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INTERVALS OF N-EXPANSIONS WITH MANY GAPS

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Abstract

In this paper the number of gaps $G(N,\alpha)$ in N-expansions is investigated for values of α such that α is smaller but very close to the maximum value $\alpha_{\max} = \sqrt{N} - 1$. As α decreases from this maximum value, there is a large volatility of G. It is shown that for some N and α, the maximum value of $G(N,\alpha)$ may be more than 6 times as large as $G(N, \sqrt{N} - 1)$. A method for determining the maximum value of G for all $N \geq 33$ and α smaller than but very close to $\sqrt{N} - 1$ is presented, and the maximum value of G for all N up to $N = 5252$, and α smaller than but very close to $\sqrt{N} - 1$, is given explicitly.

1. Preliminaries

This paper is a follow-up of the investigation of the number of gaps of N-expansions in [1], with $\alpha = \alpha_{\max} = \sqrt{N} - 1$. In this paper we will show that for α close to $\alpha_{\max}$ the number of gaps may be considerably larger. Before we present our findings, we will provide an overview of the most important concepts and notations that were used in [1].

Definition 1. For $N \in \mathbb{N}_{\geq 2}$ and $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq \sqrt{N} - 1$, let $I_\alpha := [\alpha, \alpha+1]$ and $I_\alpha^- := [\alpha, \alpha+1)$. The *$N$-expansion map* $T_\alpha : I_\alpha \to I_\alpha^-$ is defined as

$$T_\alpha(x) := \frac{N}{x} - d(x),$$

where $d : I_\alpha \to \mathbb{N}$ is defined by

$$d(x) := \left\lfloor \frac{N}{x} - \alpha \right\rfloor, \text{ if either } x \in (\alpha, \alpha+1] \text{ or both } x = \alpha \text{ and } \frac{N}{\alpha} - \alpha \notin \mathbb{Z}.$$

If $\frac{N}{\alpha} - \alpha \in \mathbb{Z}$, then

$$d(\alpha) = \left\lfloor \frac{N}{\alpha} - \alpha \right\rfloor - 1, \text{which prevents cylinders from consisting of only one point.}$$

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Definition 2. For $N \in \mathbb{N}_{\geq 2}$ and a fixed $\alpha \in (0, \sqrt{N}-1]$ and $x \in I_\alpha$, the numbers $d_n = d_n(x) := d(T_\alpha^{n-1}(x))$ for $n \in \mathbb{N}$ are called the *partial quotients* or *digits* of the N-expansion T_α. Since $0 \notin I_\alpha$, this expansion is infinite for every $x \in I_\alpha$.

As in [1], we write $x = [d_1, d_2, d_3, \ldots]_{N,\alpha}$. For reasons of legibility, we will usually omit suffixes such as '(N)', '(N, α)' or '(N, d)'. Furthermore, we will usually write β instead of $\alpha + 1$. We write $x_n := T_\alpha^n(x)$, with $n \in \mathbb{N} \cup \{0\}$. The sequence x_n, $n = 0, 1, 2, \ldots$, is called the *orbit of* x *under* T_α. When $x = [d_1, d_2, d_3, \ldots]$ and there are smallest $h, k \in \mathbb{N}$ such that $d_{h+i} = d_{h+nk+i}$ for all $n \in \mathbb{N}$ and $i \in \{0, \ldots, k-1\}$, we call the expansion *eventually periodic* with *period length* $|x| = k$ and denote the periodic part as $\overline{d_h, \ldots, d_{h+k-1}}$. If $h = 1$, we write $x = [\overline{d_1, \ldots, d_k}]$. In this latter case, x is called a *periodic point* with a *purely periodic expansion* and *periodic orbit*.

Let $N \in \mathbb{N}_{\geq 2}$ and $\alpha \in (0, \sqrt{N}-1]$. The interval I_α is divided into *cylinder sets* $\Delta_i := \{x \in I_\alpha; d(x) = i\}$ *of rank* 1, with $d_{\min} \leq i \leq d_{\max}$, where $d_{max} := d(\alpha)$ is the largest partial quotient[1] and $d_{min} := d(\beta)$ the smallest one. In each of these cylinder sets the map T_α obviously has one *fixed point* f_i. For these fixed points f_i we have $N/f_i - i = f_i$, so $f_i = (\sqrt{4N + i^2} - i)/2$, for $d_{\min} \leq i \leq d_{\max}$. Note that $N/\alpha - \alpha \in \mathbb{Z}$ if and only if for some $d \geq 2$ we have that $d + 1 = \max\{d_i\}$ for any $\alpha_0 < \alpha$, i.e., $\Delta_{d+1} \neq \emptyset$ for $\alpha_0 < \alpha$, and $\alpha = f_{d+1}$. Moreover, we can write $f_i = [\overline{i}]$. Note that fixed points are periodic points with period length 1.

Each pair of consecutive cylinder sets (Δ_i, Δ_{i-1}) is separated by a *discontinuity point* $p_i(N, \alpha)$ of T_α, satisfying $N/p_i - i = \alpha$, so $p_i = N/(\alpha + i)$. An interval I_α together with its cylinder sets, associated fixed points and discontinuity points, is called an *arrangement* of I_α, depending on N. As in [2] and in [1], we will make a lot of use of the graphs of T_α, which are drawn in the square[2] $\Upsilon_{N,\alpha} := I_\alpha \times I_\alpha^-$. This square is divided into rectangular sets of points $\square_i := \{(x, y) \in \Upsilon_\alpha : d(x) = i\}$. We identify these two-dimensional *fundamental regions* $\square_i$ with the one-dimensional cylinder sets Δ_i we already mentioned. It is obvious that the graph of T_α has one fixed point $F_i := (f_i, f_i)$ in each $\square_i$. The dividing line between $\square_i$ and $\square_{i-1}$ is the line segment $\{p_i\} \times [\alpha, \alpha + 1)$, where p_i is the discontinuity point between Δ_i and Δ_{i-1}. Finally, we will use the word "arrangement" in a similar way for Υ_α together with its cylinder sets, fixed points and dividing lines as for I_α.

Given N, we let $\alpha_{max} = \sqrt{N} - 1$ be the largest value of α we consider, so as to avoid 0 being a partial quotient as well. Since $T'_\alpha(x) = -N/x^2$ and because $0 < \alpha \leq \sqrt{N} - 1$, we have $|T'_\alpha(x)| > 1$ on I_α^-. From this, it follows that the fixed points act as *repellers* and that the maps T_α are *expanding* when $0 < \alpha \leq \sqrt{N} - 1$.

As a follow-up to [1] and [2], this paper is a study of *gaps*, defined as follows.

Definition 3. A maximal open interval $(a, b) \subset I_\alpha$ is called a *gap* of[3] I_α if for

[1] Note that the number of occurring digits is finite for $\alpha > 0$.
[2] We have $\Upsilon_{N,\alpha} := I_\alpha \times I_\alpha$ in case $N/\alpha - \alpha \in \mathbb{Z}$.
[3] We will usually omit the addition "of I_α".

almost every[4] $x \in I_\alpha$ there is an $n_0(x) \in \mathbb{N}$ such that $x_n \notin (a,b)$ for all $n \geq n_0$.

