, b_m = 0\bar{0}a_m$
	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow \bar{b}_0, \bar{b}_1, \dots, \bar{b}_m; a_m \rightsquigarrow b_0, \bar{b}_1, \dots, \bar{b}_m, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{0}$ Theorem 14
170*	$\{10, 00, 00, 01, 10\}$	
171*	$\{10, 10, 00, 00, 01\}$	
172*	$\{10, 01, 00, 00, 10\}$	
173*	$\{10, 00, 00, 01, 10\}$	
174*	$\{10, 01, 00, 00, 10\}$	
175*	$\{10, 01, 00, 00, 10\}$	
176*	$\{10, 10, 00, 01, 00\}$	
177	$\{10, 00, 01, 00, 10\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$	Theorem 15
178*	$\{10, 01, 00, 00, 10\}$	
179*	$\{10, 00, 01, 00, 10\}$	
180	$\{10, 10, 01, 00, 00\}$ $\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	Theorem 12
181	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_0)^2; b_m \rightsquigarrow (b_{m+1})^{m+3}; a_m \rightsquigarrow b_{m-1}, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = 0\bar{0}a_m$
	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_1)^2; b_m \rightsquigarrow (b_{m+1})^{m+2}; a_m \rightsquigarrow b_m, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = 0\bar{0}^m$
	$\{10, 10, 01, 00, 00\}$	$\epsilon \rightsquigarrow 0, a_1; \bar{0} \rightsquigarrow b_0; b_m \rightsquigarrow (b_{m+1})^{m+2}; a_m \rightsquigarrow \bar{0}, b_0, b_1, \dots, b_{m-1}, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = 0\bar{0}1^m$
	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow a_1, b_1; b_m \rightsquigarrow (b_{m+1})^{m+1}; a_m \rightsquigarrow \sum_{k \geq 0} k!(1 + \sum_{i=1}^k \frac{1}{i})x^k$ $a_{m+1}, b_{m+1}, (a_{m+1})^m, \text{ where } a_m = \bar{0}^m, b_m = 0^m$
182	$\{10, 01, 00, 00, 10\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$	Theorem 12
183*	$\{10, 00, 00, 00, 10\}$	
184*	$\{10, 01, 00, 00, 10\}$	
185	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; \bar{0}\bar{0} \rightsquigarrow (\bar{0}\bar{0})^2; b_m \rightsquigarrow (\bar{0}\bar{1})^m, \bar{0}\bar{0}, b_{m+1}; a_m \rightsquigarrow a_{m+1}, (\bar{0}\bar{0})^{m+1}, \text{ where } a_m = \bar{0}^m, b_m = 0^m$
	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow a_1, c_1; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; \bar{0}\bar{0} \rightsquigarrow (\bar{0}\bar{0})^2; a_m \rightsquigarrow a_{m+1}, \bar{0}\bar{0}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow \bar{0}\bar{1}, b_1, b_2, \dots, b_m; c_m \rightsquigarrow (\bar{0}\bar{1})^m, \bar{0}\bar{0}, c_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m m, c_m = 0^m$ $\frac{1-3x+5x^2-4x^3}{(1-x)^3(1-2x)}$
186	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, b_0, a_1; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; c_m \rightsquigarrow \bar{0}\bar{1}, c_0, c_1, \dots, c_m; b_m \rightsquigarrow \bar{0}\bar{1}, c_0, c_1, \dots, c_m, b_{m+1}; a_m \rightsquigarrow \bar{0}\bar{1}, c_0, c_1, \dots, c_{m-2}, b_{m-1}, a_{m+1}, \text{ where } a_m = 0^m, b_m = 0\bar{0}a_m, c_m = b_m \bar{1}$ $\frac{1-4x+7x^2-5x^3}{(1-x)^2(1-2x)^2}$
187	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{0} \rightsquigarrow (\bar{0}\bar{0})^2; \bar{0}\bar{1}\bar{0} \rightsquigarrow \bar{0}\bar{1}\bar{0}; b_m \rightsquigarrow \bar{0}\bar{0}, b_{m+1}, d_m, \dots, d_1; d_m \rightsquigarrow \bar{0}\bar{1}\bar{0}, d_{m+1}, d_m, \dots, d_1; c_m \rightsquigarrow c_1, c_2, \dots, c_{m+1}; a_m \rightsquigarrow c_1, c_2, \dots, c_m, a_{m+1}, \bar{0}\bar{0}, \text{ where } a_m = \bar{0}^m, b_m = 0^m, c_m = \bar{0}\bar{1}^m, d_m = a_m \bar{1}$ $\frac{1+x^2-x^3}{1-x}C(x) - \frac{1-3x+x^2}{(1-x)^2(1-2x)}$
188*	$\{10, 10, 01, 01, 10\}$	
189	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, c_0, b_1; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; d_m \rightsquigarrow \bar{0}\bar{1}, d_0, d_1, \dots, d_m; c_m \rightsquigarrow \bar{0}\bar{1}, d_0, d_1, \dots, d_m, c_{m+1}; a_m \rightsquigarrow \bar{0}\bar{1}, d_0, d_1, \dots, d_{m-2}, c_{m-1}, a_{m+1}; b_m \rightsquigarrow \bar{0}\bar{1}, d_0, d_1, \dots, d_{m-2}, c_{m-1}, c_m, b_{m+1}, \text{ where } a_m = 0^m, b_m = \bar{0}\bar{1}^m, c_m = 0\bar{0}a_m, d_m = c_m \bar{1}$ $\frac{1-3x+4x^2}{(1-x)(1-2x)^2}$
190*	$\{10, 10, 01, 00, 10\}$	
191	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; \bar{0} \rightsquigarrow a_1, b_1, 0; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}, b_1; b_m \rightsquigarrow \bar{0}\bar{1}, b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow \bar{0}\bar{1}, b_1, b_2, \dots, b_{m-1}, a_{m+1}, b_m, \text{ where } a_m = \bar{0}^m, b_m = 0a_m$ $(1+x)C^2(x) - \frac{x}{1-x}$
192	$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; \bar{0} \rightsquigarrow (0)^2; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, 0, b_1, b_2, \dots, b_m; b_m \rightsquigarrow 0, c_1, c_2, \dots, c_m, b_m; c_m \rightsquigarrow c_1, c_2, \dots, c_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m, c_m = b_m(m-1)$ $\frac{(1-3x+3x^2)C^2(x)}{(1-x)(1-2x)}$

193	{10, 10, 01, 00, 10}	$\epsilon \rightsquigarrow a_1, b_0; \bar{0}1 \rightsquigarrow a_1, \bar{0}1; b_m \rightsquigarrow (\bar{0})^{m+2}, b_{m+1};$ $a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_1, \bar{0}1, \text{ where } a_m = \bar{0}^m,$ $b_m = 01 \dots m$	$\frac{(1-x)C^2(x)-x}{(1-x)^2}$
194	{10, 10, 01, 00, 10}	$\epsilon \rightsquigarrow a_1, c_0; \bar{0}1 \rightsquigarrow \bar{0}1, b_1; b_m \rightsquigarrow \bar{0}1, b_1, b_2, \dots, b_{m+1};$ $c_m \rightsquigarrow \bar{0}, b_1, b_2, \dots, b_{m+1}, c_{m+1}; a_m \rightsquigarrow$ $\bar{0}1, b_1, b_2, \dots, b_{m-1}, a_{m+1}, b_m, \text{ where } a_m = \bar{0}^m,$ $b_m = 0a_m, c_m = 01 \dots m$	$\frac{C(x)+2x^2C^4(x)}{1-x}$
195*	{10, 10, 01, 01, 10}		
196	{10, 10, 01, 01, 10}	$\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1}, b_0, b_1, \dots, b_m; a_m b_m \rightsquigarrow$ $b_0, b_1, \dots, b_m, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m$	$(1+x C^3(X))C(x)$
197	{10, 10, 00, 01, 10}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, b_0, a_1; \bar{0}02 \rightsquigarrow b_0, \bar{0}02;$ $b_m \rightsquigarrow b_{m+1}, b_m, \dots, b_0, \bar{0}02; a_m \rightsquigarrow$ $b_{m-1}, a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = \bar{0}^m,$ $b_m = 00a_m$	$\frac{(1-2x)\sqrt{1-4x-4x^2+6x-1}}{2x(1-x)\sqrt{1-4x}}$
198*	{10, 10, 01, 01, 10}		
199	{10, 10, 01, 01, 10}	$\epsilon \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow b_1, (a_1)^2; \bar{0} \rightsquigarrow b_1, (a_1)^2;$ $b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow$ $b_1, b_2, \dots, b_{m+1}, (a_{m+1})^2, \text{ where } a_m = 00^m,$ $b_m = 01^m$	$\frac{1}{\sqrt{1-4x}}$
200*	{10, 10, 00, 01, 10}		
201*	{10, 10, 01, 00, 10}		
202	{10, 10, 00, 01, 10}	$\epsilon \rightsquigarrow a_1, b_1; \bar{0}1 \rightsquigarrow \bar{0}1, b_1; b_m \rightsquigarrow 01, b_1, b_2, \dots, b_{m+1};$ $a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, (b_m)^m, \text{ where } a_m = \bar{0}^m,$ $b_m = 0^m$	$\frac{1-6x+9x^2-2x^3-(x^2-4x+1)\sqrt{1-4x}}{2x^2(1-4x)}$
203*	{10, 10, 01, 00, 10}		
204*	{10, 10, 00, 01, 10}		
205*	{10, 10, 00, 01, 10}		
206*	{10, 10, 00, 01, 10}		
207	{10, 10, 00, 01, 10}	$\epsilon \rightsquigarrow a_1, b_1; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1}, (b_{m+1})^{m+1}, \text{ where } a_m = \bar{0}^m,$ $b_m = 0^m$	
	{10, 10, 00, 01, 10}	$\epsilon \rightsquigarrow a_1, 0; \bar{0} \rightsquigarrow 0, b_1; b_m \rightsquigarrow b_0, b_1, \dots, b_{m+1}; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1}, (b_{m+1})^{m+1}, \text{ where } a_m = \bar{0}^m,$ $b_m = 0a_m$	$\frac{1-4x+2x^2-(1-4x)\sqrt{1-4x}}{2x(1-4x)}$
208*	{10, 10, 01, 00, 10}		
209*	{10, 10, 00, 01, 10}		
210*	{10, 10, 01, 00, 10}		
211*	{10, 10, 00, 01, 10}		
212*	{10, 10, 00, 01, 10}		
213*	{10, 10, 01, 00, 10}		
214*	{10, 10, 01, 00, 10}		
215*	{10, 10, 00, 00, 10}		
216*	{10, 10, 00, 00, 10}		
217*	{10, 10, 01, 00, 10}		
218*	{10, 10, 00, 01, 10}		
219*	{10, 10, 00, 01, 10}		
220*	{10, 10, 01, 00, 10}		
221	{10, 10, 01, 01, 10}	$\epsilon \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow (a_1)^2, a_0; \bar{0} \rightsquigarrow (a_1)^2, a_0; a_m \rightsquigarrow$ $a_{m+1}, a_{m+1}, a_m, \dots, a_0, \text{ where } a_m = 00^m$	$\frac{1-x-\sqrt{1-6x+x^2}}{2x}$
222*	{10, 10, 01, 00, 10}		
223*	{10, 10, 00, 01, 10}		
224*	{10, 10, 00, 00, 10}		
225*	{10, 10, 00, 00, 10}		
226*	{10, 10, 01, 00, 10}		
227*	{10, 10, 01, 00, 10}		
228*	{10, 10, 01, 01, 10}		
229*	{10, 10, 01, 00, 10}		
230*	{10, 10, 00, 00, 10}		
231*	{10, 10, 00, 00, 10}		
232*	{10, 10, 00, 00, 10}		
233*	{10, 10, 00, 00, 10}		
234*	{10, 10, 00, 00, 10}		
235*	{10, 10, 00, 00, 10}		
236*	{10, 10, 00, 00, 10}		

237*	$\{\bar{1}0, \bar{1}0, 00, 00, 10\}$	
238	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{1}0, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (b_1)^2, \bar{0}\bar{1}; \bar{0}\bar{1} \rightsquigarrow c_1, \bar{0}\bar{1};$ $b_m \rightsquigarrow b_{m+1}, b_{m+1}, c_m, c_{m-1}, \dots, c_1, \bar{0}\bar{1};$ $c_m \rightsquigarrow c_{m+1}, c_m, \dots, c_1, \bar{0}\bar{1}; \quad a_m \rightsquigarrow$ $\bar{0}, b_1, b_2, \dots, b_m, a_{m+1}, a_m, \dots, a_1, \quad \text{where}$ $\underline{a_m = 0^m, b_m = 0\bar{0}^m, c_m = b_m \bar{1}}$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{1}0, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}\bar{1}, (b_1)^2; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}, c_1; \frac{1-3x}{1-4x} C(x)$ $c_m \rightsquigarrow \bar{0}\bar{1}, c_1, c_2, \dots, c_{m+1}; \quad b_m \rightsquigarrow$ $\bar{0}\bar{1}, c_1, c_2, \dots, c_m, b_{m+1}, b_{m+1}; \quad a_m \rightsquigarrow$ $\bar{0}, b_1, b_2, \dots, b_m, a_{m+1}, a_m, \dots, a_1, \quad \text{where}$ $\underline{a_m = 0^m, b_m = 0\bar{0}^m, c_m = 0\bar{0}\bar{1}^m}$
239	$\{\bar{1}0, \bar{1}0, 00, \bar{1}0, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, b_1, a_1; \quad b_m \rightsquigarrow$ $\bar{0}, b_1, b_2, \dots, b_{m+1}, a_{m+1}, a_m, \dots, a_1;$ $a_m \rightsquigarrow \bar{0}, b_1, b_2, \dots, b_m, a_{m+1}, a_m, \dots, a_1,$ $\text{where } \underline{a_m = 0^m, b_m = 0\bar{0}^m}$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{1}0, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, b_0, a_1; \quad b_m \rightsquigarrow \sum_{k \geq 0} \frac{1}{n+1} \binom{3n+1}{n} x^k$ $\bar{0}, b_0, b_1, \dots, b_{m+1}, a_{m+2}, a_{m+1}, \dots, a_1;$ $a_m \rightsquigarrow \bar{0}, b_0, b_1, \dots, b_{m-1}, a_{m+1}, a_m, \dots, a_1,$ $\text{where } \underline{a_m = 0^m, b_m = 00a_m}$

Table 6: Succession rules of $\mathcal{T}[B]$ and generating function $F_B(x)$, where $|B| = 5$.

Theorem 12. *We have*

$$\begin{aligned} \{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}1, \bar{0}0\} &\sim \{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}1, \bar{0}0\} \sim \{\bar{1}0, \bar{0}\bar{1}, \bar{0}1, \bar{0}0, \bar{1}0\}; \\ \{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0\} &\sim \{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0\}; \\ \{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{1}0\} &\sim \{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{1}0\}; \\ \{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0\} &\sim \{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0\}. \end{aligned}$$

Proof. Let $A = \{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}1, \bar{0}0\}$, $B = \{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}1, \bar{0}0\}$, and $C = \{\bar{1}0, \bar{0}\bar{1}, \bar{0}1, \bar{0}0, \bar{1}0\}$. Any signed inversion sequence that avoids A with at least one barred letter \bar{a} can be presented as $\pi^{(0)}\bar{a}\pi^{(1)} \dots \bar{a}\pi^{(s)}$ such that $\pi^{(0)}\pi^{(1)} \dots \pi^{(s)}$ has no barred letters. On the another hand, any signed inversion sequence that avoids B with at least one barred letter \bar{a} can be presented as $\pi^{(0)}\bar{a}\pi^{(1)}a\pi^{(2)} \dots a\pi^{(s)}$ such that $\pi^{(0)}\pi^{(1)} \dots \pi^{(s)}$ has no barred letters. We define α from $\bar{\mathcal{I}}_n(A)$ to $\bar{\mathcal{I}}_n(B)$ by letting $\alpha(pi)$ be the same as π , except that we change all barred letters other than the leftmost letter (if any such barred letters exist) to unbarred letters. Clearly, $\pi \in \bar{\mathcal{I}}_n(A)$ if and only if $\alpha(\pi) \in \bar{\mathcal{I}}_n(B)$.

By arguments similar to those used in defining the bijection α , we see that there is a bijection between $\bar{\mathcal{I}}_n(A)$ and $\bar{\mathcal{I}}_n(C)$ and we obtain

$$\begin{aligned} \{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0\} &\sim \{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0\}, \\ \{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{1}0\} &\sim \{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{1}0\}, \\ \{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0\} &\sim \{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0\}. \end{aligned}$$

□

Theorem 13. *We have* $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}1, \bar{1}0\} \sim \{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}1, \bar{1}0\}$.

Proof. Let $A = \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ and $B = \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$. Any signed inversion sequence π that avoid A either (1) π has no barred letters, or (2) contains a subsequence $aa \cdots a\bar{a}\bar{a} \cdots \bar{a}$ for some a , where the absolute value of each letter in π is at most a and no letter smaller than a is barred.

Also, any signed inversion sequence π that avoids B either (1) has no barred letters, or (2) contains a subsequence $aa \dots a\bar{a}aa \dots a$ for some a , where the absolute value of each letter in π is at most a and no letter smaller than a is barred.

Thus, there is a bijection α between $\tilde{\mathcal{I}}_n(A)$ and $\tilde{\mathcal{I}}_n(B)$ defined as follows. Let $\pi \in \tilde{\mathcal{I}}_n(A)$. We define $\alpha(\pi) = \pi$ if π has no barred letters, and $\alpha(\pi) = \pi'$ where π' is obtained from π after barring all the letters a on the rightmost of the letter \bar{a} in π . □

Theorem 14. *We have*

$$\begin{aligned} \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\} &\sim \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\} \sim \{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}\} \sim \{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\} \\ &\sim \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\} \sim \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\} \sim \{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}. \end{aligned}$$

Proof. Note that if $\mathcal{T}(A) = \mathcal{T}(B)$ then $A \sim B$. Thus, by Table 6 (see Class 169), we have that

$$\begin{aligned} \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\} &\sim \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\}, \\ \{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}\} &\sim \{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\}, \\ \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\} &\sim \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}. \end{aligned}$$

The succession rules of $\mathcal{T}(\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}\})$ are given by

$$\epsilon \rightsquigarrow a_0, b_1; \quad b_m \rightsquigarrow (b_{m+1})^{m+1}; \quad a_m \rightsquigarrow a_{m+1}, (b_{m+2})^{m+2},$$

where $a_m = \bar{0}\bar{1} \cdots \bar{m}$ and $b_m = 0^m$. The succession rules of $\mathcal{T}(\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\})$ are given by

$$\epsilon \rightsquigarrow c_1, d_1; \quad d_m \rightsquigarrow (d_{m+1})^{m+1}; \quad c_m \rightsquigarrow c_{m+1}, (d_{m+1})^{m+1},$$

where $c_m = \bar{0}^m$ and $d_m = 0^m$. By mapping $\mathcal{T}(\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}\})$ to $\mathcal{T}(\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\})$ with $a_i \mapsto c_{i+1}$ and $b_i \mapsto d_i$, we obtain that

$$\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}\} \sim \{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\}.$$

Very similarly, we can construct a bijection between the generating trees $\mathcal{T}(\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\})$ and $\mathcal{T}(\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\})$. To prove the last equivalence, we find the corresponding generating functions. Note that the rules $\mathcal{T}(\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\})$ can be translated to $A_\epsilon(x) = 1 + xA_{a_0}(x) + xA_{b_1}(x)$, $A_{a_m}(x) = A_{b_{m+1}}(x) + xA_{a_{m+1}}(x)$, and $A_{b_m}(x) = 1 + (m + 1)xA_{b_{m+1}}(x)$. By iterating, we see that

$$A_{b_m}(x) = \sum_{j \geq 0} \frac{(m + j)!}{m!} x^j, \quad A_{a_0}(x) = \sum_{k \geq 1} \sum_{j=1}^k \frac{k!}{j!} x^{k-1}.$$

Hence, $A_\epsilon(x) = \sum_{k \geq 0} \sum_{j=0}^k \frac{k!}{j!} x^k$.