The union of gapless intervals of I_α is called the *attractor* of I_α, denoted by A_α. The maximal intervals constituting A_α are called the *components* of A_α. When studying gaps and attractors, we are mainly interested in the *characteristic part* of continued fraction expansions and their orbits, defined as follows.

Definition 4. Let $x = [d_1, d_2, d_3, \ldots]_{N,\alpha}$ be the N-expansion of x. The *characteristic part* of this expansion, CPE in short, is $x_n = T_\alpha^n(x) = [d_{n+1}, d_{n+2}, \ldots]_{N,\alpha}$, where $n \in \mathbb{N} \cup \{0\}$ is the smallest number such that $x_n \in A_\alpha$.

Figure 1 is an example of a two-dimensional arrangement.

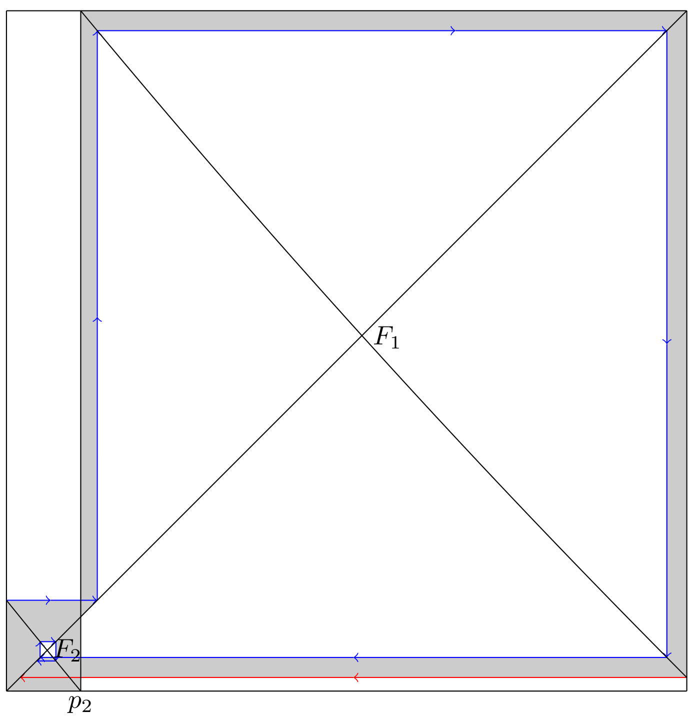


Figure 1: An arrangement with two gaps. Here $N = 100$ and $\alpha = 8.99$.

In Figure 1, the area in grey is the domain for *cobweb plots* of points in A_α. In this example, there is one large gap containing F_1 and a much smaller one containing F_2.

[4]Here we use "for almost all x" (and not "for all x") because we want to exclude fixed points and pre-images of fixed points, i.e., points that T_α maps to a fixed point, which may never leave an interval (a,b).

The boundaries of the components of A_α are α, $\alpha_1, \ldots, \alpha_4$, and β. In this paper, we confine ourselves to arrangements $\Upsilon_{N,\alpha} = \square_2 \cup \square_1$, with a large gap containing F_1. Moreover, we will mostly focus on arrangements with more than two gaps, the smallest of which would hardly be visible in the complete arrangement. Therefore, we will usually only draw a small part of $\Upsilon_{N,\alpha}$ around F_2 only, including p_2, since the gaps in $\square_2$ fully determine those in $\square_1$.

In [1] and [2], a complete overview of gapless arrangements is given. As far as arrangements with gaps are concerned, the main theorem in [1] is the following.

Theorem 1. *Let $N \in \mathbb{N}_{\geq 4}$. Set $\alpha := \sqrt{N} - 1$ and $\alpha_i := T_\alpha^i(\alpha)$, for $i \geq 1$. Define $a := 2, 1, 1$ and $b := 2, 2$ as strings of partial quotients so as to recursively define the N-expansions c_n as follows. First, $c_0 := [\overline{1, 2}]$, $c_1 := [\overline{1}]$, $c_2 := [\overline{2}]$, $c_3 := [\overline{a}]$, and $c_4 := [\overline{a, b}]$. For $n \geq 5$, if $c_{n-1} = [\overline{j, k}]$, then $c_n = [\overline{j, k, j, \ell}]$, with j a string of partial quotients and $\{k, \ell\} = \{a, b\}$. Let $m \in \mathbb{N}$ be such that*

$$\alpha_{|c_i|} < \alpha_{2^{i-1}} \text{ for } i \in \{2, \ldots, m\} \text{ and } \alpha_{|c_{m+1}|} > \alpha_{2^m}.$$

Let $\mathcal{G}_N$ be the collection of gaps of I_α. Then every gap in $\mathcal{G}_N$ contains exactly one point from one of the orbits in $\bigcup_{i=1}^m \{c_i\}$. Conversely, each of the points of the orbits[5] in $\bigcup_{i=1}^m \{c_i\}$ is contained in some (unique) gap in $\mathcal{G}_N$. The number of gaps $G(N) = |\mathcal{G}_N|$ is therefore $\sum_{i=1}^m |c_i|$, which can be expressed more explicitly as

$$G(N) = \begin{cases} \dfrac{2^{m+1} - 1}{3}, & \text{when } m \text{ is odd;} \\[2ex] \dfrac{2^{m+1} - 2}{3}, & \text{when } m \text{ is even.} \end{cases}$$

In particular, G is a finite, monotonically non-decreasing and unbounded function of N.

From here, we proceed with the findings about gaps in N-expansions for $\alpha < \alpha_{\max}$.

2. Introduction

This paper is the second one of a triple in which the number of gaps is investigated in arrangements that are not gapless. In [1], the first one, we only had to deal with N, since there we had $\alpha = \alpha_{\max} = \sqrt{N} - 1$. In the current paper and the third one, we investigate the number of gaps $\Upsilon_{N,\alpha}$ in arrangements with $\alpha < \alpha_{\max}$. For α small enough, G cannot be larger than 3 and will eventually become constantly 0 as

[5] For $N = 2, \ldots, 8$ the sets $\mathcal{G}_N$ and $\bigcup_{i=1}^m \{c_i\}$ are empty.

α decreases further; see [2]. However, as α decreases from $\alpha_{\max}$, for most $N_{\geq 2}$ there will be an initial rapid alternation of $G(N,\alpha)$, and for some N and $\alpha < \alpha_{\max}$ it is even possible that $G(N,\alpha) > 6G(N,\alpha_{\max})$. This first alternation is the subject of the current paper; in the third one, we will look at alternations of G for α that are less close to $\alpha_{\max}$. We will use a notation that largely builds upon that of Theorem 1, writing ab for the sequence $2,1,1,2,2$ and abb for the sequence $2,1,1,2,2,2,2$. To avoid confusion caused by too many notation conventions, we will drop the c_i notation, and write $[\overline{1}]$, $[\overline{2}]$, $[\overline{a}]$, and $[\overline{ab}]$ instead of c_1, c_2, c_3, and c_4.

An example with $N = 2116$ illustrates quite well the initial rapid alternation of gaps when α decreases from α_{max}. From Theorem 1 it follows that $G(2116,45) = 10$, with gaps around the orbit points of $[\overline{1}]$, $[\overline{2}]$, $[\overline{a}]$, and $[\overline{ab}]$. Table 1 shows that for α only slightly smaller than $\alpha_{\max}$ much larger values of G occur.

α	G	periodic points other than $[\overline{1}]$ and $[\overline{2}]$ in gaps
44.9997	10	$[\overline{a}]$, $[\overline{ab}]$
44.99967	7	$[\overline{ab}]$
44.9996	10	$[\overline{a}]$, $[\overline{ab}]$
44.99958	17	$[\overline{a}]$, $[\overline{ab}]$, $[\overline{abb}]$
44.99956	14	$[\overline{ab}]$, $[\overline{abb}]$
44.99955	17	$[\overline{a}]$, $[\overline{ab}]$, $[\overline{abb}]$
44.99954	34	$[\overline{a}]$, $[\overline{ab}]$, $[\overline{abb}]$, $[\overline{abb,ab,ab}]$
44.999538	31	$[\overline{ab}]$, $[\overline{abb}]$, $[\overline{abb,ab,ab}]$
44.9995376	62	$[\overline{ab}]$, $[\overline{abb}]$, $[\overline{abb,ab,ab}]$, $[\overline{abb,ab,ab,abb,abb}]$
44.999537	31	$[\overline{ab}]$, $[\overline{abb}]$, $[\overline{abb,ab,ab}]$
44.999533	14	$[\overline{ab}]$, $[\overline{abb}]$
44.999532	45	$[\overline{ab}]$, $[\overline{abb}]$, $[\overline{abb,ab,ab,abb,abb}]$
44.9995317	48	$[\overline{a}]$, $[\overline{ab}]$, $[\overline{abb}]$, $[\overline{abb,ab,ab,abb,abb}]$
44.9995314	45	$[\overline{ab}]$, $[\overline{abb}]$, $[\overline{abb,ab,ab,abb,abb}]$
44.9995312	14	$[\overline{ab}]$, $[\overline{abb}]$

Table 1: The volatility of $G(2116)$ as α decreases from $\alpha_{\max} = 45$

Remark 1. Some of the values for G in Table 1 were found by the author by merely inspecting $G(2116,\alpha)$ while decreasing α in steps of 0.000001. This can be done easily using a technical computing system[6]: simply let it compute and sort a large number of consecutive images of α, and then search for notable gaps. However, as Table 1 shows, this is not very accurate when it comes to finding the maximum of $G(N)$. Not only may the range of α for which certain gaps exist be very small, the gaps themselves may be very small as well. In this paper a theory will be developed

[6]For almost all computations in this paper we made use of Wolfram's *Mathematica*.

with which we can determine $G(N)$ or at least find an upper bound for it; see Section 6. We used it, for instance, to find arrangements with 62 gaps.

In order to tackle the large volatility of G as α decreases, we distinguish four phases, i.e., sets of α for which all CPEs (characteristic parts of N-expansions; see Section 1) have a common characteristic. In *Phase I*, digit 1 only occurs in tuples, which are followed by an odd number of digits 2. This is the case for α such that $\alpha_3 < f_2$, which is possible for $N \geq 33$ only. In *Phase II* both digit 1 and digit 2 occur in sequences of even length only. This is the case if $\beta_1 > f_2$ and $\alpha_1 < f_1$, which is possible for $N \geq 18$ only. In *Phase III* we still have $I_\alpha = \Delta_2 \cup \Delta_1$, but now digit 2 only occurs in tuples, which are followed by an odd number of digits 1. This is the case if $\alpha_1 > f_1$. In *Phase IV* we have $I_\alpha = \Delta_3 \cup \Delta_2 \cup \Delta_1$.

In this paper we will confine ourselves to Phase I, introducing some general concepts and notions that are important for the other phases as well. In Section 3 we define the *characteristic polynomial* of an arrangement, as a means to determine for which N similar arrangements exist. In addition, we define the opening and closure of gaps as α decreases. In Section 4, we investigate the volatility of $G(N, \alpha)$ as α decreases for a relatively small value of N. In Section 5 we investigate arrangements with relatively few gaps. In Section 6 we present an approach for determining the *maximum* of $G(N, \alpha)$ in Phase I, based on arrangements that we define as *ideal*. In a forthcoming paper we will study the other three phases.

3. The Characteristic Polynomial of Arrangements in Phase I

If $I_\alpha = \Delta_2 \cup \Delta_1$, the attractor A_α is determined by the expansion under T_α of intervals containing p_2. According to Theorem 5 in [1], the endpoints of all components of A_α are images of α or β. Since α is the left endpoint of the leftmost component of A_α, and β is the right endpoint of the rightmost component of A_α, the next lemma follows immediately.

Lemma 1. *Each image of α that is a left endpoint of a component of A_α has an* even *index, while the index is* odd *if the image is a right endpoint. Similarly, each image of β that is a left endpoint of a component of A_α has an* odd *index, while the index is* even *if the image is a right endpoint.*

If I_α contains at least two gaps, one of the components of A_α is the maximal closed interval containing p_2, which we denote by P_2. We write $P_2 = J_\ell \cup J_r$, with $J_\ell = [\gamma_\ell, p_2]$ and $J_r = (p_2, \gamma_r]$, where both γ_ℓ and γ_r are an image of α or β. The maximality of P_2 implies that for some minimal $n \in \mathbb{N}_{\geq 2}$ we have either $T_\alpha^n(J_\ell) = P_2$ and $|T_\alpha^n(J_\ell)| = |P_2|$ or $T_\alpha^n(J_r) \subset P_2$ and $|T_\alpha^n(J_r)| = |P_2|$. The value of n depends on the expansion of intervals in I_α under T_α. In [1], some useful inequalities were found for the expansion factors associated with the strings $a := 2, 1, 1$ and $b := 2, 2,$

as defined in Theorem 1. These factors can easily be adjusted to the conditions of Phase I. In order to do so, let g_r be the expansion factor after three iterations of T_α on an interval K such that $K \subset (p_2, f_1]$, $T_\alpha(K) \subset [f_1, \beta)$, and $T_\alpha^2(K) \subset (\alpha, f_2]$, and let g_ℓ be the expansion factor after two iterations of T_α on an interval L such that $L \subset [f_2, p_2]$ and $T_\alpha(L) \subset [\alpha, f_2]$. Then, omitting computations such as in [1], we have the following inequalities:

$$\frac{((N+1)\alpha + 3N + 2)^2}{N^3} < g_r(K) < \frac{N(\alpha+2)^2}{(N-1-\alpha)^2}$$
$$\frac{(\sqrt{N+1}+1)^4}{N^2} < g_\ell(L) < \left(1 + \frac{2}{\alpha}\right)^2. \tag{1}$$

In [1], it is shown that in case $\alpha = \alpha_{\max} = \sqrt{N} - 1$, the factor $F(N) := (1 + 2/(\sqrt{N}-1))^2 = (1+2/\alpha)^2$ of (1) provides for very good approximations for both g_ℓ and g_r . The boundaries between classes of integers with an equal number of gaps can be found by computing the positive root of the polynomial $F^{2^{n-2}} - 2$.