Moreover, the rules $\mathcal{T}(\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1\})$ can be translated to $A_\epsilon(x) = 1 + xA_{\bar{0}}(x) + xA_{a_1}(x)$, $A_{\bar{0}}(x) = \frac{1}{1-x}$; $A_{b_m}(x) = 1 + x \sum_{j=0}^m A_{b_j}(x)$, and $A_{a_m}(x) = 1 + x \sum_{j=0}^m A_{b_j}(x) + (m+1)xA_{a_{m+1}}(x)$. Clearly, by induction on m , we have that $A_{b_m}(x) = \frac{1}{(1-x)^{m+1}}$. Thus, $A_{a_m}(x) = \frac{1}{(1-x)^m} + (m+1)xA_{a_{m+1}}(x)$, which (by iterating) implies that $A_{a_1}(x) = \sum_{j \geq 1} \frac{j!x^{j-1}}{(1-x)^{j+1}}$. Hence,

$$A_{a_1}(x) = \sum_{j \geq 1} \sum_{k \geq 0} j! \binom{j+k}{k} x^{j+k-1} = \sum_{k \geq 0} \sum_{j=0}^k \frac{k!}{j!} x^{k-1} - \frac{1}{x(1-x)},$$

which implies $A_\epsilon(x) = \sum_{k \geq 0} \sum_{j=0}^k \frac{k!}{j!} x^k$.

Since $F_{\{\bar{1}0, \bar{0}1, \bar{0}0, 0\bar{1}, 1\bar{0}\}}(x) = F_{\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1\}}(x)$, we complete the proof. □

Theorem 15. *We have $A = \{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{0}, 1\bar{0}\} \sim B = \{\bar{1}0, \bar{0}0, \bar{0}1, 0\bar{0}, 1\bar{0}\}$.*

Proof. By our method, we see that the generating tree $\mathcal{T}(A)$ is given by

$$\begin{aligned} \epsilon &\rightsquigarrow a_1, b_1, \\ a_m &\rightsquigarrow a_1, a_2, \dots, a_{m+1}, \\ b_m &\rightsquigarrow c_{m,1}, c_{m-1,2}, \dots, c_{1,m}, b_{m+1}, d_{m,1}, d_{m,2}, \dots, d_{m,m}, \\ c_{m,j} &\rightsquigarrow c_{m,1}, c_{m,2}, \dots, c_{m,j}, (c_{m,j+1})^{m+1}, \\ d_{m,j} &\rightsquigarrow c_{m+1,1}, c_{m,2}, \dots, c_{j+1, m+1-j}, (d_{m+1,j})^{j+1}, \\ &\quad d_{m+1,j+1}, d_{m+1,j+2}, \dots, d_{m+1, m+1}, \end{aligned}$$

where $a_m = \bar{0}^m$, $b_m = 0^m$, $c_{m,j} = a_m \bar{m}^j$, and $d_{m,j} = a_m j$. Also, the generating tree $\mathcal{T}(B)$ is given by

$$\begin{aligned} \epsilon &\rightsquigarrow e_0, f_1, \\ e_m &\rightsquigarrow e_1, e_2, \dots, e_{m+1}, \\ f_m &\rightsquigarrow g_{m,0}, g_{m-1,1}, \dots, g_{1, m-1}, f_{m+1}, h_{m,1}, h_{m,2}, \dots, h_{m,m}, \\ g_{m,j} &\rightsquigarrow g_{m,0}, g_{m,1}, \dots, g_{m,j}, (g_{m,j+1})^{m+1}, \\ h_{m,j} &\rightsquigarrow g_{m+1,0}, g_{m,1}, \dots, g_{j+1, m-j}, (h_{m+1,j})^{j+1}, \\ &\quad h_{m+1,j+1}, h_{m+1,j+2}, \dots, h_{m+1, m+1}, \end{aligned}$$

where $e_m = \bar{0}^m$, $f_m = 0^m$, $g_{m,j} = a_m \bar{m} a_j$, and $h_{m,j} = a_m j$.

By mapping $a_m \mapsto e_{m-1}$, $b_m \mapsto f_m$, $c_{m,j} \mapsto g_{m,j-1}$, and $d_{m,j} \mapsto h_{m,j}$, we obtain that there is a bijection between the generating trees $\mathcal{T}(A)$ and $\mathcal{T}(B)$. Thus, $A \sim B$. □

Case $k = 6$

	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, \bar{0}; \bar{0} \rightsquigarrow \bar{0}, \bar{0}; \bar{0} \rightsquigarrow \bar{0}$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, \bar{0}; \bar{0} \rightsquigarrow \bar{0}; \bar{0} \rightsquigarrow \bar{0}, \bar{0}$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; a_m \rightsquigarrow (\bar{0})^{m+1}, a_{m+1}, a_m = \bar{0}\bar{1} \dots m$	$\frac{1}{(1-x)^2}$
14	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_1;$ $a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, a_{m+1},$ where $a_m = 0^m, b_m = a_m \bar{m}$	$\frac{1-x^2-x^3}{(1-x)(1-x^2)}$
15	$\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, \bar{0}; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}\bar{0}$	$\frac{1-x-x^2+x^3-2x^4}{(1-x)(1-2x)}$
16	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow (\bar{0})^m, a_{m+1},$ where $a_m = 0^m$	
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; a_m \rightsquigarrow (\bar{0}\bar{1})^m, \bar{0}, a_{m+1},$ where $a_m = 0^m$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, \bar{0}; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}$	$\frac{1-x+x^3}{(1-x)^3}$
17	$\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_0, b_0; a_m \rightsquigarrow a_{m+1}, b_0, b_1, \dots, b_{m+1};$ $b_m \rightsquigarrow b_0, b_1, \dots, b_{m-1},$ where $a_m = \bar{0}\bar{1} \dots m,$ $b_m = a_{m-1} \bar{m}$	
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow b_0, a_0; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_0;$ $a_m \rightsquigarrow b_{m+1}, b_m, \dots, b_0, a_{m+1},$ where $a_m = \bar{0}\bar{1} \dots m, b_m = a_{m-1} \bar{m}$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow b_0, a_1; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_0; a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0, a_{m+1},$ where $a_m = 0^m, b_m = a_m \bar{m}$	$\frac{1+x}{1-x^2}$
18	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, b_0; b_0 \rightsquigarrow b_0; a_m \rightsquigarrow a_{m+1}, b_0, b_2, b_3, \dots, b_m;$ $b_m \rightsquigarrow b_m, b_2, b_3, \dots, b_m;$ $b_m, b_0, b_2, b_3, \dots, b_{m-1},$ where $a_m = \bar{0}^m, b_m = a_m \bar{m}$	
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$		
	$\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$		
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, \bar{0}; \bar{0} \rightsquigarrow \bar{0}, \bar{0}$	
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow b_0, a_1; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0; a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0, a_{m+1},$ where $a_m = 0^m, b_m = a_m \bar{m}$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, \bar{0}; \bar{0} \rightsquigarrow \bar{0}, \bar{0}$	$\frac{1-x-x^2}{(1-x)(1-2x)}$
19	$\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; a_1 \rightsquigarrow \bar{0}, a_2; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow (b_m)^m; a_m \rightsquigarrow (b_m)^m, a_{m+1},$ where $a_m = 0^m, b_m = a_m \bar{m}$	$\frac{1}{1-x} + \sum_{k \geq 1} \frac{x^k}{1-x^k}$
20	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, \bar{0}; a_m \rightsquigarrow a_{m+1}, \bar{0}, b_1, b_2, \dots, b_m;$ $b_m \rightsquigarrow b_m, \bar{0}, b_1, b_2, \dots, b_{m-1},$ where $a_m = 0^m, b_m = a_m \bar{m}$	$1 + 2x + \frac{3x^2}{1-2x}$
21	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; \bar{0}\bar{1} \rightsquigarrow (\bar{0}\bar{1})^2; a_m \rightsquigarrow (\bar{0}\bar{1})^m, a_{m+1},$ where $a_m = 0^m$	$\frac{1-2x+2x^3}{(1-x)^2(1-2x)}$
22	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, \bar{0}; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow a_{m+1}, b_1, b_2, \dots, b_m;$ $b_m \rightsquigarrow b_m, b_1, b_2, \dots, b_m,$ where $a_m = \bar{0}^m, b_m = a_m \bar{m}$	$\frac{x(1-2x-x^2+x^3)}{(1-x)(1-3x+x^2)}$
23	$\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, \bar{0}; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1},$ where $a_m = \bar{0}^m$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m$	$\frac{x}{1-x} + C(x)$
24	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, \bar{0}; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow a_{m+1}, (a_m)^m,$ where $a_m = \bar{0}^m$	$\frac{x}{1-x} + \sum_{k \geq 0} \frac{x^k}{\prod_{j=1}^k (1-jx)}$
25	$\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, \bar{0}; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1},$ where $a_m = \bar{0}^m$	
	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$	$\epsilon \rightsquigarrow a_1, \bar{0}; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \bar{0},$ where $a_m = \bar{0}^m$	$(1+x)C(x)$
26	$\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}\}$		
	$\{\bar{1}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$		

	$\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow (\bar{0})^m, 0\bar{0}, a_{m+1},$ where $a_m = 0^m$	
	$\{10, 01, 00, 00, 01, 10\}$	$\epsilon \rightsquigarrow 0, a_1; 0 \rightsquigarrow (01)^2; 01 \rightsquigarrow 01; a_m \rightsquigarrow$ $(01)^m, a_{m+1},$ where $a_m = 0^m$	$\frac{1-x+x^2-x^3+x^4}{(1-x)^3}$
43	$\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, 01, 10\}$	$\epsilon \rightsquigarrow 0, a_1; \bar{0} \rightsquigarrow 00, \bar{0}; \bar{0}0 \rightsquigarrow \bar{0}0;$ $b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_1; a_m \rightsquigarrow$ $b_m, b_{m-1}, \dots, b_1, a_{m+1},$ where $a_m = 0^m,$ $b_m = a_m \bar{m}$	$\frac{1-x-x^3}{(1-x)^2(1-x-x^2)}$
44	$\{10, 01, 00, 00, 01, 10\}$	$\epsilon \rightsquigarrow 0, a_1; 0 \rightsquigarrow 0; b_m \rightsquigarrow b_{m-1}, b_{m-2}, \dots, b_1, 00;$ $a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, 00, a_{m+1},$ where $a_m = 0^m, b_m = a_m \bar{m}$	
	$\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow$ $a_{m+1}, 00, b_1, b_2, \dots, b_m; b_m \rightsquigarrow$ $00, b_1, b_2, \dots, b_{m-1},$ where $a_m = \bar{0}^m,$ $b_m = a_m m$	$\frac{1-x^3-x^4}{(1-x)(1-x-x^2)}$
45	$\{\bar{1}0, \bar{0}0, \bar{0}1, \bar{0}0, 01, 10\}$	$\epsilon \rightsquigarrow a_0, 0; 0 \rightsquigarrow 00; a_0 \rightsquigarrow a_1, 00, \bar{0}1; \bar{0}1 \rightsquigarrow 0, 00;$ $a_m \rightsquigarrow a_{m+1}, 00, 0, b_2, b_3, \dots, b_{m+1};$ $b_m \rightsquigarrow 00, 0, b_2, b_3, \dots, b_{m-1}; c_m \rightsquigarrow$ $b_m, 00, 0, b_2, b_3, \dots, b_{m-1},$ where $a_m =$ $\bar{0}1 \dots \bar{m}, b_m = a_m m, c_m = a_{m-1} m$	$\frac{1+x+x^2}{1-x-x^2}$
46	$\{\bar{1}0, 10, \bar{0}1, \bar{0}0, 00, 01\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (0\bar{1})^2; 0\bar{1} \rightsquigarrow 0\bar{1};$ $a_m \rightsquigarrow a_{m+1}, 0\bar{1}, b_2, b_3, \dots, b_m; b_m \rightsquigarrow$ $b_m, 0\bar{1}, b_2, b_3, \dots, b_{m-1},$ where $a_m = \bar{0}^m,$ $b_m = a_m m$	$\frac{1-x-2x^3}{(1-x)(1-2x)}$
47	$\{10, 01, 00, 01, 00, 10\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, 00, 10\}$ $\{\bar{1}0, 10, \bar{0}0, \bar{0}0, 00, 01\}$ $\{\bar{1}0, \bar{0}0, \bar{0}1, 00, 01, 10\}$ $\{\bar{1}0, \bar{0}0, \bar{0}0, 00, 01, 10\}$ $\{\bar{1}0, 10, \bar{0}1, \bar{0}1, 00, 01\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, \bar{0}1; \bar{0}1 \rightsquigarrow \bar{0}1$	
	$\{\bar{1}0, \bar{0}0, \bar{0}1, 00, 01, 10\}$ $\{\bar{1}0, 10, \bar{0}1, \bar{0}1, 00, 01\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{1}, 0; \bar{0} \rightsquigarrow \bar{0}, 0; 0\bar{1} \rightsquigarrow 0\bar{1}$	
	$\{\bar{1}0, 10, \bar{0}1, \bar{0}0, 01, 01\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, 01, 10\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow 0\bar{1}, \bar{0}; 0\bar{1} \rightsquigarrow 0\bar{1}; a_m \rightsquigarrow$ $(0\bar{1})^m, a_{m+1},$ where $a_m = 0^m$	
	$\{\bar{1}0, 10, \bar{0}1, \bar{0}0, 01, 01\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, 01, 10\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow (\bar{0})^{m+1}, a_{m+1},$ where $a_m = 0^m$	
	$\{\bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, 00, 10\}$ $\{\bar{1}0, 01, 00, 01, 00, 10\}$ $\{\bar{1}0, 10, \bar{0}1, \bar{0}0, \bar{0}1, 00\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 00, 0; \bar{0} \rightsquigarrow \bar{0}; 00 \rightsquigarrow \bar{0}, 00$	
	$\{\bar{1}0, 10, \bar{0}0, \bar{0}1, 00, 00\}$ $\{\bar{1}0, 01, 00, 01, 00, 10\}$ $\{\bar{1}0, 10, \bar{0}0, \bar{0}1, 00, 01\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}; 012345 \rightsquigarrow (\bar{0})^{m+2}, a_{m+2},$ where $a_m = 01 \dots m$	
	$\{\bar{1}0, 10, \bar{0}0, \bar{0}1, 00, 00\}$ $\{\bar{1}0, 01, 00, 01, 00, 10\}$ $\{\bar{1}0, 10, \bar{0}0, \bar{0}1, 00, 01\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow 0\bar{1}, 0; 0\bar{1} \rightsquigarrow 0\bar{1}$ $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0\bar{1}, \bar{0}, 0; \bar{0} \rightsquigarrow 0; 0\bar{1} \rightsquigarrow 0\bar{1}, 0$	
	$\{\bar{1}0, 10, \bar{0}0, \bar{0}1, 00, 01\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0; \bar{0} \rightsquigarrow \bar{0}, 0, \bar{0}1; \bar{0}1 \rightsquigarrow 0, \bar{0}1$	$\frac{1-x+x^2}{(1-x)^3}$
48	$\{10, 01, 00, 01, 01, 10\}$	$\epsilon \rightsquigarrow 0, a_1; 0 \rightsquigarrow 0; b_m \rightsquigarrow$ $b_{m-1}, b_{m-2}, \dots, b_1, 010; a_m \rightsquigarrow$ $b_m, b_{m-1}, \dots, b_1, \bar{0}, a_{m+1},$ where $a_m = 0^m,$ $b_m = a_m \bar{m}$	
	$\{\bar{1}0, \bar{0}1, \bar{0}1, 00, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; \bar{0}1 \rightsquigarrow \bar{0}10;$ $a_m \rightsquigarrow a_{m+1}, 0, b_1, b_2, \dots, b_m; b_m \rightsquigarrow$ $\bar{0}10, b_1, b_2, \dots, b_{m-1},$ where $a_m = \bar{0}^m,$ $b_m = a_m m$	$\frac{1-x-x^4}{(1-x)^2(1-x-x^2)}$
49	$\{10, 00, 00, 01, 00, 10\}$ $\{\bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, 00, 10\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, 00, 10\}$ $\{10, 01, 00, 01, 01, 10\}$ $\{10, 01, 00, 01, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 00, 01; \bar{0} \rightsquigarrow \bar{0}, 0; 00 \rightsquigarrow 00, 00; 01 \rightsquigarrow 01$ $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 00, 0; \bar{0} \rightsquigarrow 0; 00 \rightsquigarrow 00, 00$	
	$\{10, 01, 00, 01, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 00, 0; a_1 \rightsquigarrow a_2, 00; 00 \rightsquigarrow$ $00; a_m \rightsquigarrow a_{m+1}, 00, 0, b_3, b_4, \dots, b_m; b_m \rightsquigarrow$ $00, 0, b_3, b_4, \dots, b_m,$ where $a_m = \bar{0}^m, b_m =$ $a_m m$	$\frac{1-2x+x^2-x^3}{(1-x)^2(1-2x)}$
50	$\{10, 01, 00, 01, 00, 10\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 01; \bar{0} \rightsquigarrow (0\bar{1})^2; 01 \rightsquigarrow 01$	
	$\{\bar{1}0, 10, \bar{0}1, \bar{0}0, 00, 01\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow$ $a_{m+1}, 00, b_1, b_2, \dots, b_m; b_m \rightsquigarrow$ $b_m, 00, b_1, b_2, \dots, b_{m-1},$ where $a_m = \bar{0}^m,$ $b_m = a_m m$	$\frac{1-x-x^3}{(1-x)(1-2x)}$
51	$\{10, 10, 01, 01, 00, 00\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1};$ $a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, 00,$ where $a_m =$ $0^m, b_m = 01^m$	