In Phase I the notion of such a characteristic polynomial for arrangements is even more important. It is based on the notion that for an arrangement $\Upsilon_{N,\alpha}$ with at least two gaps there are minimal $s, t \in \mathbb{N}$ such that

$$\begin{cases} T_\alpha^s(J_\ell) \subset P_2 \text{ and } T_\alpha^t(J_r) \subset P_2; \\ T_\alpha^s(J_\ell) \cup T_\alpha^t(J_r) = P_2. \end{cases} \tag{2}$$

Both T_α^s and T_α^t involve products F_s, and F_t, respectively, of expansion factors g_ℓ and g_r, respectively, as defined above. In terms of F_s and F_t, the first statement of (2) can be read as

$$F_s|J_\ell| \le |P_2| \text{ and } F_t|J_r| \le |P_2|.$$

Since $|P_2| = |J_\ell| + |J_r|$, from $F_s|J_\ell| \le |P_2|$ it follows that $(F_s - 1)|J_\ell| \le |J_r|$, yielding $|J_\ell| \le |J_r|/(F_s - 1)$. If we combine this with $F_t|J_r| \le |P_2|$, we obtain

$$F_t|J_r| \le |J_r| + \frac{|J_r|}{F_s - 1}, \text{ so } F_t \le 1 + \frac{1}{F_s - 1} = \frac{F_s}{F_s - 1}, \text{ yielding}$$
$$F_s F_t - F_t - F_s \le 0. \tag{3}$$

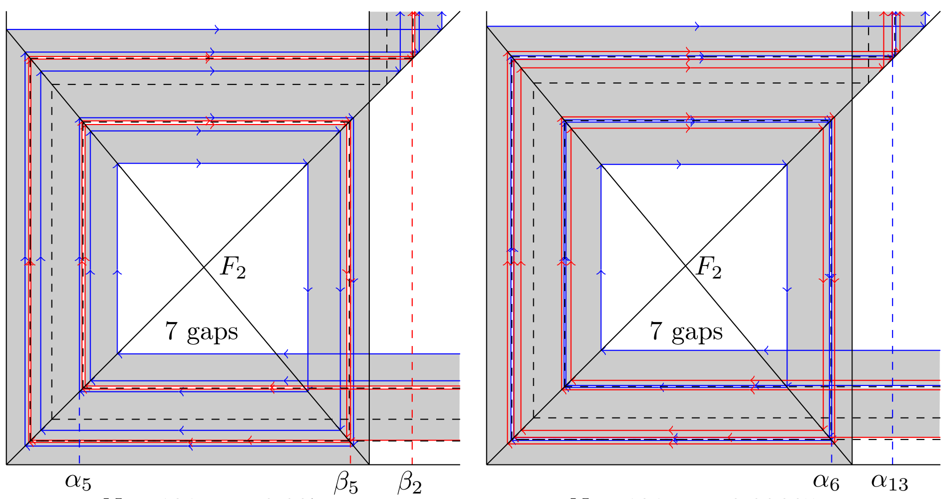
For the computations in the rest of this paper it is very convenient if we approximate both g_ℓ and g_r in Phase I by $F = F(N) = (1 + 2/(\sqrt{N}-1))^2$. If m and n are the number of expansion factors g_ℓ and g_r of F_s and F_r, respectively, it follows from Inequality (3) that

$$F^n F^m - F^n - F^m < 0 \text{ or } F^n F^m - F^n - F^m \approx 0. \tag{4}$$

If $n = m$, we can write (4) as $F^n - 2 < 0$ or $F^n - 2 \approx 0$; if $n \ne m$, we can write (4) as $F^{\max\{n,m\}} - F^{|n-m|} - 1 < 0$ or $F^{\max\{n,m\}} - F^{|n-m|} - 1 \approx 0$. Depending on m and n, either $F^n - 2$ or $F^{\max\{n,m\}} - F^{|n-m|} - 1$ is called the *characteristic polynomial*[7] for $\Upsilon_{N,\alpha}$. Two arrangements are called *similar* if they have the same characteristic polynomial and contain the same periodic points.

Remark 2. We use F in three ways: it may be the variable in a characteristic polynomial, or the positive real root of a characteristic polynomial. For legibility, we will also sometimes use F *instead* of F_s or F_t. We trust that this will not cause confusion and will prevent an unnecessary expansion of notations.

We introduce the use of characteristic polynomials by inspection of several arrangements. We generally take N to be a square, which may be useful to compare cases as α decreases from $\alpha_{\max} = \sqrt{N} - 1$. As before, we write[8] $P_2 = J_\ell \cup J_r$, with $J_\ell = [\gamma_\ell, p_2]$ and $J_r = (p_2, \gamma_r]$ for $\gamma_\ell, \gamma_r \in I_\alpha$. As a first example, we consider $N = 121$; see Figure 2. From [1], we know that for $N = 121$ both $[\overline{a}]$ and $[\overline{ab}] = [\overline{2,1,1,2,2}]$ are contained in $A_{\alpha_{\max}}$, and $G(121, \alpha_{\max}) = 2$. For some α smaller than $\alpha_{\max}$, an arrangement exists such that $\beta_1 < [\overline{ab}]$ and $\beta_{1+|[\overline{ab}]|} = \beta_6 > [\overline{ab}]$, implying $[\overline{ab}] \in I_\alpha \setminus A_\alpha$.



$N = 121; \alpha = 9.997$ $N = 121; \alpha = 9.99665$

Figure 2: Two arrangements for $N = 121$ where I_α has seven gaps

The left arrangement[9] in Figure 2, with $\alpha = 9.997$, illustrates this. We have $J_\ell = [\beta_5, p_2]$ and $J_r = (p_2, \beta_2]$. It is not hard to see that $|T_\alpha^3(J_r)| = |[\beta_5, \beta_2)| =$

[7]We will often write 'CP' for "characteristic polynomial" and 'CPE' for "characteristic part of an/the N-expansion".

[8]We will use these notations throughout this paper.

[9]For visual reasons, we will often only draw only the bottom left part of arrangements. Images of α are in blue, those of β are in red.

$|P_2|$, while $T_\alpha^7(J_\ell) = [\alpha_6, \beta_2] \subsetneq P_2$. In terms of F, we have $F|J_r| = |P_2|$ and $F^3|J_\ell| \approx |P_2|$, yielding $F^3 - F^2 - 1$ as a CP. Although the gaps containing orbit points $[\overline{ab}]$ are relatively small for all α such that $\Upsilon_{121,\alpha}$ contains 7 gaps, one might suspect that there are smaller N with similar arrangements, with even smaller gaps containing the orbit points of $[\overline{ab}]$. For the smallest of these N, the difference between $F^3|J_\ell|$ and $|P_2|$ will be very small, so we have $F^3 - F^2 - 1 \approx 0$. The real root of $F^3 - F^2 - 1$ is $1.4655\cdots$, and applying [10] $N = (1 + 2/(\sqrt{F} - 1))^2$, we find $N \approx (1 + 2/(\sqrt{1.4656} - 1))^2 \approx 110$.

Remark 3. We want to stress that the approximation 110 for N in the above example is only an indication of the real value of N we want to determine, albeit a useful one. Since N is an integer, F does not have to be very precise. For $N = 110$, for instance, we have $\alpha_{max} = 9.4880\cdots$, while α as small as 9.4624, with $F(\alpha) = (1 + 2/9.4624)^2 = 1.4673\cdots$, would still yield $N \approx 110$. We use CPs to classify arrangements in the first place, but they are also useful for approximating the smallest N for which an arrangement exists that is similar to a given arrangement.

The following two straightforward lemmas[11] are useful for proving that $N = 110$ is actually the smallest N in Phase I for which an α exists such that I_α has 7 gaps.

Lemma 2. *Let N be fixed, and α, α' in Phase I, with $\alpha < \alpha'$. If $d(\alpha_j) = d(\alpha'_j)$ for all $j \in \{0, \ldots, m\}$, with $m \in \mathbb{N}$, then $\alpha_m > \alpha'_m$ if and only if m is odd. Similarly, if $d(\beta_k) = d(\beta'_k)$ for all $k \in \{0, \ldots, n\}$, with $n \in \mathbb{N}$, then $\beta_n > \beta'_n$ if and only if n is odd.*

The next lemma is a consequence of Lemma 2.

Lemma 3. *Let N be fixed, and let $\Upsilon_{N,\alpha}$ and $\Upsilon_{N,\alpha'}$ be similar arrangements in Phase I, with $\alpha < \alpha'$. Let $m, n \in \mathbb{N}$ have opposite parity. If $d(\alpha_j) = d(\alpha'_j)$, with $j \in \{0, \ldots, \max\{m, n\}\}$, then $|(\alpha_m, \alpha_n)| < |(\alpha'_m, \alpha'_n)|$ if and only if m is odd. Similarly, let $s, t \in \mathbb{N}$ have opposite parity. If $d(\beta_k) = d(\beta'_k)$, with $k \in \{0, \ldots, \max\{s, t\}\}$, then $|(\beta_s, \beta_t)| < |(\beta'_s, \beta'_t)|$ if and only if s is odd.*

Finally, the following definition is useful.

Definition 5. Let N be fixed and let $\Upsilon_{N,\alpha}$ and $\Upsilon_{N,\alpha'}$ be arrangements, with $\alpha < \alpha'$. Let x be a periodic point such that $x \in A_\alpha$ for $\alpha \in \{\alpha, \alpha'\}$ and $x \in I_\alpha \setminus A_\alpha$ for $\alpha \in (\alpha, \alpha')$. Then α' is called the *opening* $\alpha_o(x)$ and α is called the *closure* $\alpha_c(x)$ of the gaps around the orbit points of x.

Remark 4. Gaps around the orbit points of a certain periodic point may have multiple openings and closures; see page Section 4 for a first example, with $N = 324$.

[10]Given that $F > 1$ and $N \geq 2$, the equations $F = (1+2/(\sqrt{N}-1))^2$ and $N = (1+2/(\sqrt{F}-1))^2$ are equivalent.

[11]In both lemmas we use the notation $d(x)$ for the digit of x in an N-expansion; see [1].

The bottom left part of the left arrangement of Figure 2 shows that $[\overline{ab}] \in (\beta_6, \beta_1) \subset (\alpha_7, \alpha_{12})$. Now we let α decrease to α', while maintaining similarity of arrangement. Then, as a result of Lemma 3, we have $|(\beta_6, \beta_1)| > |(\beta_6', \beta_1')|$, while $|(\alpha_7, \alpha_{12})| < |(\alpha_7', \alpha_{12}')|$. Eventually, we obtain the right arrangement in Figure 2, where $[\overline{ab}] \in (\alpha_7, \alpha_{12}) \subset (\beta_6, \beta_1)$. In order to find for which α the arrangements of $N = 121$ have 7 gaps, we have to solve $\beta_1 = \beta_6$ (with root α_o) and $\alpha_7 = a_{12}$ (with root α_c), yielding $(\alpha_c, \alpha_o) \subset (9.9965\cdots, 9.9971\cdots)$. For an arrangement with $[\overline{1}]$, $[\overline{2}]$ and $[\overline{ab}]$ in gaps (and $[\overline{a}]$ in the attractor) it is necessary that $\alpha_o > \alpha_c$. We suspect that the smallest N for which this is the case is about 110; see Remark 3. For $N = 110$, we compute the root of $\beta_1 = \beta_6$, which is $9.4848\cdots$, while the root of $\alpha_7 = \alpha_{12}$ is $9.4847\cdots$, satisfying $\alpha_o > \alpha_c$. For $N = 109$, we find $\alpha_o = 9.43701\cdots < 9.43702\cdots = \alpha_c$, so the smallest N with a similar arrangement as $\Upsilon_{121,9.997}$ is indeed $N = 110$.

Remark 5. For technical computing systems, the equations $\beta_1 = \beta_6$ and $\alpha_7 = a_{12}$ are very easy to solve. However, we will also come across opening and closure equations with indices above 100, which are demanding for such systems as well.

Remark 6. The characteristic polynomial for arrangements with only 2 gaps is $F^2 - F - 1$, with positive root $\frac{1}{2}\sqrt{5} + \frac{1}{2}$. This value of F is related to $N = 69$, which is the smallest N such that $2, 2, 2, 2$ cannot occur in CPEs if $\alpha = \alpha_{\max}$. Although we will shortly see that arrangements in Phase I with more than 2 gaps only exist for $N \geq 110$, $N = 69$ can be considered as the theoretical lower bound for the application of CPs in Phase I.

Remark 7. If an arrangement has at least three gaps, while $[\overline{a}] \in A_\alpha$, there is a largest point $x < [\overline{a}]$ in the orbit of a periodic point such that $x \in I_\alpha \setminus A_\alpha$. In that case, $T_\alpha(J_\ell) \subset [\alpha, x)$ and $T_\alpha^2(J_\ell) \subset (x_1, \alpha_1]$, with $x_1 > [\overline{1, 1, 2}]$, yielding $T_\alpha^2(J_\ell) \cap P_2 = \emptyset$. As a consequence, the polynomial $F^n - F^{n-1} - 1$, with $n \geq 3$, applies to arrangements with $T^3(J_r) \subset P_2$ only. In a similar way we find that the polynomial $F^n - F^{n-2} - 1$, with $n \geq 3$, applies to arrangements with $T^3(J_r) \cap P_2 = \emptyset$ and $T^6(J_r) \subset P_2$ only.