	$\{\bar{1}0, \bar{0}0, \bar{0}1, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (0\bar{0})^2; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow x + \frac{2x^2}{1-x} + C(x)$ $a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = \bar{0}^m$
52	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}0, \bar{0}1\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0; \bar{0} \rightsquigarrow \bar{0}, \bar{0}0, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow 0, \bar{0}0$ $\frac{1-x+x^2+x^3}{(1-x)^3}$
53	$\{\bar{1}0, \bar{0}0, \bar{0}1, \bar{0}0, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, 0; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}\bar{0}\bar{2}; \bar{0}\bar{0}\bar{2} \rightsquigarrow \bar{0}\bar{0}\bar{2}$ $\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}; a_m \rightsquigarrow \frac{1-2x+2x^2}{(1-x)^4}$ $\bar{0}, (\bar{0}\bar{0})^{m+1}, a_{m+1}, \text{ where } a_m = \bar{0}1 \dots m$
54	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}\bar{1}, \bar{0}\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{0}, \bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}\bar{0}$ $\frac{1-2x+x^2-x^4}{(1-x)^2(1-2x)}$
55	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}0, \bar{0}1, \bar{0}0\}$ $\{\bar{1}0, \bar{0}\bar{0}, \bar{0}0, \bar{0}0, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}1, \bar{0}0\}$ $\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{0}, \bar{0}0\}$ $\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}0\}$ $\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}1\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}1, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}1, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}1, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0, 0; \bar{0} \rightsquigarrow 0, 0$ $\epsilon \rightsquigarrow \bar{0}, \bar{0}; 0 \rightsquigarrow \bar{0}\bar{1}, \bar{0}\bar{1}, 0; \bar{0} \rightsquigarrow \bar{0}; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}, \bar{0}\bar{1}$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0$ $a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0, a_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{m}$ $\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow a_{m+1}, (\bar{0}\bar{1})^m, \text{ where } a_m = \bar{0}^m$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}\bar{1}, \bar{0}\bar{0}; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; b_m \rightsquigarrow b_1, b_2, \dots, b_m; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{1}$ $\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}; \bar{0}\bar{0} \rightsquigarrow (\bar{0}\bar{0})^2; a_m \rightsquigarrow \bar{0}, (\bar{0}\bar{0})^{m+1}, a_{m+1}, \text{ where } a_m = \bar{0}1 \dots m$ $\epsilon \rightsquigarrow a_0, b_0; a_m \rightsquigarrow a_{m+1}, b_0, b_1, \dots, b_{m+1}; b_m \rightsquigarrow b_0, b_1, \dots, b_m, \text{ where } a_m = \bar{0}\bar{1} \dots m, b_m = a_m - 1^m$ $\epsilon \rightsquigarrow c_1, a_1; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1; a_m \rightsquigarrow b_m, b_{m-1}, \dots, b_1, a_{m+1}; c_m \rightsquigarrow c_{m+1}, d_1, d_2, \dots, d_m; d_m \rightsquigarrow d_1, d_2, \dots, d_m, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{m}, c_m = \bar{0}^m, d_m = c_m \bar{m}$ $\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow a_{m+1}, b_0, b_1, \dots, b_m; b_m \rightsquigarrow b_m, b_0, b_1, \dots, b_{m-1}, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{m}$ $\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow \bar{0}, b_0, b_1, \dots, b_m; a_m \rightsquigarrow \bar{0}, b_0, b_1, \dots, b_m, a_{m+1}, \text{ where } a_m = \bar{0}1 \dots m, b_m = a_m \bar{0}$ $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0; \bar{0} \rightsquigarrow \bar{0}, (\bar{0}\bar{0})^2; \bar{0}\bar{0} \rightsquigarrow (\bar{0}\bar{0})^2$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; \bar{0}\bar{1} \rightsquigarrow (\bar{0}\bar{1})^2; a_m \rightsquigarrow (\bar{0}\bar{1})^m, \bar{0}, a_{m+1}, \text{ where } a_m = \bar{0}^m$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}; \bar{0}\bar{1} \rightsquigarrow (\bar{0}\bar{1})^2; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow (\bar{0}\bar{1})^m, a_{m+1}, \text{ where } a_m = \bar{0}^m$ $\epsilon \rightsquigarrow b_0, a_1; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow b_0, b_1, \dots, b_m; a_m \rightsquigarrow b_0, b_1, \dots, b_m, a_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{0}$ $\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow a_{m+1}, b_0, b_1, \dots, b_m; b_m \rightsquigarrow b_0, b_1, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m$
56	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}1\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow a_{m+1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow b_m, b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m$ $\frac{1-3x+2x^2-x^3}{(1-x)^2(1-3x+x^2)}$
57	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_{m+1}, (b_m)^m; b_m \rightsquigarrow \frac{1}{1-x} + \frac{1}{1-x} \sum_{j \geq 1} \frac{x^j}{1-jx}$ $(b_m)^m, \text{ where } a_m = \bar{0}^m, b_m = a_m 1$
58	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow a_1, b_0; a_m \rightsquigarrow a_{m+1}, (b_m)^{m+1}; 1 + \sum_{k \geq 1} \frac{x^k}{1-kx}$ $b_m \rightsquigarrow (b_m)^{m+1}, \text{ where } a_m = \bar{0}\bar{0}^m, b_m = a_m 1$

59	$\{10, 00, 01, 01, 01, 10\}$ $\{10, 10, 00, 01, 01, 01\}$ $\{10, 10, 00, 01, 01, 00\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0\bar{0}, 0; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = \bar{0}^m$
	$\{10, 10, 01, 01, 00, 01\}$ $\{10, 10, 01, 01, 00, 01\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, 0, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}\bar{1}^m$
	$\{10, 00, 01, 01, 00, 10\}$ $\{10, 01, 01, 01, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, 0\bar{0}; 0\bar{0} \rightsquigarrow 0\bar{0}; a_m \rightsquigarrow \frac{x}{(1-x)^2} + C(x)$ $a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = 0^m$
60	$\{10, 00, 00, 01, 01, 10\}$	$\epsilon \rightsquigarrow 0, a_1; 0 \rightsquigarrow 0; a_m \rightsquigarrow 0\bar{0}, a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = 0^m$
	$\{10, 01, 01, 00, 00, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \bar{0}\bar{0}, \text{ where } a_m = \bar{0}^m$ $\frac{x}{1-x} + (1+x^2C(x))C(x)$
61	$\{10, 00, 00, 01, 01, 10\}$ $\{10, 00, 00, 01, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow 0\bar{0}, a_{m+1}, (b_m)^m; b_m \rightsquigarrow (b_{m+1})^{m+2}, \text{ where } a_m = 0^m, b_m = a_m 1$
	$\{10, 00, 01, 01, 00, 10\}$ $\{10, 10, 00, 01, 01, 00\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0}\bar{1})^2; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; a_m \rightsquigarrow x + \frac{2x^2}{1-x} + \sum_{k \geq 0} k!x^k$ $(a_{m+1})^{m+1}, \text{ where } a_m = 0^m$
62	$\{10, 01, 00, 00, 01, 10\}$	$\epsilon \rightsquigarrow a_1, c_1; 0\bar{1} \rightsquigarrow (0\bar{1})^2; c_m \rightsquigarrow (0\bar{1})^m, c_{m+1}; a_m \rightsquigarrow a_{m+1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m, c_m = 0^m$ $\frac{1-2x+x^2+x^3}{(1-x)^2(1-2x)}$
63	$\{10, 10, 01, 00, 00, 01\}$	$\epsilon \rightsquigarrow a_1, c_1; 0\bar{1} \rightsquigarrow 0\bar{1}; c_m \rightsquigarrow (0\bar{1})^m, c_{m+1}; a_m \rightsquigarrow a_{m+1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow b_m, b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m, c_m = 0^m$ $\frac{1-4x+5x^2-2x^3-2x^4+x^5}{(1-x)^3(1-3x+x^2)}$
64*	$\{10, 00, 00, 00, 01, 10\}$	
65	$\{10, 10, 01, 01, 00, 01\}$	$\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow a_{m+1}, b_0, b_1, \dots, b_m; b_m \rightsquigarrow b_m, b_0, b_1, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = \frac{1-2x}{(1-x)(1-3x+x^2)}$ $\frac{1-2x}{(1-x)(1-3x+x^2)}$
66	$\{10, 00, 00, 01, 01, 10\}$	$\epsilon \rightsquigarrow 0, a_1; 0 \rightsquigarrow 0; b_m \rightsquigarrow b_0, b_1, \dots, b_{m+1}; c_m \rightsquigarrow c_0, c_1, \dots, c_m; a_m \rightsquigarrow b_0, b_1, \dots, b_{m-1}, c_m, a_{m+1}, \text{ where } a_m = 0^m, b_m = 0\bar{1}a_m, c_m = a_m \bar{0}$
	$\{10, 01, 01, 00, 00, 10\}$	$\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow a_{m+1}, b_m, c_m, c_{m-1}, \dots, c_1; b_m \rightsquigarrow b_m, b_{m-1}, \dots, b_0; c_m \rightsquigarrow c_{m+1}, c_m, \dots, c_1, \text{ where } a_m = 0^m, b_m = a_m \bar{0}, c_m = a_m 1$ $\frac{x}{1-2x} + C(x)$
67	$\{10, 01, 00, 01, 01, 10\}$	$\epsilon \rightsquigarrow 0, a_1; 0 \rightsquigarrow 0; a_m \rightsquigarrow 0, a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \rightsquigarrow 01\bar{0}, b_{m+1}, b_m, \dots, b_1, \text{ where } a_m = 0^m, b_m = a_m 1$
	$\{10, 01, 01, 00, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}, \bar{0}\bar{1}\bar{0}; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, 0, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}\bar{1}^m$ $\frac{x^2}{(1-x)^2} + (1+x)C(x)$
68	$\{10, 01, 01, 00, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, 0, \text{ where } a_m = \bar{0}^m$
	$\{10, 00, 00, 00, 01, 10\}$	$\epsilon \rightsquigarrow 0, a_1; \bar{0} \rightsquigarrow 0, a_1; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = 0^m$
	$\{10, 00, 00, 01, 00, 10\}$	$\epsilon \rightsquigarrow 0, a_1; 0 \rightsquigarrow 0, a_1; a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = 0^m$
	$\{10, 00, 01, 00, 00, 10\}$ $\{10, 10, 00, 01, 00, 00\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = \bar{0}^m$
	$\{10, 01, 00, 01, 01, 10\}$	$\epsilon \rightsquigarrow 0, a_1; \bar{0} \rightsquigarrow 0; a_m \rightsquigarrow 0, a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = 0^m$ $\frac{C(x)}{1-x}$
69	$\{10, 00, 01, 00, 00, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_{m+1}, (b_m)^{m+2}; b_m \rightsquigarrow (b_m)^{m+2}, \text{ where } a_m = \bar{0}\bar{1} \dots \bar{m}, b_m = a_m \bar{0}$
	$\{10, 01, 01, 00, 00, 10\}$ $\{10, 01, 01, 00, 00, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; b_m \rightsquigarrow (b_m)^{m+1}; a_m \rightsquigarrow \sum_{k \geq 0} \frac{x^k}{1-(k+1)x}$ $a_{m+1}, (b_m)^{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{0}$
70	$\{10, 10, 01, 00, 00, 00\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_{m+1}, (a_m)^m, \text{ where } a_m = \bar{0}^m$ $\frac{1}{1-x} \sum_{k \geq 0} \frac{x^j}{i^k (1-ix)}$
71	$\{10, 01, 00, 01, 01, 10\}$	

	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow \bar{0}, a_{m+1}, (b_m)^m; b_m \rightsquigarrow (b_{m+1})^{m+2}, \text{ where } a_m = 0^m, b_m = a_m 1$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0\}$ $\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, \bar{0}0; \bar{0}0 \rightsquigarrow \bar{0}0; a_m \rightsquigarrow \frac{x}{(1-x)^2} + \sum_{k \geq 0} k!x^k$ $(a_{m+1})^{m+1}, \text{ where } a_m = 0^m$	
72	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow \bar{0}0, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m$	$1 + 2x + \sum_{k \geq 1} (1 + (k+1)(k-1)!)x^k$
73	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{1} \rightsquigarrow (0\bar{1})^2; b_m \rightsquigarrow (0\bar{1})^m, b_{m+1}; a_m \rightsquigarrow a_{m+1}, (0\bar{1})^m, \text{ where } a_m = \bar{0}^m, b_m = 0^m$	$\frac{1-2x+x^2+2x^3}{(1-x)^2(1-2x)}$
74	$\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}\}$ $\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{1}0\}$ $\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, b_1; b_m \rightsquigarrow b_{m+1}, b_m, \dots, b_1; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = 0^m$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow a_1, a_2; a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = 0^m$ $\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow a_1, a_0; a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_0, \text{ where } a_m = \bar{0}0^m$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow a_1, a_2; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = 0^m$ $\epsilon \rightsquigarrow a_1, b_0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, b_m, \text{ where } a_m = \bar{0}^m, b_m = 0\bar{1}^m$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow a_0, a_1; a_m \rightsquigarrow a_0, a_1, \dots, a_{m+1}, 2C(x) - 1$ $\text{where } a_m = \bar{0}0^m$
75*	$\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{1}0\}$		
76*	$\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{1}0\}$		
77	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}0\}$ $\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0, \bar{0}0\}$	$\epsilon \rightsquigarrow a_1, a_0; \bar{0} \rightsquigarrow a_1, a_0; a_m \rightsquigarrow a_{m+1}, (a_m)^{m+1}, \text{ where } a_m = \bar{0}0^m$ $\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow a_{m+1}, (b_m), (a_m)^m; b_m \rightsquigarrow (b_m)^{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m 0$	$1 + 2 \sum_{k \geq 1} \frac{x^k}{i!k-1(1-ix)}$
78*	$\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{1}0\}$		
79	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}0, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0\}$ $\{\bar{1}0, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow (a_{m+1})^{m+1}, \text{ where } a_m = 0^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}\}$ $\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow \bar{0}, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m$	$\frac{1}{1-x} \sum_{k \geq 0} k!x^k$
80*	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$		
81	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, b_0; b_0 \rightsquigarrow b_0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow b_0, b_1, \dots, b_m, b_1, b_2, \dots, b_{m-1}, \text{ where } a_m = \bar{0}^m, b_m = a_m m$	$\frac{1-4x^2-x^3-x(3-4x-x^2)C(x)}{(1-4x-x^2)(1-x)}$
82	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{1}0\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow c_1; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, c_m; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}, c_{m+1}; c_m \rightsquigarrow c_1, c_2, \dots, c_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}\bar{1}^m, c_m = 0\bar{1}^m$	
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow b_0; b_m \rightsquigarrow b_{m+1}, b_m, \dots, b_0; a_m \rightsquigarrow b_{m-1}, a_{m+1}, c_m, c_{m-1}, \dots, c_1; c_m \rightsquigarrow b_m, c_{m+1}, c_m, \dots, c_1, \text{ where } a_m = 0^m, b_m = 0\bar{0}\bar{1}^m, c_m = a_m 1$	$\frac{x(3x-5)C(x)+5x-1}{x\sqrt{1-4x}}$
83*	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$		
84	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow b_1, b_2, \dots, b_m, b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m$	$\frac{1-3x-4x^2-(x-1)\sqrt{1-4x}}{2(1-4x)(1-x)}$
85*	$\{\bar{1}0, \bar{0}0, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{1}0\}$		
86	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{1}0\}$ $\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{1}0\}$	$\epsilon \rightsquigarrow b_1, a_1; a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_1; b_m \rightsquigarrow b_{m+1}, b_{m+1}, b_m, \dots, b_2, \text{ where } a_m = 0^m, b_m = \bar{0}^m$	$C(x) + \frac{1-x-\sqrt{1-6x+x^2}}{2}$

87*	{10, 00, 00, 01, 00, 10}	
88*	{10, 10, 00, 01, 00, 00}	
89	{10, 00, 01, 01, 00, 10}	
	{10, 10, 00, 01, 01, 00}	$\epsilon \sim b_1, a_1; a_m \sim (a_{m+1})^{m+1}; b_m \sim C(x) + \sum_{k \geq 1} k!x^k$ $b_1, b_2, \dots, b_{m+1}, \text{ where } a_m = 0^m, b_m = 0^m$
90	{10, 01, 00, 01, 00, 10}	
	{10, 01, 00, 01, 00, 10}	$\epsilon \sim b_1, a_1; a_m \sim (a_{m+1})^{m+1}; b_m \sim b_{m+1}, (a_m)^m, \text{ where } a_m = 0^m, b_m = 0^m$
	{10, 01, 00, 00, 01, 10}	
	{10, 01, 00, 00, 01, 10}	$\epsilon \sim \bar{0}, a_1; a_1 \sim 0\bar{0}, a_2, b_1; \bar{0} \sim 0\bar{0}; 0\bar{0} \sim \sum_{k \geq 0} k!x^k + x \sum_{k \geq 0} \sum_{j=0}^k \frac{k!}{j!} x^k$ $(b_1)^2; a_m \sim b_{m-1}, a_{m+1}, (b_m)^m; b_m \sim (b_{m+1})^{m+2}, \text{ where } a_m = 0^m, b_m = a_m 1$
91*	{10, 01, 00, 01, 00, 10}	
92*	{10, 01, 00, 01, 00, 10}	
93	{10, 10, 01, 00, 01, 00}	
	{10, 10, 01, 00, 01, 00}	Theorem 16
94*	{10, 00, 00, 01, 00, 10}	
95*	{10, 10, 00, 00, 01, 00}	
96*	{10, 10, 01, 00, 00, 01}	
97*	{10, 01, 00, 00, 01, 10}	
98*	{10, 01, 00, 00, 01, 10}	
99*	{10, 10, 00, 00, 00, 01}	
100*	{10, 01, 00, 00, 00, 10}	
101*	{10, 01, 00, 00, 01, 10}	
102*	{10, 01, 00, 00, 00, 10}	
103	{10, 10, 01, 00, 01, 00}	
	{10, 00, 01, 01, 00, 10}	
	{10, 10, 01, 00, 01, 00}	
	{10, 01, 00, 01, 00, 10}	
	{10, 01, 00, 01, 00, 10}	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim (a_2)^2; a_m \sim (a_{m+1})^{m+1}, \text{ where } a_m = 0^m$
	{10, 10, 01, 00, 00, 01}	$\epsilon \sim \bar{0}, a_1; a_m \sim (\bar{0})^{m+1}, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m$
	{10, 10, 01, 00, 00, 01}	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim b_0; b_m \sim (b_{m+1})^{m+2}; a_m \sim b_{m-1}, (a_{m+1})^{m+1}, \text{ where } a_m = 0^m, b_m = 0\bar{0}1^m$
	{10, 01, 00, 01, 00, 10}	
	{10, 01, 00, 01, 00, 10}	
	{10, 01, 00, 00, 01, 10}	
		$\epsilon \sim \bar{0}, a_1; a_m \sim (a_{m+1})^{m+1}; b_m \sim b_m, \dots, b_0, a_{m+1}, c_{m,1}, \dots, c_{m,m}; b_m \sim (b_m)^m; c_{m,j} \sim b_{m+1}, \dots, b_j, (c_{m+1,j})^j, c_{m+1,j}, c_{m+1,j+1}, \dots, c_{m+1,m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{m}, c_{m,j} = a_m j$
	{10, 10, 01, 00, 00, 00}	Theorem 17, $1 + 2 \sum_{k \geq 1} k!x^k$
104*	{10, 00, 00, 01, 00, 10}	
105*	{10, 01, 00, 00, 00, 10}	
106	{10, 10, 01, 00, 01, 10}	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim (0\bar{0})^2; 0\bar{0} \sim 0\bar{0}; a_m \sim \frac{1+2x^2-x^3}{(1-x)^2}$ $(0\bar{1})^m, 0\bar{0}, a_{m+1}, \text{ where } a_m = 0^m$
107	{10, 10, 01, 00, 01, 10}	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim 0\bar{0}, 0; 0\bar{0} \sim 0\bar{0}; a_m \sim \frac{1+2x^2}{(1-x)^2}$ $(0\bar{1})^m, 0\bar{0}, a_{m+1}, \text{ where } a_m = 0^m$
108	{10, 10, 01, 00, 01, 10}	$\epsilon \sim a_1, 0; 0 \sim (0\bar{1})^2; 0\bar{1} \sim 0\bar{1}; a_m \sim \frac{1-x+2x^2-2x^3+x^4}{(1-x)^3}$ $a_{m+1}, (0\bar{1})^{m+1}, \text{ where } a_m = 0^m$
109	{10, 10, 01, 00, 01, 10}	$\epsilon \sim a_1, 0; 0 \sim (0\bar{1})^2; 0\bar{1} \sim 0\bar{1}; 0\bar{1} \sim \frac{1-x+x^2-2x^3-x^4+x^5}{(1-x)^2(1-x-x^2)}$ $0\bar{1}0; a_m \sim a_{m+1}, 0\bar{1}, b_1, b_2, \dots, b_m; b_m \sim 0\bar{1}0, b_1, b_2, \dots, b_{m-1}, \text{ where } a_m = 0^m, b_m = a_m \bar{m}$
110	{10, 10, 01, 01, 00, 10}	$\epsilon \sim 0, 0; 0 \sim 0, 0\bar{0}, 0; 0 \sim 0, 0\bar{0}; 0\bar{0} \sim 0\bar{0}$
	{10, 10, 00, 01, 01, 10}	$\epsilon \sim 0, 0; 0 \sim 0\bar{0}, 0; \bar{0} \sim 0, 0\bar{0}, 0; 0\bar{0} \sim 0\bar{0}$ $\frac{1-x+2x^2-x^3}{(1-x)^3}$
111	{10, 10, 01, 01, 00, 10}	$\epsilon \sim a_1, 0; 0 \sim (0\bar{0})^2; 0\bar{0} \sim 0\bar{0}; \frac{x(1+x-x^2)}{(1-x)^2} + C(x)$ $b_m \sim b_1, b_2, \dots, b_{m+1}; a_m \sim b_1, b_2, \dots, b_m, a_{m+1}, 0\bar{0}, \text{ where } a_m = 0^m, b_m = 0\bar{1}^m$
112	{10, 10, 01, 01, 00, 10}	$\epsilon \sim 0, a_0; \bar{0} \sim 0, 0\bar{0}; 0\bar{0} \sim 0\bar{0}; a_m \sim \bar{0}, (0\bar{0})^{m+1}, a_{m+1}, a_m = 01 \dots m$