Remark 8. With regard to the endpoints of components of A_α in arrangements similar to those in Figure 2, we can actually distinguish three stages. This is because the first image of α that is a potential endpoint of a gap around an orbit point of $[\overline{ab}]$ is α_5; it replaces β_4 as an endpoint as soon as α decreases beyond the root of $\alpha_5 = \beta_4$ (which is $9.9968\cdots$ for $N = 121$). As a result, in the second stage all endpoints of gaps are images of α, except β_1, β_2, and β_3. These three are replaced by α_{12}, α_{13}, and α_{14}, respectively, as soon as α decreases beyond the root of $\alpha_{12} = \beta_1$ (which is $9.9966\cdots$ for $N = 121$). In the first stage, A_α consists of the components, from leftmost to rightmost,

$$[\alpha, \beta_6],\ [\beta_1, \beta_4],\ [\beta_9, \alpha_3],\ [\alpha_4, \beta_{10}],\ [\beta_5, \beta_2],\ [\beta_7, \alpha_1],\ [\alpha_2, \beta_8],\ [\beta_3, \beta].$$

Note that in this first stage we have three components of A_α with images of β only, i.e., β through β_5. The other components all have both an image of α and one of β as an endpoint, their index numbers always differing by 6. This is so in a large fraction of all arrangements in Phase I; see Section 6 for an explanation. In the second stage, between the roots of $\alpha_5 = \beta_4$ and $\alpha_{12} = \beta_1$, we have

$$[\alpha, \alpha_7],\ [\beta_1, \alpha_5],\ [\alpha_{10}, \alpha_3],\ [\alpha_4, \alpha_{11}],\ [\alpha_6, \beta_2],\ [\alpha_8, \alpha_1],\ [\alpha_2, \alpha_9],\ [\beta_3, \beta].$$

In the third stage, between the root of $\alpha_{12} = \beta_1$ and α_c, the components are

$$[\alpha, \alpha_7],\ [\alpha_{12}, \alpha_5],\ [\alpha_{10}, \alpha_3],\ [\alpha_4, \alpha_{11}],\ [\alpha_6, \alpha_{13}],\ [\alpha_8, \alpha_1],\ [\alpha_2, \alpha_9],\ [\alpha_{14}, \beta].$$

The left arrangement of Figure 2 may serve as an illustration of this. Henceforth, we will generally not distinguish between these stages in the interval (α_c, α_o).

The question arises whether even more gaps are possible in I_α for N near 121. Since the positive real root of $F^n - F^m - 1$ is an increasing function of m, it is clear that $F^n - F^{n-1} - 1$ allows for smaller values of N than $F^n - F^m - 1$, with $1 \leq m \leq n-1$, let alone $F^n - 2$. Table 2 shows the smallest approximate N associated with these possibly CPs.

polynomial	positive real root	related N
$F^2 - 2$	$1.4142\cdots$	133
$F^2 - F - 2$	$1.1618\cdots$	69
$F^3 - 2$	$1.2599\cdots$	300
$F^3 - F - 1$	$1.3247\cdots$	203
$F^3 - F^2 - 1$	$1.4655\cdots$	110
$F^4 - 2$	$1.1892\cdots$	533
$F^4 - F - 1$	$1.2207\cdots$	403
$F^4 - F^2 - 1$	$1.2720\cdots$	277
$F^4 - F^3 - 1$	$1.3802\cdots$	155

Table 2: Polynomials of a low degree with related N

Although the values of N in Table 2 are approximations, the table implies that to have arrangements in Phase I with more than 7 gaps, N has to be at least about 155.

Before we look at arrangements with at least 7 gaps, two things are noteworthy. The first is that for α larger than the root $\mathcal{A}_2$ of $\alpha_6 = p_2$, the sequence $2, 2, 2, 2, 2$ cannot occur in CPEs. The second is that for α smaller than the root $\mathcal{A}_1$ of $\beta_2 = p_2$, the sequence $1, 2, 1$ cannot occur in CPEs; see Figure 3. So, it is only for $\alpha \in (\mathcal{A}_1, \mathcal{A}_2)$ that CPEs can contain the sequences $1, 2, 1$ and $1, 2, 2, 2, 1$ and $1, 2, 2, 2, 2, 2, 1$.

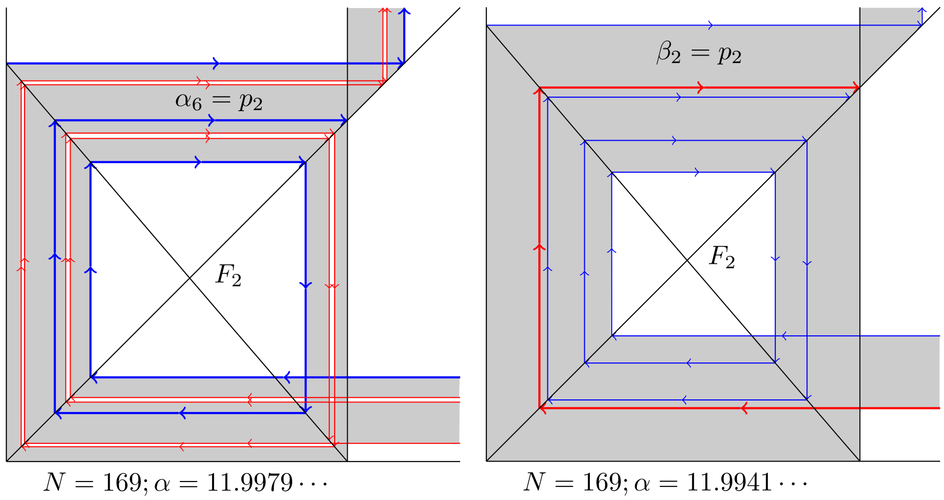


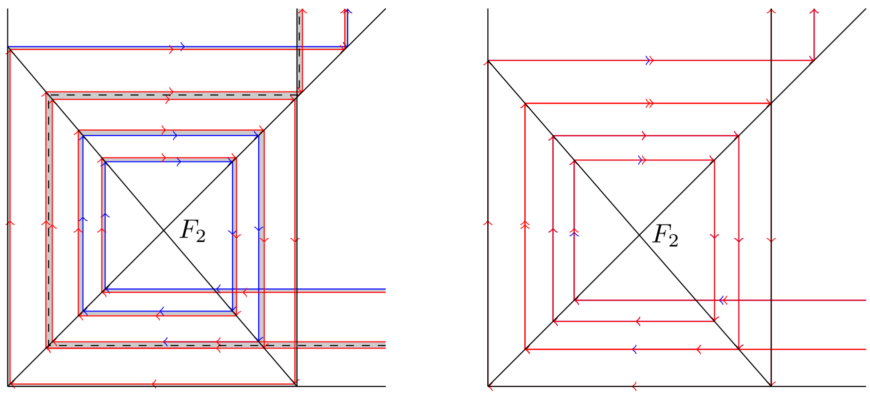
Figure 3: Arrangements for $N = 169$ when $\alpha_6 = p_2$ and when $\beta_2 = p_2$

The right arrangement in Figure 3 shows that for $N = 169$ the CPEs may even contain the sequence $2, 2, 2, 2, 2, 2, 2$. This particular circumstance, where CPEs can contain both the sequence $1, 2, 1$ and the sequence with seven consecutive 2s, requires that the root of $\alpha_8 = p_2$ is smaller than the root of $\beta_2 = p_2$, which is the case for $N \leq 202$ only. For this arrangement, with wide gaps around the orbit points of $[\overline{abb}] = [\overline{2, 1, 1, 2, 2, 2, 2}]$, we find α_0 where $\beta_1 = \beta_{13}$, and α_c where $\alpha_7 = \alpha_{19}$. The smallest N for which $\alpha_o([\overline{abb}]) > \alpha_c([\overline{abb}])$ is $N = 154$, so we have similar arrangements with 9 gaps for $154 \leq N \leq 202$; see Figure 4.

In the case of $N = 202$, the components of A_α are all very small. The 9 gaps exist for $\alpha_c = 13.20776940435\cdots < \alpha < 13.20776944205\cdots = \alpha_0$ only, while $\alpha_8 = p_2$ for $\alpha = 13.20776944207\cdots$. Moreover, we have

$$\begin{aligned} \beta_5 &= 13.28268417 \\ \alpha_8 &= 13.28268426 \\ p_2 &= 13.28268428 \\ [\overline{1, 1, 2}] &= 13.28268441 \\ \beta_2 &= 13.28268458 \end{aligned}$$

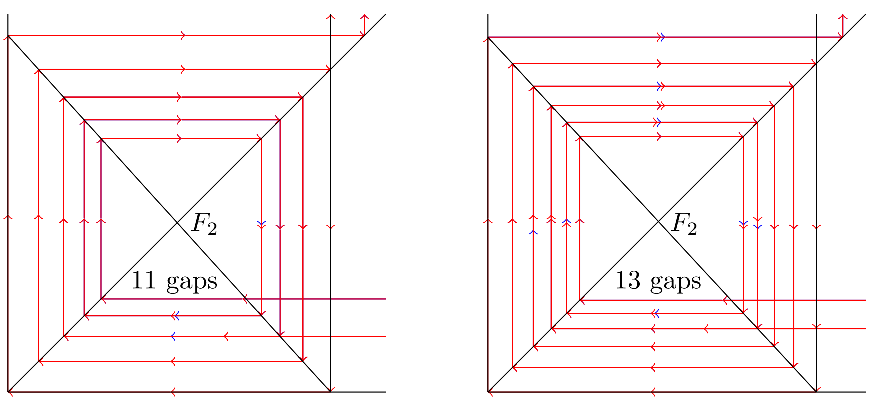
For $N > 202$, similar components which are very small exist for N such that all three sequences $1, \underbrace{2, \ldots\ldots, 2}_{2n-1 \text{ digits } 2}, 1$, and $1, \underbrace{2, \ldots\ldots, 2}_{2n+1 \text{ digits } 2}, 1$, and $1, \underbrace{2, \ldots\ldots, 2}_{2n+3 \text{ digits } 2}, 1$, with $n \geq 2$, may occur. This is possible only if an α exists such that $\alpha_{2n} < p_2$ and $\beta_{2n-6} > p_2$. The arrangements with $(N, \alpha) = (401, 19.020007318)$ and $(N, \alpha) =$



$N = 154; \alpha = 11.404057$ $N = 202; \alpha = 13.20776944$

Figure 4: Two similar arrangements with 9 gaps

$(668, 24.84128889)$, for instance, have 11 and 13 gaps, respectively, and very small components; see Figure 5. These numbers, however, are no maxima of G, as we will shortly show. It is not hard to see that the CP of arrangements like these is $F^n - F^3 - 1$, with $n = 5$ in case $N = 401$, and $n = 6$ in case $N = 668$.



$N = 401; \alpha = 19.02007318$ $N = 668; \alpha = 24.84128889$

Figure 5: Two arrangements with very small components

Remark 9. Figure 3 illustrates that the arrangements where $\alpha_6 = p_2$ or $\beta_6 = p_2$ will generally not have a notably large number of gaps. It merely implies that no more than five images of α can be endpoints of gaps in the first case. In the second case, the only image of β that might be an endpoint is β_1, although one can argue that in that case $\beta_3 = \alpha$ and therefore $\beta_{n+3} = \alpha_n$, with $n \in \mathbb{N}$.

4. Development of Arrangements

For a further introduction of the link between CPs and G, we refer to nine arrangements for $N = 324$, as shown in Figures 6 through 8. Our aim is twofold: with this abundance of arrangements we want to give a detailed view on the process of rapidly changing gaps for α close to $\alpha_{\max}$, and we also want to show why we will eventually focus on arrangements with a maximal number of gaps. For every N there is a unique progression of arrangements as α decreases, but we will show how a classification of N according to this maximum may be feasible.

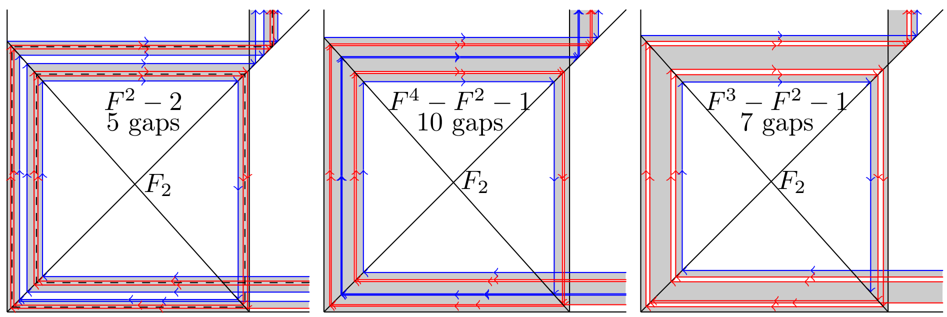
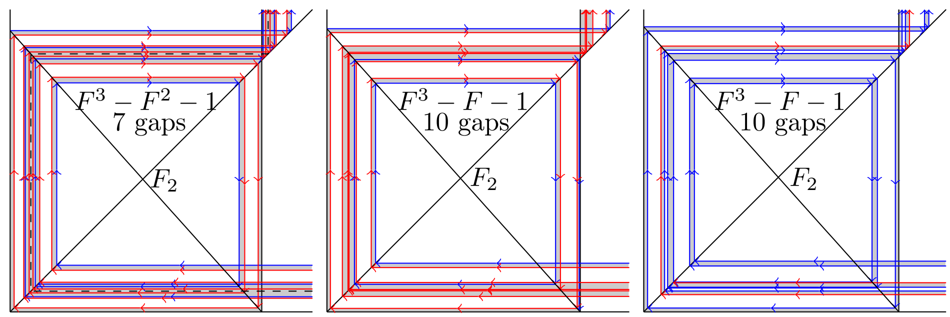