	$\{\bar{1}0, 10, \bar{0}0, 00, 01, 10\}$	$\bar{\epsilon} \sim \bar{0}, 0; 0 \sim \bar{0}\bar{1}, 0; \bar{0} \sim \bar{0}, 0, 0; 0\bar{1} \sim \bar{0}\bar{1}$	
	$\{\bar{1}0, 10, \bar{0}0, 0\bar{1}, 01, 10\}$	$\bar{\epsilon} \sim \bar{0}, 0; 0 \sim \bar{0}\bar{0}, 0; \bar{0} \sim \bar{0}, 0\bar{0}, 0\bar{1}; 0\bar{0} \sim 0\bar{0}; 0\bar{1} \sim 0\bar{0}, 0\bar{0}, 0\bar{1}$	
	$\{\bar{1}0, 10, \bar{0}0, \bar{0}1, 00, 10\}$	$\bar{\epsilon} \sim \bar{0}, 0; 0 \sim \bar{0}, \bar{0}, 0; \bar{0} \sim \bar{0}\bar{1}, \bar{0}; \bar{0}\bar{1} \sim \bar{0}\bar{1}$	$\frac{1-x+2x^2}{(1-x)^3}$
113	$\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}0, 01, 10\}$	$\bar{\epsilon} \sim a_1, c_1; 0\bar{1} \sim 0\bar{1}; c_m \sim 0\bar{1}, (0\bar{1})^m, c_{m+1}; a_m \sim a_{m+1}, 0\bar{1}, b_2, b_3, \dots, b_m; b_m \sim 0\bar{1}, b_2, b_3, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m, c_m = 0^m$	$\frac{1-3x+4x^2-4x^3+x^4}{(1-x)^3(1-2x)}$
114	$\{\bar{1}0, 10, \bar{0}1, 0\bar{1}, 00, 10\}$	$\bar{\epsilon} \sim a_1, 0; 0 \sim 0\bar{0}, 0; 0\bar{0} \sim 0\bar{0}; b_m \sim b_1, b_2, \dots, b_{m+1}; a_m \sim b_1, b_2, \dots, b_m, a_{m+1}, 0\bar{0}, \text{ where } a_m = \bar{0}^m, b_m = 0\bar{1}^m$	$\frac{x(1-x^2)}{(1-x)^3} + C(x)$
115	$\{\bar{1}0, 10, \bar{0}0, \bar{0}1, 0\bar{1}, 10\}$	$\bar{\epsilon} \sim \bar{0}, a_1; \bar{0} \sim (0\bar{0})^2; 0\bar{0} \sim 0\bar{0}; a_m \sim 0\bar{0}, a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \sim 0\bar{1}\bar{0}, b_{m+1}, b_m, \dots, b_1, \text{ where } a_m = 0^m, b_m = a_m 1$	$\frac{x^2(2-x)}{(1-x)^2} + (1+x)C(x)$
116	$\{\bar{1}0, 10, \bar{0}0, \bar{0}1, 0\bar{1}, 10\}$	$\bar{\epsilon} \sim \bar{0}, a_1; \bar{0} \sim (0\bar{0})^2; 0\bar{0} \sim 0\bar{0}; a_m \sim 0\bar{0}, a_{m+1}, (b_m)^m; b_m \sim (b_{m+1})^{m+2}, \text{ where } a_m = 0^m, b_m = a_m 1$	$\frac{x(1+x-x^2)}{(1-x)^2} + \sum_{k \geq 0} k!x^k$
117	$\{\bar{1}0, 10, 0\bar{1}, 00, 00, 10\}$	$\bar{\epsilon} \sim \bar{0}, a_0; 0 \sim 0, 0\bar{1}; 0\bar{1} \sim 0\bar{1}; a_m \sim (0)^{m+2}, a_{m+1}, \text{ where } a_m = 0\bar{1} \dots m$	
	$\{\bar{1}0, 10, \bar{0}0, 00, 01, 10\}$	$\bar{\epsilon} \sim \bar{0}, 0; 0 \sim 0\bar{1}, 0; \bar{0} \sim \bar{0}, 0, 0\bar{1}; 0\bar{1} \sim 0\bar{1}; 0\bar{1} \sim 0\bar{1}, 0, 0\bar{1}$	
	$\{\bar{1}0, 10, \bar{0}0, 0\bar{1}, 00, 10\}$	$\bar{\epsilon} \sim \bar{0}, 0; 0 \sim 0\bar{0}, 0; \bar{0} \sim \bar{0}, 0\bar{0}, 0; 0\bar{0} \sim 0\bar{0}, 0\bar{0}2; 0\bar{0}2 \sim 0\bar{0}2$	$\frac{1-2x+3x^2-x^3}{(1-x)^4}$
118	$\{\bar{1}0, 10, 0\bar{1}, 0\bar{1}, 01, 10\}$	$\bar{\epsilon} \sim \bar{0}, a_1; \bar{0} \sim (0)^2; 0\bar{1} \sim 0\bar{1}; a_m \sim (0\bar{1})^m, \bar{0}, a_{m+1}, \text{ where } a_m = 0^m$	
	$\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}0, 01, 10\}$		
	$\{\bar{1}0, 10, 0\bar{1}, 0\bar{0}, 01, 10\}$	$\bar{\epsilon} \sim a_1, b_1; b_m \sim (0\bar{1})^m, b_{m+1}; 0\bar{1} \sim 0\bar{1}; 0\bar{1} \sim (0\bar{1})^2; a_m \sim a_{m+1}, 0\bar{1}, (0\bar{1})^m, \text{ where } a_m = \bar{0}^m, b_m = 0^m$	
	$\{\bar{1}0, 10, 0\bar{1}, 0\bar{0}, 01, 10\}$	$\bar{\epsilon} \sim a_1, c_1; 0\bar{1} \sim 0\bar{1}; 0\bar{1} \sim 0\bar{1}, b_1; c_m \sim (0\bar{1})^m, c_{m+1}; a_m \sim a_{m+1}, 0\bar{1}, b_1, b_2, \dots, b_m; b_m \sim 0\bar{1}, b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m, c_m = 0^m$	
	$\{\bar{1}0, 10, \bar{0}0, \bar{0}1, 00, 10\}$	$\bar{\epsilon} \sim \bar{0}, a_0; \bar{0} \sim 0\bar{1}, \bar{0}; 0\bar{1} \sim 0\bar{1}; b_m \sim 0\bar{1}, \bar{0}, b_1, b_2, \dots, b_m; a_m \sim (0)^2, b_1, b_2, \dots, b_m, a_{m+1}, \text{ where } a_m = 0\bar{1} \dots m, b_m = a_m \bar{0}$	$\frac{1-3x+4x^2-3x^3}{(1-x)^2(1-2x)}$
119	$\{\bar{1}0, 10, 00, 01, 00, 10\}$	$\bar{\epsilon} \sim 0, 0; 0 \sim 0\bar{0}, 0\bar{1}; 0 \sim 0, 0\bar{0}, 0\bar{1}; 0\bar{0} \sim 0\bar{0}, 0\bar{0}; 0\bar{1} \sim 0\bar{1}; 0\bar{1} \sim 0\bar{0}, 0\bar{0}, 0\bar{0}$	
	$\{\bar{1}0, 10, 0\bar{1}, \bar{0}0, 00, 10\}$	$\bar{\epsilon} \sim \bar{0}, 0; 0 \sim \bar{0}, \bar{0}, 0; \bar{0} \sim \bar{0}, \bar{0}$	$\frac{1-x+x^2}{(1-x)(1-2x)}$
120	$\{\bar{1}0, 10, 0\bar{1}, 0\bar{1}, 01, 10\}$	$\bar{\epsilon} \sim a_1, 0; 0 \sim (0)^2; 0\bar{1}0 \sim 0\bar{1}0; a_m \sim a_{m+1}, 0, b_1, b_2, \dots, b_m; b_m \sim 0\bar{1}0, b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m m$	
	$\{\bar{1}0, 10, \bar{0}0, 0\bar{0}, 01, 10\}$	$\bar{\epsilon} \sim \bar{0}, a_1; \bar{0} \sim \bar{0}, a_2, a_1; b_m \sim b_1, b_2, \dots, b_m; a_m \sim b_1, b_2, \dots, b_m, a_{m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{1}$	
	$\{\bar{1}0, 10, 00, 00, 01, 10\}$		
	$\{\bar{1}0, 10, \bar{0}0, \bar{0}0, 01, 10\}$	$\bar{\epsilon} \sim \bar{0}, a_1; \bar{0} \sim \bar{0}, a_1; 0\bar{1} \sim 0\bar{1}; b_m \sim 0\bar{1}, b_1, b_2, \dots, b_m; a_m \sim 0\bar{1}, b_1, b_2, \dots, b_m, a_{m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{0}$	
	$\{\bar{1}0, 10, 0\bar{1}, 0\bar{1}, 01, 10\}$	$\bar{\epsilon} \sim a_1, 0; 0 \sim (0)^2; a_m \sim a_{m+1}, (0)^{m+1}, \text{ where } a_m = \bar{0}^m$	$\frac{1-2x+2x^2}{(1-x)^2(1-2x)}$
121	$\{\bar{1}0, 10, 00, 0\bar{1}, 01, 10\}$	$\bar{\epsilon} \sim \bar{0}, a_1; \bar{0} \sim 0\bar{1}, b_0; 0\bar{1} \sim 0\bar{1}; c_m \sim 0\bar{1}, c_0, c_1, \dots, c_m; b_m \sim 0\bar{1}, c_0, c_1, \dots, c_m, b_{m+1}; a_m \sim 0\bar{1}, c_0, c_1, \dots, c_{m-2}, b_{m-1}, a_{m+1}, \text{ where } a_m = 0^m, b_m = 0\bar{0}a_m, c_m = b_m \bar{1}$	$\frac{(1-x)^2}{(1-2x)^2}$
122	$\{\bar{1}0, 10, 0\bar{1}, 0\bar{1}, 01, 10\}$		
	$\{\bar{1}0, 10, \bar{0}\bar{1}, 0\bar{1}, 01, 10\}$	$\bar{\epsilon} \sim a_1, 0; 0 \sim (0)^2; b_m \sim b_1, b_2, \dots, b_{m+1}; a_m \sim b_1, b_2, \dots, b_m, a_{m+1}, 0, \text{ where } a_m = \bar{0}^m, b_m = 0\bar{1}^m$	$\frac{x}{(1-x)(1-2x)} + C(x)$

123	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, a_1, 0; \bar{0} \rightsquigarrow a_1, \bar{0}1; \bar{0}1 \rightsquigarrow \bar{0}1;$ $a_m \rightsquigarrow a_{m+1}, (b_m)^{m+1}; b_m \rightsquigarrow (b_m)^{m+1},$ where $a_m = 0\bar{0}^m, b_m = a_m 1$	$\frac{2}{1-x} \sum_{k \geq 0} \frac{x^k}{1-kx} - \frac{1}{1-x}$
124	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}\bar{1}, \bar{0}0, 10\}$	$\epsilon \rightsquigarrow a_1, c_1; \bar{0}\bar{1} \rightsquigarrow \bar{0}1, \bar{0}\bar{1}1; \bar{0}0 \rightsquigarrow \bar{0}0; c_m \rightsquigarrow$ $c_{m+1}, c_m, \dots, c_1; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1};$ $a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, \bar{0}0,$ where $a_m =$ $\bar{0}^m, b_m = \bar{0}\bar{1}^m, c_m = 0^m$	$\frac{2x-1}{(1-x)^2} + 2C(x)$
125	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, 1\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_2, c_1; b_m \rightsquigarrow$ $b_1, b_2, \dots, b_m; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1};$ $c_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, c_{m+1},$ where $a_m = 0^m, b_m = a_m \bar{1}, c_m = \bar{0}\bar{1}^m$	$\frac{1-4x+6x^2-3x^3-x^4}{(1-x)^2(1-2x)^2}$
126	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}1, \bar{0}\bar{1}, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0})^2; \bar{0}1\bar{0} \rightsquigarrow \bar{0}1\bar{0};$ $a_m \rightsquigarrow \bar{0}, a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \rightsquigarrow$ $\bar{0}1\bar{0}, b_{m+1}, b_m, \dots, b_1,$ where $a_m = 0^m, b_m =$ $a_m \bar{1}$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, 00, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; \bar{0}02 \rightsquigarrow \bar{0}02;$ $a_m \rightsquigarrow a_{m+1}, c_m, b_m, b_{m-1}, \dots, b_1;$ $c_m \rightsquigarrow c_m, c_{m-1}, \dots, c_1, \bar{0}02; b_m \rightsquigarrow$ $b_{m+1}, b_m, \dots, b_1,$ where $a_m = \bar{0}^m,$ $b_m = a_m \bar{1}, c_m = a_m 0$	$\frac{x^2}{(1-x)^2(1-2x)} + \frac{1}{1-x} C(x)$
127	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_{m+1}, (b_m)^{m+1};$ $b_m \rightsquigarrow (b_m)^{m+1},$ where $a_m = \bar{0}^m, b_m = a_m 0$	$\frac{1}{1-x} + \frac{1}{1-x} \sum_{k \geq 1} \frac{x^k}{1-(k+1)x}$
128	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}0, 10\}$	$\epsilon \rightsquigarrow 0, a_1; 0 \rightsquigarrow 0, a_2, a_1; a_m \rightsquigarrow$ $a_{m+1}, a_m, \dots, a_1,$ where $a_m = 0^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, a_1, 0; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow$ $\bar{0}, a_1, a_2, \dots, a_{m+1},$ where $a_m = 0\bar{0}^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 10\}$	$\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; b_m \rightsquigarrow$ $\bar{0}\bar{0}, b_{m+1}, b_m, \dots, b_1; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m, b_m = 0^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1};$ $a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, b_m,$ where $a_m =$ $\bar{0}^m, b_m = \bar{0}\bar{1}^m$	$\frac{(2-x)C(x)-1}{1-x}$
129	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow c_1, \bar{0}1; \bar{0}1 \rightsquigarrow \bar{0}1; c_m \rightsquigarrow$ $c_{m+1}, (b_m)^{m+1}; a_m \rightsquigarrow a_{m+1}, c_m, (b_m)^m;$ $b_m \rightsquigarrow (b_m)^{m+1},$ where $a_m = \bar{0}^m, b_m = a_m 1,$ $c_m = 0a_m$	$\frac{1}{1-x} + \sum_{k \geq 1} \frac{(2k-1)x^k}{1-kx}$
130*	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}1, 10\}$		
131*	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0, 10\}$		
132	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0})^2; a_m \rightsquigarrow \bar{0}, a_{m+1}, (b_m)^m;$ $b_m \rightsquigarrow (b_{m+1})^{m+2},$ where $a_m = 0^m, b_m =$ $a_m \bar{1}$	$\frac{1}{1-2x} + x \sum_{i \geq 1} \sum_{j=1}^i \frac{j(i+1)!}{(j+1)!} x^i$
133	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 10\}$	$\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow a_1, a_2, \dots, a_m;$ $b_m \rightsquigarrow \bar{0}\bar{0}, b_{m+1}, (c_m)^m; c_m \rightsquigarrow (c_{m+1})^{m+2},$ where $a_m = \bar{0}^m, b_m = 0^m, c_m = b_m \bar{1}$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; b_m \rightsquigarrow (b_{m+1})^{m+1};$ $c_m \rightsquigarrow c_1, c_2, \dots, c_{m+1}; a_m \rightsquigarrow$ $c_1, c_2, \dots, c_m, a_{m+1}, \bar{0}\bar{0},$ where $a_m = \bar{0}^m,$ $b_m = 0^m, c_m = \bar{0}\bar{1}^m$	$\frac{x^2}{(1-x)^2} + xC^2(x) + \sum_{k \geq 0} k!x^k$
134*	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}1, 10\}$		
135	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow b_1, \bar{0}; b_m \rightsquigarrow b_{m+1}, b_m, \dots, b_1, \bar{0};$ $a_m \rightsquigarrow \bar{0}, (b_1)^{m+1}, a_{m+1},$ where $a_m =$ $\bar{0}1 \dots m, b_m = 0\bar{0}^m$	$\frac{2(1-x)C(x)-1}{(1-x)^2}$
136	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}1, 10\}$	$\epsilon \rightsquigarrow a_1, b_0; b_m \rightsquigarrow b_0, b_1, \dots, b_{m+1}; a_m \rightsquigarrow$ $b_0, b_1, \dots, b_{m-1}, a_{m+1}, b_m,$ where $a_m = \bar{0}^m,$ $b_m = 0a_m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}0, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}, a_0; a_m \rightsquigarrow \bar{0}, a_0, a_1, \dots, a_{m+1},$ where $a_m = \bar{0}1 \dots m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}1, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1}, b_0, b_1, \dots, b_m; b_m \rightsquigarrow$ $b_0, b_1, \dots, b_m,$ where $a_m = \bar{0}^m, b_m = a_m m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}, a_0; a_m \rightsquigarrow \bar{0}, a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 00, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow$ $a_{m+1}, a_m, \dots, a_1, 0,$ where $a_m = \bar{0}^m$	

	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, b_1; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow C^2(x)$
	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$b_1, b_2, \dots, b_m, a_{m+1}, b_{m+1}, \text{ where } a_m = 0^m, b_m = 0^m$
137	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow \frac{2-x^2+(2x^2-1)C(x)}{(1-x)\sqrt{1-4x}}$
		$(\bar{0})^{m+1}, a_{m+1}, b_{m,1}, b_{m,2}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow (\bar{0})^{m+2-j}, (01\bar{0})^j, b_{m+1,j},$ $b_{m+1,j+1}, \dots, b_{m+1,m+1}, \text{ where } a_m = 0^m,$ $b_{m,j} = a_m j$
138	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow \frac{(1-4x+5x^2-x^3)C(x)-(1-2x)(1-x)}{x^2(1-x)}$
		$a_1, a_2, \dots, a_{m+1}, 0, b_1, b_2, \dots, b_m;$ $b_m \rightsquigarrow b_1, b_2, \dots, b_m, b_m, \text{ where } a_m = \bar{0}^m,$ $b_m = a_m m$
139	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow \frac{2-x+(x-1)C(x)}{(1-x)\sqrt{1-4x}}$
		$(\bar{0})^{m+1}, a_{m+1}, b_{m,1}, b_{m,2}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow (\bar{0})^{m+2}, b_{m+1,j},$ $b_{m+1,j+1}, \dots, b_{m+1,m+1}, \text{ where } a_m = 0^m,$ $b_{m,j} = a_m j$
140*	$\{10, 10, 00, 00, 00, 10\}$	
141*	$\{10, 10, 00, 01, 00, 10\}$	
142*	$\{10, 10, 00, 00, 00, 10\}$	
143	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0\bar{0}, 0; 0\bar{0} \rightsquigarrow 0\bar{0};$
		$a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, b_1 b_2, \dots, b_m;$ $b_m \rightsquigarrow 0\bar{0}, b_1, b_2, \dots, b_m, b_m, \text{ where } a_m = \bar{0}^m,$ $b_m = a_m m$
144*	$\{10, 10, 00, 01, 00, 10\}$	
145	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; c_m \rightsquigarrow c_{m+1}, c_m, \dots, c_0;$
		$a_m \rightsquigarrow c_{m-1}, a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \rightsquigarrow$ $c_m, b_{m+1}, b_m, \dots, b_1, \text{ where } a_m = 0^m, b_m =$ $a_m 1, c_m = 0\bar{0}1^m$
146	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; a_1 \rightsquigarrow 0\bar{0}, a_2, b_1; \bar{0} \rightsquigarrow \bar{0}, a_1; 0\bar{0} \rightsquigarrow$
		$(b_1)^2; a_m \rightsquigarrow b_{m-1}, a_{m+1}, (b_m)^m; b_m \rightsquigarrow$ $(b_{m+1})^{m+2}, \text{ where } a_m = 0^m, b_m = a_m 1$
147*	$\{10, 10, 01, 01, 00, 10\}$	
148	$\{10, 10, 01, 00, 01, 10\}$	
	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow$
		$(\bar{0})^{m+1}, a_{m+1}, b_{m,1}, \dots, b_{m,m};$ $b_{m,j} \rightsquigarrow (\bar{0})^{m+2-j}, (b_{m+1,j})^j, b_{m+1,j},$ $b_{m+1,j+1}, \dots, b_{m+1,m+1}, \text{ where } a_m = 0^m,$ $b_{m,j} = a_m j$
149	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow b_0, 0; b_m \rightsquigarrow b_{m+1}, b_m, \dots, b_0, \bar{0};$
		$a_m \rightsquigarrow b_{m-1}, a_{m+1}, a_m, \dots, a_1, \text{ where } a_m =$ $0^m, b_m = 0\bar{0}a_m$
150*	$\{10, 10, 00, 00, 01, 10\}$	
151	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, b_1; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow$
		$a_1, a_2, \dots, a_{m+1}, (b_m)^m, \text{ where } a_m = \bar{0}^m,$ $b_m = 0^m$
152*	$\{10, 10, 01, 00, 00, 10\}$	
153	$\{10, 10, 00, 01, 00, 10\}$	$\epsilon \rightsquigarrow 0, 0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}; b_m \rightsquigarrow$
		$a_1, a_2, \dots, a_m, b_{m+1}, b_m, c_{m,2}, \dots, c_{m,m};$ $c_{m,j} \rightsquigarrow a_1, a_2, \dots, a_m, c_{m+1,j}, c_{m,j},$ $c_{m,j+1}, \dots, c_{m,m}, \text{ where } a_m = \bar{0}^m,$ $b_m = 0^m, c_{m,j} = a_m j$
	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow \frac{2xC(x)+2-2x}{4x}$
		$a_1, a_2, \dots, a_{m+1}, c_{m,0}, c_{m,1}, \dots, c_{m,m};$ $c_{m,0} \rightsquigarrow c_{m,0}, c_{m-1,0}, \dots, c_{0,0}; c_{m,j} \rightsquigarrow$ $c_{m+2-j,1}, \dots, c_{m+1,j}, c_{m,j}, \dots, c_{j,j}, \text{ where}$ $a_m = \bar{0}^m, c_{m,j} = a_m j$
154*	$\{10, 10, 00, 00, 00, 10\}$	
155	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow \frac{1}{1-x} + xC^2(x)$
		$a_1, a_2, \dots, a_{m+1}, (b_m)^{m+1}; b_m \rightsquigarrow (b_m)^{m+1},$ $\text{ where } a_m = \bar{0}^m$
		$+ x \sum_{k \geq 1} \frac{(k+1)(xC(x))^k}{1-(k+1)x}$

156	{10, 10, 01, 01, 00, 10}	$\epsilon \sim a_1, b_1; a_m \sim \frac{-2+3x-x^2-x\sqrt{x^2-6x+1}}{2x}$ $a_{m+1}, b_{m+1}, c_m, c_{m-1}, \dots, c_1; b_m \sim \frac{(1-x)C(x)}{x} + b_{m+1}, b_m, \dots, b_1; c_m \sim c_{m+1}, c_{m+1}, c_m, \dots, c_1, \text{ where } a_m = \bar{0}^m, b_m = 0^m, c_m = a_m 1$
157*	{10, 10, 00, 00, 01, 10}	
158*	{10, 10, 00, 00, 00, 10}	
159*	{10, 10, 00, 00, 00, 10}	
160*	{10, 10, 00, 00, 00, 10}	
161*	{10, 10, 00, 01, 00, 10}	
162*	{10, 10, 01, 00, 01, 10}	
163*	{10, 10, 00, 00, 01, 10}	
164*	{10, 10, 00, 01, 00, 10}	
165*	{10, 10, 00, 00, 01, 10}	
166	{10, 10, 00, 00, 01, 10}	$\epsilon \sim a_1, b_0; b_m \sim b_1, b_2, \dots, b_{m+1}; a_m \sim \frac{x^2-4x+1}{2x(1-4x)} - \frac{4x^3+15x^2-8x+1}{2x\sqrt{1-4x^3}}$ $a_1, a_2, \dots, a_{m+1}, (b_m)^{m+1}, \text{ where } a_m = \bar{0}^m, b_m = 01^m$
167	{10, 10, 00, 00, 01, 10}	$\epsilon \sim a_1, b_0; b_m \sim b_0, b_1, \dots, b_{m+1}; a_m \sim \frac{(6x^2-9x+2)\sqrt{1-4x-28x^2+15x-2}}{2x\sqrt{1-4x^3}}$ $a_1, a_2, \dots, a_{m+1}, (b_m)^m, \text{ where } a_m = \bar{0}^m, b_m = 0a_m$
168*	{10, 10, 01, 00, 00, 10}	
169*	{10, 10, 00, 00, 01, 10}	
170*	{10, 10, 00, 01, 00, 10}	
171*	{10, 10, 01, 00, 00, 10}	
172*	{10, 10, 00, 01, 00, 10}	
173	{10, 10, 01, 00, 01, 10}	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim (b_1)^2; b_m \sim \frac{x^2-6x+5+(x-3)\sqrt{x^2-6x+1}}{2}$ $b_{m+1}, a_{m+1}, b_{m+1}, b_m, \dots, b_1; a_m \sim b_{m+1}, a_{m+1}, b_{m+1}, b_m, \dots, b_1, \text{ where } a_m = 0^m, b_m = 00^m$
174	{10, 10, 00, 01, 00, 10}	$\epsilon \sim a_1, b_1; b_m \sim (b_m+1)^{m+1}; a_m \sim C(x) + \sum_{j \geq 1} j! x^j$ $a_1, a_2, \dots, a_{m+1}, (b_m)^m, \text{ where } a_m = \bar{0}^m, b_m = 0^m + \sum_{j \geq 1} j! x^j \sum_{m=1}^j \frac{x^{1-m} (x C(x))^m}{(m-1)!}$
175*	{10, 10, 00, 01, 00, 10}	
176*	{10, 10, 00, 00, 00, 10}	
177*	{10, 10, 01, 00, 00, 10}	
178*	{10, 10, 00, 00, 00, 10}	
179*	{10, 10, 01, 00, 00, 10}	
180	{10, 10, 01, 00, 01, 10}	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim (b_0)^2; b_m \sim (b_m+1)^{m+3};$ $a_m \sim b_{m-1}, a_{m+1}, (b_{m-1})^m, \text{ where } a_m = 0^m, b_m = 00a_m$
	{10, 10, 01, 01, 00, 10}	$\epsilon \sim a_1, b_1; b_m \sim (b_m+1)^{m+1}; a_m \sim \frac{1}{1-x} + \sum_{j \geq 1} \sum_{i=1}^j \frac{i \cdot j!}{i!} x^j$ $a_{m+1}, (b_m+1)^{m+1}, \text{ where } a_m = \bar{0}^m, b_m = 0^m$
181*	{10, 10, 01, 00, 00, 10}	
182	{10, 10, 01, 01, 10, 10}	$\epsilon \sim a_1, b_1; 0\bar{1} \sim 0\bar{1}; 0\bar{0} \sim (0\bar{0})^2; b_m \sim \frac{1-3x+5x^2-5x^3}{(1-x)^3(1-2x)}$ $(0\bar{1})^m, 0\bar{0}, b_{m+1}; a_m \sim a_{m+1}, (0\bar{0})^{m+1}, \text{ where } a_m = \bar{0}^m, b_m = 0^m$
183	{10, 10, 01, 01, 10, 10}	$\epsilon \sim a_1, b_1; 0\bar{0} \sim (0\bar{0})^2; b_m \sim \frac{1}{1-2x} + \frac{2x^2}{1-x} C^3(x)$ $0\bar{0}, b_{m+1}, d_m, d_{m-1}, \dots, d_1; d_m \sim d_{m+1}, d_m, \dots, d_1; c_m \sim c_1, c_2, \dots, c_{m+1};$ $a_m \sim c_1, c_2, \dots, c_m, b_{m+1}, 0\bar{0}, \text{ where } a_m = 0^m, b_m = 0^m, c_m = 0\bar{1}^m, d_m = b_m 1$
184	{10, 10, 01, 00, 10, 10}	$\epsilon \sim a_1, 0; 0 \sim a_1, b_1, 0; 0\bar{1} \sim 0\bar{1}; \frac{1-4x+7x^2-5x^3}{(1-x)^2(1-2x)^2}$ $b_m \sim b_{m+1}, c_m, c_{m-1}, \dots, c_1, 0\bar{1}; c_m \sim c_m, c_{m-1}, \dots, c_1, 0\bar{1}; a_m \sim a_{m+1}, b_m, c_{m-1}, c_{m-2}, \dots, c_1, 0\bar{1}, \text{ where } a_m = \bar{0}^m, b_m = 0a_m, c_m = b_m 1$
185	{10, 10, 01, 01, 10, 10}	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim (0)^2; a_m \sim \frac{2(1-x)C(x)-1}{1-2x}$ $(0)^{m+1}, a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = 0^m$
186	{10, 10, 00, 01, 10, 10}	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}, b_0, a_1; 0\bar{1} \sim 0\bar{0}, 0\bar{1}; C(x) + C^2(x) - \frac{1}{1-x}$ $b_m \sim b_{m+1}, b_m, \dots, b_0, 0\bar{1}; a_m \sim b_{m-1}, a_{m+1}, b_{m-2}, \dots, b_0, 0\bar{1}, \text{ where } a_m = 0^m, b_m = 00a_m$

187	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{1}0, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}\bar{1}, b_0; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1};$ $c_m \rightsquigarrow \bar{0}\bar{1}, c_0, c_1, \dots, c_m; b_m \rightsquigarrow$ $\bar{0}\bar{1}, c_0, c_1, \dots, c_m, b_{m+1}; a_m \rightsquigarrow$ $\bar{0}, b_0, b_1, \dots, b_{m-1}, a_{m+1}, a_m, \dots, a_1,$ where $a_m = 0^m, b_m = 0\bar{0}a_m, c_m = b_m\bar{1}$	$\frac{(x^2+x-1)\sqrt{1-4x+x^2}-3x+1}{2x^2(1-2x)}$
188	$\{\bar{1}0, 10, 0\bar{1}, 00, \bar{1}0, 10\}$ $\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}0, \bar{1}0, 10\}$ $\{\bar{1}0, 10, \bar{0}0, \bar{0}\bar{1}, \bar{1}0, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, b_1; b_m \rightsquigarrow \bar{0}, b_1, b_2, \dots, b_{m+1};$ $a_m \rightsquigarrow 0, b_1, b_2, \dots, b_m, a_{m+1}, a_m, \dots, a_1,$ where $a_m = 0^m, b_m = 0\bar{0}^m$ $\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{1} \rightsquigarrow a_1, \bar{0}\bar{1}; b_m \rightsquigarrow$ $a_1, a_2, \dots, a_m, b_{m+1}, b_m, \dots, b_1; a_m \rightsquigarrow$ $a_{m+1}, a_m, \dots, a_1, \bar{0}\bar{1},$ where $a_m \bar{0}^m, b_m = 0^m$ $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow (a_1)^2, b_0; \bar{0} \rightsquigarrow (a_1)^2, b_0;$ $a_m \rightsquigarrow a_m, a_m, b_m, b_{m-1}, \dots, b_0; b_m \rightsquigarrow$ $b_{m+1}, b_m, \dots, b_0,$ where $a_m = 0\bar{0}^m, b_m =$ $a_m\bar{1}$ $\epsilon \rightsquigarrow a_1, b_1; \bar{0}\bar{1} \rightsquigarrow c_1, \bar{0}\bar{1}; c_m \rightsquigarrow$ $c_{m+1}, c_m, \dots, c_1, \bar{0}\bar{1}; b_m \rightsquigarrow$ $c_m, b_{m+1}, c_{m-1}, \dots, c_1, \bar{0}\bar{1}; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1}, b_m, b_{m-1}, \dots, b_1,$ where $a_m = 0^m, b_m = 0^m, c_m = 0a_m$	$\frac{1}{\sqrt{1-4x}}$
189*	$\{\bar{1}0, 10, \bar{0}0, \bar{0}0, \bar{1}0, 10\}$		
190	$\{\bar{1}0, 10, \bar{0}0, \bar{0}0, \bar{1}0, 10\}$ $\{\bar{1}0, 10, \bar{0}0, \bar{0}0, \bar{1}0, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow a_1, a_2, a_1; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_m, a_{m+1}, a_m, \dots, a_1,$ where $a_m = 0^m$ $\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow$ $0, a_1, a_2, \dots, a_m, a_{m+1}, a_m, \dots, a_1,$ where $a_m = 0^m$	$\frac{1-x-\sqrt{1-6x+x^2}}{2x}$

Table 7: Succession rules of $\mathcal{T}[B]$ and generating function $F_B(x)$, where $|B| = 6$.

Theorem 16. We have $\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}1, \bar{0}0\} \sim \{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}1, \bar{0}0\}$.

Proof. Let $A = \{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}1, \bar{0}0\}$ and $B = \{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}1, \bar{0}0\}$. Any signed inversion sequence that avoids A with at least one barred letter \bar{a} can be presented as $\pi' \bar{a} \bar{a} \dots \bar{a}$, where π' has no barred letters. On the another hand, any signed inversion sequence that avoids B with at least one barred letter \bar{a} can be presented as $\pi'' \bar{a} a a \dots a$, where π'' has no barred letters. We define α from $\tilde{\mathcal{I}}_n(A)$ to $\tilde{\mathcal{I}}_n(B)$ by $\alpha(\pi)$ is the same as π , where we change the barred letters (if exist) except the leftmost letter to unbarred letters. Clearly, $\pi \in \tilde{\mathcal{I}}_n(A)$ if and only if $\alpha(\pi) \in \tilde{\mathcal{I}}_n(B)$. \square

Theorem 17. Let A be either $\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{0}\bar{1}\}$, $\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{0}0\}$, or $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{0}1, \bar{1}0\}$. Then $|\tilde{\mathcal{I}}_n(A)| = 2n!$, for all $n \geq 1$.

Proof. Any signed inversion sequence avoids A contains at most one barred letter, which says that the corresponding signed permutation contains at most one barred letter. So, $|\tilde{\mathcal{I}}_n(A)| = 2n!$, for all $n \geq 1$. \square

Case $k = 7$

No	B	Succession rules of $\mathcal{T}[B]$	Generating function $F_B(x)$
1	$\{\bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 00, 01\}$ $\{\bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}0, 00, 01\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}$	$1 + 2x + x^2$
2	$\{\bar{0}\bar{1}, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 00, 00\}$		

	$\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, \bar{0}1, 00, 10\}$ $\{\bar{1}0, 10, \bar{0}0, \bar{0}1, 00, 00, 01\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, 01, 00, 01, 10\}$ $\{\bar{1}0, 10, \bar{0}1, \bar{0}0, \bar{0}1, 00, 01\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, 01, 00, 01, 10\}$ $\{\bar{1}0, 10, \bar{0}1, \bar{0}0, \bar{0}1, 00, 01\}$ $\{\bar{1}0, 10, \bar{0}1, \bar{0}0, \bar{0}0, \bar{0}0, 01\}$	$\epsilon \sim \bar{0}, a_0; a_m \sim (\bar{0})^{m+2}, a_{m+1}, \text{ where } a_m =$ $\frac{01 \dots m}{\dots \dots \dots}$ $\epsilon \sim \bar{0}, 0; 0 \sim 0; \bar{0} \sim 0, \bar{0}$ $\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}; a_m \sim (0\bar{1})^m, a_{m+1}, \text{ where } \frac{1}{(1-x)^2}$ $a_m = 0^m$
19	$\{10, 01, 00, 01, 00, 01, 10\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, 00, 01, 10\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{0}0, 01, 10\}$ $\{10, \bar{0}1, \bar{0}0, \bar{0}0, \bar{0}0, 01, 10\}$	$\epsilon \sim a_1, 0; 0 \sim 0; a_m \sim a_{m+1}, b_1, b_2, \dots, b_m;$ $b_m \sim b_1, b_2, \dots, b_{m-1}, \text{ where } a_m = \bar{0}^m,$ $b_m = a_m m$ $\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}; a_m \sim \frac{1-x^2-x^3}{(1-x)(1-x^2)}$ $b_m, b_{m-1}, \dots, b_1, a_{m+1}; b_m \sim$ $b_{m-1}, b_{m-2}, \dots, b_1, \text{ where } a_m = 0^m,$ $b_m = a_m m$
20	$\{10, 01, 00, 00, 01, 00, 10\}$ $\{10, 01, 00, 00, 01, 00, 10\}$	$\epsilon \sim 0, 0; 0 \sim 00, \bar{0}; \bar{0} \sim 0; 00 \sim 00, 00$ $\frac{1-x-x^2-x^3}{(1-x)(1-2x)}$
21	$\{10, 01, 00, 00, 00, 01, 10\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, 00, 01, 10\}$ $\{\bar{1}0, 01, 00, \bar{0}1, 00, 01, 10\}$ $\{\bar{1}0, 10, \bar{0}1, \bar{0}0, \bar{0}1, 00, 01\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, 00, 01, 10\}$	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}; a_m \sim (\bar{0})^m, a_{m+1}, \text{ where}$ $a_m = 0^m$ $\epsilon \sim \bar{0}, 0; 0 \sim 0; \bar{0} \sim \bar{0}, \bar{0}0; \bar{0}0 \sim 0, \bar{0}0$ $\epsilon \sim \bar{0}, 0; 0 \sim 00, 0; \bar{0} \sim 0; \bar{0}0 \sim 00, \bar{0}$ $\epsilon \sim \bar{0}, a_1; 0\bar{1} \sim 0\bar{1}; a_m \sim (0\bar{1})^m, \bar{0}, a_{m+1},$ $\text{where } a_m = 0^m$ $\frac{1-x+x^3}{(1-x)^3}$
22	$\{10, 00, 01, 00, 00, 01, 10\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, \bar{0}1, 01, 10\}$ $\{\bar{1}0, \bar{0}1, 0\bar{1}, 0\bar{0}, 00, 01, 10\}$	$\epsilon \sim a_0, b_0; a_m \sim a_{m+1}, b_0, b_1, \dots, b_m; b_m \sim$ $b_0, b_1, \dots, b_{m+1}, \text{ where } a_m = \bar{0}\bar{1} \dots \bar{m},$ $b_m = a_{m-1} m$ $\epsilon \sim b_0, a_1; b_m \sim b_{m-1}, b_{m-2}, \dots, b_0; a_m \sim$ $b_m, b_{m-1}, \dots, b_0, a_{m+1}, \text{ where } a_m = 0^m,$ $b_m = a_m m$ $\epsilon \sim a_1, b_0; a_m \sim a_{m+1}, b_0, b_1, \dots, b_m;$ $b_m \sim b_0, b_1, \dots, b_{m-1}, \text{ where } a_m = 0^m,$ $b_m = a_m m$ $\frac{1+x}{1-x-x^2}$
23	$\{10, 01, 00, 01, 00, 00, 10\}$ $\{\bar{1}0, 10, \bar{0}1, \bar{0}0, 01, 00, 00\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, 01, 00, 00, 10\}$ $\{\bar{1}0, 10, \bar{0}1, \bar{0}0, \bar{0}0, 01, 00\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, 00, 01, 10\}$ $\{\bar{1}0, 10, \bar{0}1, \bar{0}0, 00, 00, 01\}$ $\{\bar{1}0, 10, \bar{0}1, \bar{0}0, 01, 00, 01\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, 01, 00, 01, 10\}$	$\epsilon \sim \bar{0}, 0; 0 \sim 0; \bar{0} \sim \bar{0}, \bar{0}$ $\epsilon \sim \bar{0}, 0; 0 \sim 0, 0; \bar{0} \sim \bar{0}$ $\epsilon \sim b_0, a_1; b_m \sim b_m, b_{m-1}, \dots, b_0; a_m \sim$ $b_m, b_{m-1}, \dots, b_0, a_{m+1}, \text{ where } a_m = 0^m,$ $b_m = a_m m$ $\epsilon \sim a_1, b_0; a_m \sim a_{m+1}, b_0, b_1, \dots, b_m;$ $b_m \sim b_0, b_1, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m =$ $\frac{a_m m}{a_m m}$ $\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}; b_m \sim \bar{0}, b_2, b_3, \dots, b_m;$ $a_m \sim \bar{0}, b_2, b_3, \dots, b_m, a_{m+1}, \text{ where } a_m =$ $0^m, b_m = a_m \bar{1}$ $\frac{1-x-x^2}{(1-x)(1-2x)}$
24	$\{10, 01, 00, 01, 00, 00, 10\}$ $\{\bar{1}0, \bar{0}1, \bar{0}0, 01, 00, 00, 10\}$	$\epsilon \sim a_1, 0; 0 \sim 0; a_1 \sim a_2, 0; a_m \sim \frac{1}{1-x} + \sum_{k \geq 0} \frac{x^k}{1-kx}$ $a_{m+1}, (b_m)^m; b_m \sim (b_m)^m, \text{ where } a_m =$ $0^m, b_m = a_m \bar{1}$
25	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, 0\bar{1}, 00, 00, 01\}$	$\epsilon \sim a_1, 0; a_m \sim a_{m+1}, 0, b_1, b_2, \dots, b_m; 1 + 2x + \frac{3x^2}{1-2x}$ $b_m \sim b_m, 0, b_1, b_2, \dots, b_{m-1}, \text{ where } a_m =$ $0^m, b_m = a_m m$
26	$\{10, 01, 00, 01, 00, 01, 10\}$ $\{10, 01, 00, 01, 00, 01, 10\}$	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}; 0\bar{1} \sim (0\bar{1})^2; a_m \sim \frac{1-2x+2x^3}{(1-x)^2(1-2x)}$ $(0\bar{1})^m, a_{m+1}, \text{ where } a_m = 0^m$
27	$\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}0, 0\bar{1}, 00, 01\}$	$\epsilon \sim a_1, 0; 0 \sim 0; a_m \sim a_{m+1}, b_1, b_2, \dots, b_m; \frac{1-2x-x^2+x^3}{(1-x)(1-3x+x^2)}$ $b_m \sim b_m, b_1, b_2, \dots, b_m, \text{ where } a_m = \bar{0}^m,$ $b_m = a_m m$
28	$\{10, 10, 01, 01, 00, 00, 01\}$	$\epsilon \sim a_1, 0; a_m \sim b_1, b_2, \dots, b_m, a_{m+1}, 0;$ $b_m \sim b_1, b_2, \dots, b_{m+1}, \text{ where } a_m = \bar{0}^m,$ $b_m = \bar{0}\bar{1}^m$

	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 10\}$ $\{\bar{1}0, 10, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 00\}$ $\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 10\}$ $\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$ $\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$ $\{\bar{1}0, 10, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01\}$ $\{\bar{1}0, 10, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01\}$	$\epsilon \sim a_1, 0; 0 \sim 0; a_m \sim a_{m+1}, a_m, \dots, a_1,$ where $a_m = \bar{0}^m$	
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$ $\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}; a_m \sim a_{m+1}, a_m, \dots, a_1,$ where $a_m = 0^m$	
	$\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}; a_m \sim a_1, a_2, \dots, a_m,$ where $a_m = 0^m$	$\frac{x}{1-x} + C(x)$
29	$\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 00\}$	$\epsilon \sim a_1, 0; 0 \sim 0; a_m \sim a_{m+1}, (a_m)^m,$ where $a_m = \bar{0}^m$	$\frac{x}{1-x} + \sum_{k \geq 0} \frac{x^k}{\prod_{j=1}^k (1-jx)}$
30	$\{\bar{1}0, 01, \bar{0}\bar{1}, \bar{0}0, \bar{0}0, 01, 10\}$ $\{\bar{1}0, 10, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 01\}$ $\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 01, 10\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 10\}$	$\epsilon \sim a_1, 0; a_m \sim a_1, a_2, \dots, a_{m+1}, 0,$ where $a_m = \bar{0}^m$	
	$\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 10\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$	$\epsilon \sim a_1, 0; 0 \sim a_1; a_m \sim a_1, a_2, \dots, a_{m+1},$ where $a_m = \bar{0}^m$	
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 10\}$	$\epsilon \sim a_1, 0; a_m \sim \bar{0}, a_{m+1}, a_m, \dots, a_1,$ where $a_m = 0^m$	$(1+x)C(x)$
31	$\{\bar{1}0, 10, \bar{0}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$ $\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$ $\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$ $\{\bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim \bar{0}; a_m \sim (a_{m+1})^{m+1},$ where $a_m = 0^m$	
	$\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, 10\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, \bar{0}\bar{1}, \bar{0}\bar{1}, 10\}$	$\epsilon \sim \bar{0}, a_1; a_m \sim \bar{0}, a_{m+1}, (b_m)^m; b_m \sim$ $(b_{m+1})^{m+2},$ where $a_m = 0^m, b_m = a_m 1$	$\frac{x}{1-x} + \sum_{k \geq 0} k!x^k$
32	$\{\bar{1}0, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 01, 10\}$	$\epsilon \sim c_{m,1}, c_{m-1,2}, \dots, c_{2,m-1}, a_m, a_m \sim$ $a_{m+1}, 0, b_2, b_3, \dots, b_m; c_{m,j} \sim c_{m+j-1,1},$ $c_{m+j-2,2}, \dots, c_{m,j}, c_{m,j+1}, 0,$ $b_2, b_3, \dots, b_{m+j-1}; b_m \sim$ $0, b_2, b_3, \dots, b_{m-1},$ where $a_m = 0^m,$ $b_m = a_m m, c_{m,j} = a_m m^j$	$\frac{1+x^2 - \sqrt{1-4x-2x^2+x^4}}{2x}$
33	$\{\bar{1}0, 01, \bar{0}0, 00, 01, 00, 10\}$ $\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, 00\}$ $\{\bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$	$\epsilon \sim \bar{0}, a_1; \bar{0} \sim a_1; a_m \sim (a_{m+1})^{m+1},$ where $a_m = 0^m$	
	$\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, 01\}$	$\epsilon \sim \bar{0}, a_1; a_m \sim \bar{0}, (a_{m+1})^{m+1},$ where $a_m =$ 0^m	$(1+x)\sum_{k \geq 0} k!x^k$
34	$\{\bar{1}0, 10, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 01\}$	$\epsilon \sim a_1, a_2, \dots, a_{m+1}, b_1, b_2, \dots, b_m; a_m \sim$ $b_m \sim b_1, b_2, \dots, b_m, b_1, b_2, \dots, b_{m-1},$ where $a_m = \bar{0}^m, b_m = a_m m$	$x + \frac{1}{x + \sqrt{1-4x}}$
35	$\{\bar{1}0, 01, \bar{0}0, 00, 01, 00, 10\}$	$\epsilon \sim b_m, b_{m-1}, \dots, b_1, a_{m+1}, a_m, c_{m,2},$ $c_{m,3}, \dots, c_{m,m}; b_m \sim$ $b_m, b_{m-1}, \dots, b_1; c_{m,j} \sim$ $b_{m+1-j}, b_{m-j}, \dots, b_1, (\bar{0})^{j-1}, c_{m+1,j},$ $c_{m,j}, c_{m,j+1}, \dots, c_{m,m},$ where $a_m = 0^m,$ $b_m = a_m m, c_{m,j} = a_m j$	
36*	$\{\bar{1}0, 01, \bar{0}0, 00, 01, 00, 10\}$		
37*	$\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}0, 00, 01, 00\}$		
38*	$\{\bar{1}0, 01, \bar{0}0, 00, 01, 00, 10\}$		
39	$\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 01, 10\}$	$\epsilon \sim \bar{0}, 0; 0 \sim 0\bar{1}, 0\bar{1}; \bar{0} \sim 0\bar{1}, 0\bar{1}$	$1 + 2x + 4x^2$
40	$\{\bar{1}0, 10, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 01, 10\}$	$\epsilon \sim \bar{0}, 0; 0 \sim 0\bar{1}, 0\bar{1}; \bar{0} \sim 0\bar{1}, 0\bar{1}; 0\bar{1} \sim 0\bar{1}$	$1 + 2x + 4x^2 + x^3$
41	$\{\bar{1}0, 10, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, 00, 10\}$	$\epsilon \sim \bar{0}, 0; 0 \sim 0\bar{0}, 01; \bar{0} \sim 01, 0\bar{0}; 01 \sim 01$	$1 + 2x^2 + \frac{2x}{1-x}$
42	$\{\bar{1}0, 10, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, 00, 10\}$	$\epsilon \sim \bar{0}, 0; 0 \sim 0\bar{0}, 0; \bar{0} \sim 0\bar{1}, 0\bar{0}; 0\bar{1} \sim 0\bar{1}$	$1 + 2x + x^2 + \frac{3x^2}{1-x}$
43	$\{\bar{1}0, 10, 01, 00, 01, 00, 10\}$	$\epsilon \sim 0, 0; 0 \sim 0, 00, 0; 0 \sim 00$	

52	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}0, \bar{0}\bar{1}, 01, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0; a_1 \rightsquigarrow a_2, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow a_{m+1}, \bar{0}\bar{0}, 0, b_3, b_4, \dots, b_m; b_m \rightsquigarrow \bar{0}\bar{0}, 0, b_3, b_4, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0\bar{1}; \bar{0} \rightsquigarrow \bar{0}, 0; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}\bar{0}; 0\bar{1} \rightsquigarrow \bar{0}\bar{1}$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, \bar{0}\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}\bar{0}$	$\frac{1-2x+x^2-x^3}{(1-x)^2(1-2x)}$
53	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, 0\bar{1}; \bar{0} \rightsquigarrow \bar{0}, \bar{0}; 0\bar{1} \rightsquigarrow 0\bar{1}$	$\frac{1-x-x^3}{(1-x)(1-2x)}$
54	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow (\bar{0}\bar{0})^2; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = \bar{0}^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (\bar{0}\bar{1})^2; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = 0^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow x + \frac{2x^2}{1-x} + C(x)$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 10\}$	$b_1, b_2, \dots, b_m, a_{m+1}, \bar{0}\bar{0}; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; \text{ where } a_m = \bar{0}^m, b_m = \bar{0}1^m$	
55	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, 00, 01, 1\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow 0, \bar{0}\bar{0}$	$\frac{1-x+x^2+x^3}{(1-x)^3}$
56	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, 0\bar{1}, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow 0, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}, 0; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}\bar{0}\bar{2}; \bar{0}\bar{0}\bar{2} \rightsquigarrow \bar{0}\bar{0}\bar{2}$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow \bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}; a_m \rightsquigarrow 0, (\bar{0}\bar{0})^{m+1}, a_{m+1}, \text{ where } a_m = 01 \dots m$	$\frac{1-2x+2x^2}{(1-x)^4}$
57	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, 01, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, c_1; 0\bar{1} \rightsquigarrow 0\bar{1}; a_m \rightsquigarrow a_{m+1}, 0\bar{1}, b_2, \dots, b_m; b_m \rightsquigarrow 0\bar{1}, b_2, b_3, \dots, b_m; c_m \rightsquigarrow (0\bar{1})^m, c_{m+1}, \text{ where } a_m = \bar{0}^m, b_m = a_m^m, c_m = 0^m$	$\frac{(1-x)^3-x^4}{(1-x)^3(1-2x)}$
58	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, 0\bar{1}, 01, 1\bar{0}\}$	$\epsilon \rightsquigarrow b_0, a_1; b_m \rightsquigarrow b_0, b_1, \dots, b_m; a_m \rightsquigarrow b_0, b_1, \dots, b_m, a_{m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{0}$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; \bar{0}\bar{1} \rightsquigarrow (\bar{0}\bar{1})^2; a_m \rightsquigarrow a_{m+1}, (\bar{0}\bar{1})^m, \text{ where } a_m = \bar{0}^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}\bar{0}$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, 00, 1\bar{0}\}$		
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, 01, 10\}$		
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, 01, 10\}$		
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, 00, 1\bar{0}\}$		
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0, 0; \bar{0} \rightsquigarrow 0, 0$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow a_{m+1}, b_0, b_1, \dots, b_m; b_m \rightsquigarrow b_0, b_1, \dots, b_m, \text{ where } a_m = \bar{0}^m, b_m = a_m^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, 00, 10\}$	$\epsilon \rightsquigarrow b_0, a_0; b_m \rightsquigarrow b_0, b_1, \dots, b_m; a_m \rightsquigarrow b_0, b_1, \dots, b_m, 0123456, \text{ where } a_m = 01 \dots m, b_m = a_m - 1\bar{0}$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, 00, 01, 1\bar{0}\}$		
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, 00, 01, 10\}$		
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, 01, 00, 01, 10\}$	$\epsilon \rightsquigarrow b_0, a_1; b_m \rightsquigarrow b_1, b_2, \dots, b_m; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, \text{ where } a_m = 0^m, b_m = a_m \bar{1}$	$\frac{1}{1-2x}$
59	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_{m+1}, (b_m)^m; b_m \rightsquigarrow (b_m)^m, \text{ where } a_m = \bar{0}^m, b_m = a_m \bar{1}$	$\frac{1}{1-x} \sum_{k \geq 0} \frac{x^k}{1-kx}$
60	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, 00, 1\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow a_1, 0\bar{1}; \bar{0} \rightsquigarrow a_1, 0\bar{1}; 0\bar{1} \rightsquigarrow 0\bar{1}; a_m \rightsquigarrow a_{m+1}, (b_m)^{m+1}; b_m \rightsquigarrow (b_m)^{m+1}, \text{ where } a_m = \bar{0}\bar{0}^m, b_m = a_m \bar{1}$	$1 + 2 \sum_{k \geq 0} \frac{x^k}{1-kx}$
61	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 1\bar{0}\}$		
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$		
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, 0, \text{ where } a_m = \bar{0}^m, b_m = \bar{0}\bar{1}^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, 00, 10\}$		
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = \bar{0}^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = 0^m$	$\frac{x}{(1-x)^2} + C(x)$

62	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, \bar{0}1, \bar{0}1, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow 0\bar{0}, a_{m+1}, a_m, \dots, a_1,$ where $a_m = 0^m$	$\frac{x^2}{1-x} + (1+x)C(x)$
63	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, \bar{0}1, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow 0\bar{0}, a_{m+1}, (b_m)^m;$ $b_m \rightsquigarrow (b_{m+1})^{m+2},$ where $a_m = 0^m, b_m =$ $\frac{a_{m+1}}{a_m}$	
	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}1, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1, \bar{0} \rightsquigarrow (\bar{0}1)^2; \bar{0}1 \rightsquigarrow \bar{0}1; a_m \rightsquigarrow x + \frac{2x^2}{1-x} + \sum_{k \geq 0} k!x^k$ where $a_m = 0^m$	
64	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, \bar{0}1, 10\}$	$\epsilon \rightsquigarrow b_1, a_1; \bar{0}1 \rightsquigarrow \bar{0}1; \bar{0}1 \rightsquigarrow (\bar{0}1)^2; a_m \rightsquigarrow \frac{1-3x+3x^2-2x^4}{(1-x)^3(1-2x)}$ $(\bar{0}1)^m, a_{m+1}; b_m \rightsquigarrow b_{m+1}, (\bar{0}1)^m,$ where $a_m = 0^m, b_m = 0^m$	
65	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}1, \bar{0}0, \bar{0}0, 10\}$	$\epsilon \rightsquigarrow a_1, b_0; a_m, b_m \rightsquigarrow \frac{2-3x+x\sqrt{1-4x}}{(1-2x)(1+\sqrt{1-4x})}$ $a_{m+1}, b_m, c_m, c_{m-1}, \dots, c_1; b_m \rightsquigarrow$ $b_m, b_{m-1}, \dots, b_0; c_m \rightsquigarrow c_{m+1}, c_m, \dots, c_1,$ where $a_m = 0^m, b_m = a_m 0, c_m = a_m 1$	
66	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{0}1, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow \frac{x^2\sqrt{1-4x}+2x^3-x^2-2x+2}{(1-x)^2(1+\sqrt{1-4x})},$ $\bar{0}, a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \rightsquigarrow$ $010, b_{m+1}, b_m, \dots, b_1,$ where $a_m = 0^m,$ $b_m = a_m 1$	
67	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, \bar{0}0, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_{m+1}a_m, \dots, a_1,$ where $a_m = 0^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{0}1, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow a_{m+1}a_m, \dots, a_1,$ where $a_m = 0^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}0, \bar{0}0, \bar{1}0\}$		
	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}0, \bar{0}0, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m$	$\frac{2}{(1-x)(1+\sqrt{1-4x})}$
68	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}1, \bar{0}0, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_{m+1}, (b_m)^{m+1};$ $b_m \rightsquigarrow (b_m)^{m+1},$ where $a_m = 0^m, b_m = a_m 0$	$\sum_{k \geq 0} \frac{x^k}{1-(k+1)x}$
69	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{0}1, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow \bar{0}, a_{m+1}, (b_m)^m; b_m \rightsquigarrow$ $(b_{m+1})^{m+2},$ where $a_m = 0^m, b_m = a_m 1$	
	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, \bar{0}0; \bar{0}0 \rightsquigarrow \bar{0}0; a_m \rightsquigarrow \frac{x}{(1-x)^2} + \sum_{k \geq 0} k!x^k$ $(a_{m+1})^{m+1},$ where $a_m = 0^m$	
70	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{0}0, 10\}$	$\epsilon \rightsquigarrow 0, a_0; 0 \rightsquigarrow a_1, a_0; a_m \rightsquigarrow$ $a_{m+1}, a_m, 0, \dots, a_0,$ where $a_m = 0\bar{0}^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow a_1, a_2; a_m \rightsquigarrow$ $a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow a_0, a_1; a_m \rightsquigarrow$ $a_0, a_1, \dots, a_{m+1},$ where $a_m = 0\bar{0}^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}1, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow b_1, a_1; a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_1; b_m \rightsquigarrow$ $b_1, b_2, \dots, b_{m+1},$ where $a_m = 0^m, b_m = 0^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, b_0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow$ $b_1, b_2, \dots, b_m, a_{m+1}, b_m,$ where $a_m = 0^m,$ $b_m = 01^m$	
	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow a_2, a_1; a_m \rightsquigarrow 2C(x) - 1$ $a_{m+1}, a_m, \dots, a_1,$ where $a_m = 0^m$	
71	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow \frac{1-x-2x^2-2x^3+(1+x)\sqrt{1-4x}}{1-3x-x^2+(1-x)x^2\sqrt{1-4x}}$ $a_1, a_2, \dots, a_{m+1}, 0, b_2, b_3, \dots, b_m; b_m \rightsquigarrow$ $0, c_2, c_3, \dots, c_m; c_m \rightsquigarrow c_2, c_3, \dots, c_{m-1},$ where $a_m = 0^m, b_m = a_m m,$ $c_m = b_m(m-1)$	
72	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow \frac{1-x-2x^2-(1-x)\sqrt{1-4x}}{(1-x)(1-3x+(1-x)\sqrt{1-4x})}$ $a_1, a_2, \dots, a_{m+1}, 0, b_2, b_3, \dots, b_m;$ $b_m \rightsquigarrow 0, b_2, b_3, \dots, b_m,$ where $a_m = 0^m$	
73*	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{0}0, 10\}$		
74*	$\{\bar{1}0, \bar{1}0, \bar{0}1, \bar{0}0, \bar{0}1, \bar{0}0, 10\}$		
75*	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, \bar{0}0, 10\}$		
76	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}1, \bar{0}0, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_1 \rightsquigarrow a_1, a_2, 0; a_m \rightsquigarrow \frac{1}{1-x} + \sum_{k \geq 0} \frac{x^k C^k(x)}{1-kx}$ $a_1, a_2, \dots, a_{m+1}, (b_m)^m; b_m \rightsquigarrow (b_m)^m,$ where $a_m = 0^m, b_m = a_m 1$	
77	$\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, \bar{0}1, \bar{0}0, \bar{1}0\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow (a_{m+1})^{m+1},$ where $a_m = 0^m$	$\sum_{k \geq 0} \sum_{j=0}^k j!x^k$

78	$\{\bar{1}0, \bar{1}0, 0\bar{1}, 0\bar{0}, 00, 01, 1\bar{0}\}$	$\epsilon \sim a_1, 0; a_m \sim \frac{(1+x)C^2(x)}{1+xC(x)}$ $a_1, a_2, \dots, a_{m+1}, b_0, b_1, \dots, b_m;$ $b_m \sim b_0, b_1, \dots, b_{m-1},$ where $a_m = 0^m,$ $b_m = a_m m$
79	$\{\bar{1}0, \bar{1}0, 0\bar{1}, 0\bar{0}, 00, 01, 1\bar{0}\}$	$\epsilon \sim a_1, b_0; a_m \sim (1+x+x^2C^3(x))C(x)$ $a_1, a_2, \dots, a_{m+1}, b_0, b_1, \dots, b_m;$ $b_m \sim b_0, b_1, \dots, b_m,$ where $a_m = 0^m,$ $b_m = a_m m$
80	$\{\bar{1}0, \bar{1}0, \bar{0}0, 0\bar{1}, 00, 01, 1\bar{0}\}$	$\epsilon \sim a_1, 0; 0 \sim 0; a_m \sim \frac{(1-x-x^2)C^2(x)}{1-x}$ $a_1, a_2, \dots, a_{m+1}, b_1, b_2, \dots, b_m;$ $b_m \sim b_0, b_1, \dots, b_m,$ where $a_m = 0^m$
81*	$\{10, 10, 0\bar{1}, 00, 00, 01, 10\}$	
82	$\{\bar{1}0, \bar{1}0, \bar{0}0, 0\bar{1}, 0\bar{0}, 01, 1\bar{0}\}$	$\epsilon \sim a_1, 0; 0 \sim 0; a_m \sim \frac{x}{1-x} + (1+x^2C^4(x))C(x)$ $a_1, a_2, \dots, a_{m+1}, b_1, b_2, \dots, b_m;$ $b_m \sim b_1, b_2, \dots, b_m, b_m,$ where $a_m = 0^m,$ $b_m = a_m m$
83*	$\{10, 10, 00, 00, 00, 01, 10\}$	
84*	$\{10, 10, 01, 00, 00, 01, 10\}$	
85*	$\{10, 10, 01, 00, 00, 00, 10\}$	
86	$\{\bar{1}0, \bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{1}, 0\bar{0}, 1\bar{0}\}$	$\epsilon \sim a_1, b_1; b_m \sim b_{m+1}, b_m, \dots, b_1; a_m \sim \frac{1-x-\sqrt{1-6x+x^2}}{2} + C(x)$ $a_{m+1}, a_{m+1}, a_m, \dots, a_2,$ where $a_m = 0^m,$ $b_m = 0^m$
87*	$\{10, 10, 00, 01, 00, 00, 10\}$	
88	$\{\bar{1}0, \bar{1}0, \bar{0}0, 0\bar{1}, 0\bar{1}, 0\bar{0}, 1\bar{0}\}$	$\epsilon \sim a_1, b_1; b_m \sim (b_{m+1})^{m+1}; a_m \sim \sum_{k \geq 0} k!x^k + C(x) - 1$ $a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m, b_m = 0^m$
89	$\{\bar{1}0, \bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{1}, 0\bar{0}, 1\bar{0}\}$ $\{\bar{1}0, \bar{1}0, 0\bar{1}, \bar{0}0, 0\bar{1}, 1\bar{0}\}$	$\epsilon \sim a_1, b_1; b_m \sim (b_{m+1})^{m+1}; a_m \sim \frac{a_{m+1}, (b_m)^m,}{(b_1)^2}; a_m \sim b_{m-1}, a_{m+1}, (b_m)^m; b_m \sim (b_{m+1})^{m+2},$ where $a_m = 0^m, b_m = a_m 1$ $a_{m+1}, (b_m)^m,$ where $a_m = 0^m, b_m = 0^m$ $\bar{0} \sim 0; a_1 \sim 0\bar{0}, a_2, b_1; \bar{0} \sim 0\bar{0}; 0\bar{0} \sim \sum_{k \geq 0} k!x^k(1+x \sum_{j=0}^k \frac{1}{j!})$
90	$\{10, 10, 0\bar{1}, 00, 01, 00, 10\}$ $\{10, 10, 0\bar{1}, 00, 01, 00, 10\}$	$\epsilon \sim 0, a_1; 0 \sim 0; a_m \sim (0)^m, a_{m+1}, b_{m,1}, b_{m,2}, \dots, b_{m,m};$ $b_{m,j} \sim (0)^{m+1-j}, b_{m+1,j}, b_{m+1,j},$ where $b_{m+1,j+1}, \dots, b_{m+1,m+1},$ $a_m = 0^m, b_{m,j} = a_m j$
91	$\{10, 10, 0\bar{1}, 00, 00, 01, 10\}$ $\{\bar{1}0, \bar{1}0, \bar{0}0, \bar{0}0, 0\bar{1}, 0\bar{0}, 1\bar{0}\}$	$\epsilon \sim 0, a_1; a_m \sim (0)^{m+1}, a_{m+1}, b_{m,1}, b_{m,2}, \dots, b_{m,m};$ $b_{m,j} \sim (0)^{m+2-j}, (b_{m+1,j})^{j+1},$ where $b_{m+1,j+1}, \dots, b_{m+1,m+1},$ $a_m = 0^m, b_{m,j} = a_m j$ $\bar{0}, b_2, b_3, \dots, b_m, a_{m+1}, c_{m,1}, \dots, c_{m,m};$ $b_m \sim 0, b_2, b_3, \dots, b_m; c_{m,j} \sim \bar{0}, b_2, b_3, \dots, b_{m+1-j}, (c_{m+1,j})^{j+1},$ where $c_{m+1,j+1}, \dots, c_{m+1,m+1},$ $a_m = 0^m, b_m = a_m \bar{1}, c_{m,j} = a_m j$
92	$\{\bar{1}0, \bar{1}0, \bar{0}0, 0\bar{0}, 00, 01, 1\bar{0}\}$	$\epsilon \sim a_1, b_0; b_m \sim b_1, b_2, \dots, b_{m+1}; a_m \sim \frac{1-3x-3x^2}{1-4x} C(x),$ $a_1, a_2, \dots, a_{m+1}, (b_m)^m,$ where $a_m = 0^m,$ $b_m = 0\bar{1}^m$
93*	$\{10, 10, 0\bar{1}, 00, 00, 00, 10\}$	
94	$\{10, 10, 0\bar{1}, 00, 0\bar{1}, 0\bar{0}, 1\bar{0}\}$	$\epsilon \sim 0, a_1; \bar{0} \sim (a_2)^2; a_m \sim (a_{m+1})^{m+1}, 1 + 2 \sum_{k \geq 1} k!x^k$ $\text{where } a_m = 0^m$
95	$\{\bar{1}0, \bar{1}0, 0\bar{1}, \bar{0}0, 01, 1\bar{0}, 1\bar{0}\}$	$\epsilon \sim 0, a_1; \bar{0} \sim (0\bar{0})^2; 0\bar{0} \sim 0\bar{0}; a_m \sim \frac{1+2x^2-x^3}{(1-x)^2}$ $(0\bar{1})^{m+1}, a_{m+1},$ where $a_m = 0^m$
96	$\{\bar{1}0, \bar{1}0, 0\bar{1}, 0\bar{1}, 00, 1\bar{0}, 1\bar{0}\}$	$\epsilon \sim 0, 0; 0 \sim 0, 0\bar{0}, 0; \bar{0} \sim 0, 0\bar{0}; 0\bar{0} \sim 0\bar{0}$ $\frac{1-x+2x^2-x^3}{(1-x)^3}$
97	$\{\bar{1}0, \bar{1}0, \bar{0}0, 0\bar{1}, 0\bar{1}, 1\bar{0}, 1\bar{0}\}$	$\epsilon \sim 0, a_1; \bar{0} \sim (0\bar{0})^2; 0\bar{0} \sim 0\bar{0}; \frac{x(1+x-x^2)}{(1-x)^2} + C(x)$ $a_m \sim 0\bar{0}, a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \sim b_{m+1}, b_m, \dots, b_1,$ where $a_m = 0^m, b_m = a_m \bar{1}$
98	$\{10, 10, 0\bar{1}, 00, 00, 10, 10\}$ $\{\bar{1}0, \bar{1}0, \bar{0}0, 0\bar{1}, 00, 10, 10\}$ $\{\bar{1}0, \bar{1}0, 0\bar{1}, \bar{0}0, 01, 1\bar{0}, 1\bar{0}\}$	$\epsilon \sim 0, 0; 0 \sim 0, 0, 0; 0 \sim 0, 0\bar{1}; 0\bar{1} \sim 0\bar{1}$ $\bar{0} \sim 0, 0; 0 \sim 0, 0, 0; \bar{0} \sim 0\bar{1}, 0; 0\bar{1} \sim 0\bar{1}$ $\epsilon \sim b_1, a_1; 0\bar{1} \sim 0\bar{1}; a_m \sim (0\bar{1})^{m+1}, a_{m+1};$ $b_m \sim b_{m+1}, (0\bar{1})^m,$ where $a_m = 0^m, b_m = 0^m$ $\frac{1-x+2x^2}{(1-x)^3}$

99	{10, 10, 01, 01, 01, 10, 10}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (0)^2; 0\bar{1} \rightsquigarrow 0\bar{1}; a_m \rightsquigarrow \frac{1-3x+4x^2-3x^3}{(1-x)^3(1-2x)}$ $(0\bar{1})^2, \bar{0}, a_{m+1}, \text{ where } a_m = 0^m$
100	{10, 10, 01, 00, 00, 10, 10}	$\epsilon \rightsquigarrow a_1, 0; b_0 \rightsquigarrow a_1, 0; 0\bar{1} \rightsquigarrow 0\bar{1};$ $a_m \rightsquigarrow a_{m+1}, b_m, b_{m-1}, \dots, b_1, \bar{0}\bar{1}; b_m \rightsquigarrow$ $b_m, b_{m-1}, \dots, b_1, \bar{0}\bar{1}, \text{ where } a_m = \bar{0}^m,$ $b_m = a_m \bar{0}$
	{10, 10, 01, 00, 00, 10, 10}	$\epsilon \rightsquigarrow \bar{0}, a_0; \bar{0} \rightsquigarrow a_1, \bar{0}\bar{1}; 0\bar{1} \rightsquigarrow \bar{0}\bar{1};$ $a_m \rightsquigarrow a_{m+1}, b_m, b_{m-1}, \dots, b_1, \bar{0}\bar{1}; b_m \rightsquigarrow$ $b_m, b_{m-1}, \dots, b_1, \bar{0}\bar{1}, \text{ where } a_m = 0\bar{0}^m,$ $b_m = a_m \bar{1}$
101	{10, 10, 01, 01, 00, 10, 10}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow b_1, 0\bar{1}; 0\bar{1} \rightsquigarrow 0\bar{1};$ $b_m \rightsquigarrow b_{m+1}, c_m, c_{m-1}, \dots, c_1, 0\bar{1};$ $c_m \rightsquigarrow c_m, c_{m-1}, \dots, c_1, 0\bar{1}; a_m \rightsquigarrow$ $a_{m+1}, b_m, c_{m-1}, c_{m-2}, \dots, c_1, 0\bar{1}, \text{ where}$ $a_m = \bar{0}^m, b_m = 0a_m, c_m = b_m \bar{1}$
102	{10, 10, 01, 01, 01, 10, 10}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow (0)^2; a_m \rightsquigarrow \frac{x(2-3x+2x^2)}{(1-2x)(1-x)} + C(x)$ $\bar{0}, a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \rightsquigarrow$ $b_{m+1}, b_m, \dots, b_1, \text{ where } a_m = 0^m,$ $b_m = a_m \bar{1}$
103	{10, 10, 00, 01, 01, 10, 10}	$\epsilon \rightsquigarrow a_1, b_1; 0\bar{0} \rightsquigarrow 0\bar{0}; b_m \rightsquigarrow 2C(x) - \frac{1-2x}{(1-x)^2}$ $0\bar{0}, b_{m+1}, c_m, c_{m-1}, \dots, c_1; c_m \rightsquigarrow$ $c_{m+1}, c_m, \dots, c_1; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1},$ $\text{ where } a_m = \bar{0}^m, b_m = 0^m, c_m = b_m \bar{1}$
104	{10, 10, 01, 00, 01, 10, 10}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, \bar{0}\bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}; a_m \rightsquigarrow$ $(0)^m, a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = 0^m$
	{10, 10, 01, 00, 01, 10, 10}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}, a_2, a_1; a_m \rightsquigarrow$ $a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = 0^m$
	{10, 10, 00, 00, 01, 10, 10}	$\epsilon \rightsquigarrow 0, a_1; \bar{0} \rightsquigarrow \bar{0}, a_1; b_m \rightsquigarrow b_{m+1}, b_m, \dots, b_0;$ $a_m \rightsquigarrow b_{m-1}, a_{m+1}, b_{m-1}, b_{m-2}, \dots, b_0,$ $\text{ where } a_m = 0^m, b_m = 0\bar{0}\bar{1}^m$
105	{10, 10, 01, 00, 01, 10, 10}	$\epsilon \rightsquigarrow \bar{0}, a_1; 0 \rightsquigarrow b_0, 0; b_m \rightsquigarrow$ $b_{m+1}, b_m, \dots, b_0, \bar{0}; a_m \rightsquigarrow$ $b_{m-1}, a_{m+1}, b_{m-2}, \dots, b_0, \bar{0}, \text{ where}$ $a_m = 0^m, b_m = 0\bar{0}a_m$
	{10, 10, 01, 00, 01, 10, 10}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow b_1, \bar{0}; b_m \rightsquigarrow$ $b_{m+1}, b_m, \dots, b_1, \bar{0}; a_m \rightsquigarrow$ $b_m, a_{m+1}, b_{m-1}, b_{m-2}, \dots, b_1, \bar{0}, \text{ where}$ $a_m = 0^m, b_m = 0\bar{0}^m$
	{10, 10, 00, 00, 01, 10, 10}	$\epsilon \rightsquigarrow b_0, a_1; b_m \rightsquigarrow b_0, b_1, \dots, b_m;$ $a_m \rightsquigarrow b_0, b_1, \dots, b_m, a_{m+1}, a_m, \dots, a_1,$ $\text{ where } a_m = 0^m, b_m = a_m \bar{0}$
	{10, 10, 01, 01, 00, 10, 10}	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1, 0; a_m \rightsquigarrow$ $a_{m+1}, a_m, \dots, a_1, 0, \text{ where } a_m = \bar{0}^m$
	{10, 10, 00, 00, 00, 10, 10}	$\epsilon \rightsquigarrow \bar{0}, b_0; b_0 \rightsquigarrow \bar{0}, a_0, b_0; \bar{0} \rightsquigarrow \bar{0}, b_0;$ $a_m \rightsquigarrow \bar{0}, a_0, a_1, \dots, a_m, b_{m+1}, b_m, \dots, b_0;$ $b_m \rightsquigarrow \bar{0}, a_0, a_1, \dots, a_m, b_m, b_{m-1}, \dots, b_0,$ $\text{ where } b_m = 0\bar{0}\bar{1}\bar{1} \dots (m-1)\bar{m} - \bar{1}m, a_m =$ $b_m \bar{m}$
	{10, 10, 00, 01, 00, 10, 10}	$\epsilon \rightsquigarrow b_0, a_1; a_m \rightsquigarrow C^2(x)$ $b_0, b_1, \dots, b_{m-1}, a_{m+1}, a_m, \dots, a_1; c_m \rightsquigarrow$ $c_0, c_1, \dots, c_m; b_m \rightsquigarrow c_0, c_1, \dots, c_m, b_{m+1},$ $\text{ where } a_m = 0^m, b_m = 0a_m, c_m = b_m \bar{1}$
106	{10, 10, 01, 00, 00, 10, 10}	$\epsilon \rightsquigarrow b_0, a_1; a_m \rightsquigarrow$ $b_0, b_1, \dots, b_{m-1}, a_{m+1}, a_m, \dots, a_1;$ $b_m \rightsquigarrow b_{m+1}, b_m, \dots, b_0, \text{ where } a_m = 0^m,$ $b_m = 0a_m$
	{10, 10, 00, 01, 00, 10, 10}	$\epsilon \rightsquigarrow a_1, b_1; b_m \rightsquigarrow$ $a_1, a_2, \dots, a_m, b_{m+1}, b_m, \dots, b_1;$ $a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, \text{ where } a_m = \bar{0}^m,$ $b_m = 0^m$
	{10, 10, 01, 00, 00, 10, 10}	$\epsilon \rightsquigarrow a_1, b_1; b_m \rightsquigarrow$ $a_1, a_2, \dots, a_m, b_{m+1}, b_m, \dots, b_1;$ $a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_1, \text{ where } a_m = \bar{0}^m,$ $b_m = 0^m$
	{10, 10, 01, 00, 00, 10, 10}	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow b_0; b_m \rightsquigarrow$ $b_{m+1}, b_m, \dots, b_0; a_m \rightsquigarrow$ $\bar{0}, b_0, b_1, \dots, b_{m-1}, a_{m+1}, a_m, \dots, a_1,$ $\text{ where } a_m = 0^m, b_m = 0\bar{0}\bar{1}^m$

$$\begin{aligned}
 107 \quad \{\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{0}, \bar{1}\bar{0}, 10\} \quad \epsilon \rightsquigarrow \bar{0}, a_1; \quad \bar{0} \rightsquigarrow \bar{0}, a_1; \quad b_m \rightsquigarrow \frac{3}{2} - \frac{1}{2}\sqrt{\frac{1-5x}{1-x}} \\
 \bar{0}, b_2, b_3, \dots, b_m, a_m, a_{m-1}, \dots, a_1; \\
 a_m \rightsquigarrow \bar{0}, b_2, b_3, \dots, b_m, a_{m+1}, a_m, \dots, a_1, \\
 \text{where } a_m = 0^m, b_m = a_m \bar{1}
 \end{aligned}$$

Table 8: Succession rules of $\mathcal{T}[B]$ and generating function $F_B(x)$, where $|B| = 7$.

Theorem 18. *We have that $A \sim B$, where*

$$\begin{aligned}
 A &= \{\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{1}\bar{0}\}, \\
 B &= \{\bar{1}\bar{0}, \bar{1}0, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{1}\bar{0}\}.
 \end{aligned}$$

Proof. Note that any signed inversion sequence $\pi \in \bar{\mathcal{I}}_n(A)$ satisfies either (1) it has no barred letters or (2) there exists a such that $\pi = \pi'\bar{a}$ where each letter of π' is unbarred and smaller or equal to a . Also, any signed inversion sequence $\theta \in \bar{\mathcal{I}}_n(B)$ satisfies one of the following conditions: (1') θ has no barred letters; or (2') there exists an integer $s \geq 1$ such that $\theta = \theta'\bar{a}_1\bar{a}_2 \cdots \bar{a}_s$, where $a_1 < a_2 < \cdots < a_s$, every letter of θ' is unbarred, and every letter of θ' is smaller than a_1 .

Now, we define a map $\alpha : \bar{\mathcal{I}}_n(A) \rightarrow \bar{\mathcal{I}}_n(B)$ as follows. In Case (1), we define $\alpha(\pi) = \pi$. In Case (2), we define $\alpha(\pi) = \alpha(\pi'\bar{a}) = \pi^{(1)} \cdots \pi^{(s)} \bar{a}i_1 + \bar{1} \cdots i_{s-1} + \bar{1}$, where $\pi' = \pi^{(1)}a \cdots \pi^{(s-1)}a\pi^{(s)}$ such that any letter in $\pi^{(j)}$ is smaller than a and the positions of the letter a in π' are $i_1 < i_2 < \cdots < i_{s-1}$ with $s \geq 1$. Thus, by Cases (1') and (2'), we have that $\pi \in \bar{\mathcal{I}}_n(A)$ if and only if $\alpha(\pi) \in \bar{\mathcal{I}}_n(B)$. Hence, α is a bijection, which implies that $A \sim B$. \square

Case $k = 8$

No	B	Succession rules of $\mathcal{T}[B]$	Generating function $F_B(x)$
1	$\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, 0$	$1 + 2x$
2	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}\}$ $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}$	$1 + 2x + x^2$
3	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0\}$ $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{1}\bar{0}\}$ $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{1}\bar{0}\}$ $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{1}\bar{0}\}$ $\{\bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{1}\bar{0}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0$	$\frac{1}{1-x} + x$
4	$\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}\}$ $\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$ $\{\bar{1}\bar{0}, \bar{1}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}\bar{0}; \bar{0} \rightsquigarrow \bar{0}\bar{0}$ $\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}, \bar{0}$	

	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}1, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; a_m \rightsquigarrow (\bar{0})^{m+1}, a_{m+1},$ where $a_m = 0^m$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}1, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow (0\bar{1})^m, a_{m+1},$ where $a_m = 0^m$
21	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}1, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow \frac{1-x^2-x^3}{(1-x)(1-x-x^2)}$ $a_{m+1}, b_1, b_2, \dots, b_m; b_m \rightsquigarrow b_1, b_2, \dots, b_{m-1},$ where $a_m = 0^m,$ $b_m = a_m m$
22	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow \bar{0}\bar{0}, \bar{0}; \bar{0} \rightsquigarrow \bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}\bar{0}$ $\frac{1-x-x^2-x^3}{(1-x)(1-2x)}$
23	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}0, \bar{0}\bar{1}, 00, 10\}$	$\epsilon \rightsquigarrow 0, 0; 0 \rightsquigarrow \bar{0}\bar{0}, 0; \bar{0} \rightsquigarrow \bar{0}; \bar{0}\bar{0} \rightsquigarrow \bar{0}\bar{0}, \bar{0}$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, 00, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_{m+1}, (0)^m,$ where $a_m = 0^m$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}1, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow (\bar{0})^m, a_{m+1},$ where $a_m = 0^m$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; \bar{0}\bar{1} \rightsquigarrow \bar{0}\bar{1}; a_m \rightsquigarrow a_{m+1}, 0, (0\bar{1})^m,$ $\frac{1-x+x^3}{(1-x)^3}$ where $a_m = 0^m$
24	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 01, 10\}$	$\epsilon \rightsquigarrow a_1, b_0; a_m \rightsquigarrow a_{m+1}, b_0, b_1, \dots, b_m;$ $\frac{1+x}{1-x-x^2}$ $b_m \rightsquigarrow b_0, b_1, \dots, b_{m-1},$ where $a_m = 0^m,$ $b_m = 0^m$
25	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}1, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_{m+1}, 0, b_2, b_3, \dots, b_m; b_m \rightsquigarrow 0, b_2, b_3, \dots, b_m,$ where $a_m = 0^m,$ $b_m = 0^m m$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 10\}$	$\epsilon \rightsquigarrow \bar{0}, 0; 0 \rightsquigarrow 0; \bar{0} \rightsquigarrow \bar{0}, \bar{0}$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}1, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; b_m \rightsquigarrow \bar{0}, b_2, b, \dots, b_m;$ $\frac{1-x-x^2}{(1-x)(1-2x)}$ $a_m \rightsquigarrow \bar{0}, b_2, b_3, \dots, b_m, a_{m+1},$ where $a_m = 0^m, b_m = 0^m \bar{1}$
26	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_1 \rightsquigarrow a_2, 0; a_m \rightsquigarrow \frac{1}{1-x} + \sum_{j \geq 1} \frac{x^j}{1-jx}$ $a_{m+1}, (b_m)^m; b_m \rightsquigarrow (b_m)^m,$ where $a_m = 0^m, b_m = 0^m \bar{1}$
27	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; \bar{0}\bar{1} \rightsquigarrow (\bar{0}\bar{1})^2; a_m \rightsquigarrow \frac{1-2x+2x^3}{(1-x)^2(1-2x)}$ $a_{m+1}, (0\bar{1})^m,$ where $a_m = 0^m$
28	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 00, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; b_m \rightsquigarrow b_1, b_2, \dots, b_{m+1}; a_m \rightsquigarrow b_1, b_2, \dots, b_m, a_{m+1}, 0,$ where $a_m = 0^m, b_m = 0^m \bar{1}$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow 0; a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_1,$ where $a_m = 0^m$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}1, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow a_{m+1}, a_m, \dots, a_1, C(x) + \frac{x}{1-x}$ where $a_m = 0^m$
29	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, 01, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$	$\epsilon \rightsquigarrow a_1, 0; 0 \rightsquigarrow a_1; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1},$ where $a_m = 0^m$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}1, \bar{0}\bar{1}, 10\}$	$\epsilon \rightsquigarrow a_1, 0; a_m \rightsquigarrow a_1, a_2, \dots, a_{m+1}, 0, (1+x)C(x)$ where $a_m = 0^m$
30	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{1}, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}\bar{1}, \bar{0}\bar{1}, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; a_m \rightsquigarrow \bar{0}, a_{m+1}, (b_m)^m; b_m \rightsquigarrow (b_{m+1})^{m+2},$ where $a_m = 0^m, b_m = 0^m \bar{1}$
	$\{\bar{1}0, \bar{1}0, \bar{0}\bar{0}, \bar{0}\bar{0}, \bar{0}1, \bar{0}\bar{1}, \bar{0}\bar{0}, 10\}$	$\epsilon \rightsquigarrow \bar{0}, a_1; \bar{0} \rightsquigarrow \bar{0}; a_m \rightsquigarrow (a_{m+1})^{m+1}, \frac{x}{1-x} + \sum_{n \geq 0} n! x^n$ where $a_m = 0^m$

31	{1̄0, 1̄0, 0̄0, 0̄1, 0̄0, 00, 01, 1̄0}	$\epsilon \sim a_1, 0; a_m \sim x + \frac{1-x-(1+x)\sqrt{1-4x}}{2x^2(x+2)}$ $a_1, a_2, \dots, a_{m+1}, 0, b_2, b_3, \dots, b_m;$ $b_m \sim 0, b_2, b_3, \dots, b_{m-1},$ where $a_m = 0^m, b_m = 0^m m$
32	{1̄0, 1̄0, 0̄1, 0̄0, 0̄0, 01, 0̄0, 10}	$\epsilon \sim 0, 0; a_m \sim \frac{x^2+1}{2x} - \frac{1-4x-x^2}{2x\sqrt{1-4x}}$ $(0)^m, a_{m+1}, a_m, b_{m,2}, b_{m,3}, \dots, b_{m,m};$ $b_{m,j} \sim (0)^m, b_{m+1,j}, b_{m,j}, \dots, b_{m,m},$ where $a_m = 0^m, b_{m,j} = a_m j$
33	{1̄0, 1̄0, 0̄1, 0̄0, 0̄0, 01, 0̄0, 1̄0}	$\epsilon \sim 0, a_1; \bar{0} \sim a_1; a_m \sim (a_{m+1})^{m+1}, (1+x)\sum_{n \geq 0} n!x^n$ where $a_m = 0^m$
34	{1̄0, 1̄0, 0̄0, 0̄1, 0̄0, 00, 01, 10}	$\epsilon \sim a_1, 0; a_m \sim 1+x+x^3(x)$ $a_1, a_2, \dots, a_{m+1}, b_1, b_2, \dots, b_m;$ $b_m \sim b_1, b_2, \dots, b_m,$ where $a_m = 0^m,$ $b_m = 0^m m$
35*	{1̄0, 10, 01, 00, 00, 01, 00, 10}	$\epsilon \sim 0, a_1; a_m \sim$ $(0)^m, a_{m+1}, b_{m,1}, \dots, b_{m,m}; b_{m,j} \sim$ $(0)^{m+1-j}, (b_{m+1,j})^{j+1}, b_{m+1,j+1},$ $b_{m+1,j+2}, \dots, b_{m+1,m+1},$ where $a_m = 0^m, b_{m,j} = 0^m j$
36	{1̄0, 1̄0, 0̄1, 0̄0, 00, 01, 1̄0, 10}	$\epsilon \sim 0, 0; 0 \sim 0\bar{1}, 0\bar{1}; \bar{0} \sim 0\bar{1}, 0\bar{1}; 0\bar{1} \sim 1+2x+4x^2$
37	{1̄0, 1̄0, 0̄0, 0̄1, 0̄1, 00, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0\bar{0}, 0\bar{1}; \bar{0} \sim 0\bar{1}, 0\bar{0}; 0\bar{1} \sim 0\bar{1}$ $1+2x^2 + \frac{2x}{1-x}$
38	{1̄0, 1̄0, 0̄1, 0̄0, 0̄1, 00, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0, 0\bar{0}, 0; \bar{0} \sim 0\bar{0}$ $1+2x + \frac{4x^2}{1-x}$
39	{1̄0, 10, 01, 01, 01, 00, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0\bar{0}, 0\bar{0}; \bar{0} \sim 0, 0\bar{0}; 0\bar{0} \sim 0\bar{0}$
	{1̄0, 10, 01, 00, 00, 01, 10, 10}	$\epsilon \sim 0, a_1; \bar{0} \sim (0\bar{0})^2; 0\bar{0} \sim 0\bar{0}, a_m \sim$ $(0\bar{1})^m, a_{m+1},$ where $a_m = 0^m$
	{1̄0, 10, 01, 00, 01, 01, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0\bar{0}, 0; \bar{0} \sim 0\bar{0}, 0\bar{0}; 0\bar{0} \sim 0\bar{0}$
	{1̄0, 10, 01, 00, 00, 01, 10, 10}	$\epsilon \sim 0, a_1; \bar{0} \sim (0\bar{0})^2; 0\bar{0} \sim 0\bar{0}; a_m \sim \frac{1+x^2-x^3}{(1-x)^2}$ $(0\bar{1})^m, a_{m+1},$ where $a_m = 0^m$
40	{1̄0, 10, 01, 00, 01, 00, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0, 0, 0; \bar{0} \sim 0$
	{1̄0, 10, 01, 00, 01, 00, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0, 0\bar{1}; \bar{0} \sim 0, 0\bar{1}; 0\bar{1} \sim 0\bar{1}$
	{1̄0, 10, 01, 01, 00, 00, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0, 0; \bar{0} \sim 0, 0\bar{0}$
	{1̄0, 10, 01, 00, 01, 01, 10, 10}	$\epsilon \sim 0, a_1; \bar{0} \sim 0; a_m \sim (0\bar{1})^m, \bar{0}, a_{m+1},$ $\frac{1+x^2}{(1-x)^2}$ where $a_m = 0^m$
41	{1̄0, 10, 00, 00, 01, 00, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0, 0\bar{0}, 0; \bar{0} \sim 0; 0\bar{0} \sim 0, 0\bar{0}$
	{1̄0, 10, 01, 00, 01, 01, 10, 10}	$\epsilon \sim a_1, 0; 0 \sim 0\bar{0}, 0; 0\bar{0} \sim 0\bar{0}; a_m \sim$ $a_{m+1}, (0\bar{0})^m,$ where $a_m = 0^m$
	{1̄0, 10, 01, 00, 00, 00, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0, 0\bar{0}, 0; \bar{0} \sim 0; 0\bar{0} \sim 0\bar{0}, \bar{0}$
	{1̄0, 10, 00, 00, 01, 00, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0, 0\bar{1}; \bar{0} \sim 0, 0; 0\bar{1} \sim 0\bar{1}$
	{1̄0, 10, 01, 00, 01, 01, 10, 10}	$\epsilon \sim 0, a_1; \bar{0} \sim 0; a_m \sim (0)^{m+1}, a_{m+1},$ where $a_m = 0^m$
	{1̄0, 10, 01, 00, 00, 00, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0, 0; \bar{0} \sim 0, 0\bar{1}; 0\bar{1} \sim 0\bar{1}$ $\frac{1-x+x^2}{(1-x)^3}$
42	{1̄0, 10, 00, 01, 01, 00, 10, 10}	$\epsilon \sim 0, a_1; \bar{0} \sim (0\bar{1})^2; 0\bar{1} \sim 0\bar{1}; a_m \sim$ $a_{m+1}, a_m, \dots, a_1,$ where $a_m = 0^m$
	{1̄0, 10, 00, 00, 01, 01, 10, 10}	$\epsilon \sim 0, a_1; \bar{0} \sim 0; a_m \sim C(x) + \frac{2x^2}{1-x}$ $0\bar{0}, a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \sim$ $b_{m+1}, b_m, \dots, b_1,$ where $a_m = 0^m,$ $b_m = 0^m 1$
43	{1̄0, 10, 01, 00, 00, 01, 10, 10}	$\epsilon \sim a_1, b_1; 0\bar{1} \sim 0\bar{1}; b_m \sim (0\bar{1})^m, b_{m+1};$ $\frac{1-x+x^2+x^3}{(1-x)^3}$ $a_m \sim a_{m+1}, (0\bar{1})^m,$ where $a_m = 0^m,$ $b_m = 0^m$
44	{1̄0, 10, 01, 01, 00, 00, 10, 10}	$\epsilon \sim a_1, b_0; a_m \sim$ $a_{m+1}, b_m, b_{m-1}, \dots, b_0; b_m \sim$ $b_m, b_{m-1}, \dots, b_0,$ where $a_m = 0^m,$ $b_m = a_m 0$
	{1̄0, 10, 00, 00, 00, 00, 10, 10}	$\epsilon \sim 0, 0; 0 \sim 0, 0; \bar{0} \sim 0, 0$
	{1̄0, 10, 01, 01, 01, 01, 10, 10}	$\epsilon \sim a_1, 0; 0 \sim a_1, 0; a_m \sim$ $a_{m+1}, b_m, b_{m-1}, \dots, b_1; b_m \sim$ $b_m, b_{m-1}, \dots, b_1,$ where $a_m = 0^m,$ $b_m = 0^m 1$
	{1̄0, 10, 01, 00, 01, 00, 10, 10}	$\epsilon \sim 0, a_0; \bar{0} \sim a_1, b_0; a_m \sim \frac{1}{1-2x}$ $a_{m+1}, b_m, b_{m-1}, \dots, b_0; b_m \sim$ $b_m, b_{m-1}, \dots, b_0,$ where $a_m = 0\bar{0}^m,$ $b_m = a_m 1$

Table 11: Succession rules of $\mathcal{T}[B]$ and generating function $F_B(x)$, where $|B| = 10$.

Case $k = 11$

No	B	Succession rules of $\mathcal{T}[B]$	Generating function $F_B(x)$
1	$\{\bar{1}0, 10, 0\bar{1}, 00, 00, 0\bar{1}, 0\bar{1}, 00, 00, 01, 1\bar{0}\}$ $\{10, 10, 0\bar{1}, 00, 00, 01, 0\bar{1}, 00, 00, 01, 10\}$	$\epsilon \rightsquigarrow 0, 0$	$1 + 2x$
2	$\{10, 10, 0\bar{1}, 00, 00, 01, 0\bar{1}, 00, 01, 10, 10\}$ $\{10, 10, 0\bar{1}, 00, 00, 01, 00, 00, 01, 10, 10\}$	$\epsilon \rightsquigarrow 0, 0; 0 \rightsquigarrow 0$	$1 + 2x + x^2$
3	$\{10, 10, 0\bar{1}, 00, 00, 01, 0\bar{1}, 00, 00, 10, 10\}$ $\{10, 10, 0\bar{1}, 00, 00, 01, 0\bar{1}, 00, 01, 10, 10\}$	$\epsilon \rightsquigarrow 0, 0; 0 \rightsquigarrow 0$	$x + \frac{1}{1-x}$

Table 12: Succession rules of $\mathcal{T}[B]$ and generating function $F_B(x)$, where $|B| = 11$.

Case $k = 12$

No	B	Succession rules of $\mathcal{T}[B]$	Generating function $F_B(x)$
1	$\{\bar{1}0, 10, 0\bar{1}, 00, 00, 0\bar{1}, 0\bar{1}, 00, 00, 01, 1\bar{0}, 10\}$	$\epsilon \rightsquigarrow 0, 0$	$x + 2x^2$

Table 13: Succession rules of $\mathcal{T}[B]$ and generating function $F_B(x)$, where $|B| = 12$.

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