Figure 6: Three arrangements for $N = 324$

In the left arrangement of Figure 6 the value of α is only slightly smaller than $\alpha_{\max}$. In this case the arrangement is similar to the case $\alpha = \alpha_{\max}$, with gaps around the orbit points of $[\overline{a}]$. The orbit of $[\overline{ab}]$ is drawn with dashed line segments. Here $[\overline{ab}] \in A_\alpha$, but the intervals $[\beta_1, \beta_6], \ldots, [\beta_5, \beta_{10}]$, containing its orbit points, are small. In the arrangement in the middle, both $[\overline{a}]$ and $[\overline{ab}]$ are contained in $I_\alpha \setminus A_\alpha$. This is because the gaps around the points of $[\overline{a}]$ close when $\alpha_5 = \alpha_8$, which is for $\alpha = 16.9991\cdots$, while the gaps around $[\overline{ab}]$ open when $\beta_1 = \beta_6$, which is for $\alpha = 16.9993\cdots$, and close when $\alpha_7 = \alpha_{12}$, which is for $\alpha = 16.9972$; see the last arrangement of Figure 8. Meanwhile, the gaps around $[\overline{a}]$ open again when $\beta_{11} = \beta_{14}$, which is for $\alpha = 16.9983\cdots$, and close again when $\alpha_{12} = \alpha_{15}$, which is for $\alpha = 16.9980\cdots$. This development is illustrated with the three arrangements of Figure 7. There it is still possible, albeit difficult, to distinguish the images of β. The right arrangement of Figure 6 illustrates that for $\alpha \in (16.9983\cdots, 16.9991\cdots)$ there are only 7 gaps.

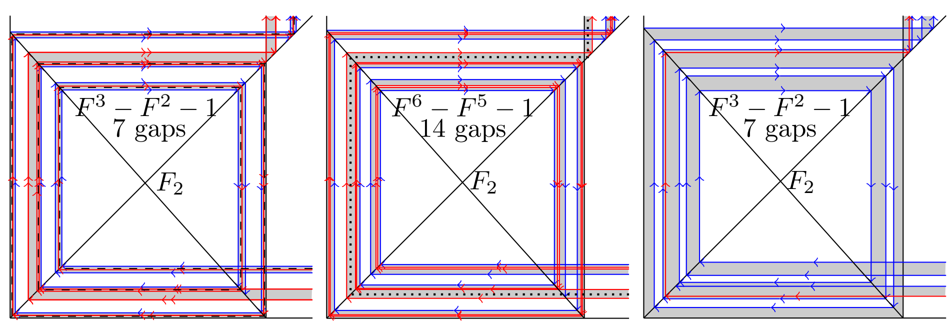
Although the 10 gaps in the middle arrangements of both Figure 6 and Figure



$N = 324; \alpha = 16.9985$ $N = 324; \alpha = 16.9983$ $N = 324; \alpha = 16.9981$

Figure 7: Three more arrangements for $N = 324$

7, and the right arrangement of Figure 7, contain the orbit points of $[\overline{1}]$, $[\overline{2}]$, $[\overline{a}]$, and $[\overline{ab}]$, the arrangements are not similar, since they have different CPs, yielding significant differences in widths of components and gaps. Another thing to observe is that $F^4 - F^2 - 1$ is related to $N = 277$ (see Table 2), while it is $N = 203$ for $F^3 - F - 1$. Solving $\alpha_{12} = \alpha_{15}$ and $\beta_{11} = \beta_{14}$ for $N \in \{202, 203\}$ yields that 203 is the smallest N with an arrangement of 10 gaps.



$N = 324; \alpha = 16.998$ $N = 324; \alpha = 16.9978$ $N = 324; \alpha = 16.9976$

Figure 8: Last three arrangements for $N = 324$

The left and middle arrangement of Figure 8 are of particular importance for the rest of this paper. In the left one, each component of A_α, except the rightmost one, contains one of the orbit points of $[\overline{abb}]$. It is useful to observe that – apart from $[\overline{a}]$ – this is the periodic point with smallest orbit length in A_α. The orbit itself is rendered with dashes. The positions of the images of β show that 16.998 is close to the value of α for which the gaps around the orbit points of $[\overline{abb}]$ open. This is when α is such that $\beta_6 = \beta_{13}$, yielding $\alpha = 16.9978\cdots$. As a consequence, the left arrangement has $|[\overline{abb}]| = 7$ gaps more than the middle one. These gaps disappear when α becomes smaller than α_c, the root of $\alpha_{17} = \alpha_{24}$, which is $16.9976\cdots$; see the right arrangement. The CP of the middle arrangement is $F^6 - F^5 - 1$, with real root $F \approx 1.2852$, which is related to $N = 254.8\cdots$. We find that $\alpha_c = 14.9347390\cdots < 14.9347398\cdots = \alpha_o$ for $N = 254$, while $\alpha_c > \alpha_o$ for $N = 253$.

For arrangements with gaps around the orbit points of $[\overline{ab}]$, $[\overline{abb}]$, and $[\overline{a}]$ (dotted in Figure 7) as well, we have $F^6 - F^4 - 1$ as iCP, with real root $438.6\cdots$. Indeed, with the root of $\beta_{11} = \beta_{14}$ as the opening and the root of $\alpha_{12} = \alpha_{15}$ as the closure, we can compute that $N = 438$ is the smallest N in Phase I for which there is an α such that $\{[\overline{1}], [\overline{2}], [\overline{a}], [\overline{ab}], [\overline{abb}]\} \subset I_\alpha \setminus A_\alpha$.

Remark 10. For both $F^3 - F - 1$ (with 10 gaps) and $F^6 - F^4 - 1$ (with 17 gaps) the equations for α_o and α_c are $\beta_{11} = \beta_{14}$ and $\alpha_{12} = \alpha_{15}$, respectively. These equations, however, depend on N. For larger values of N we find $\beta_{18} = \beta_{21}$ as the opening equation, and $\alpha_{29} = \alpha_{32}$ as the closure equation. We will elaborate on this in Section 6.

5. Arrangements with Relatively Few Gaps

So far, we have found the maximum of $G(N, \alpha)$ in Phase I for $N \in \{33, \ldots, 437\}$. In all cases, the associated CP is of the form $F^n - F^{n-1} - 1$ or $F^n - F^{n-2} - 1$. This can be understood as follows. If the CP of an arrangement is $F^n - F^{n-1} - 1$, almost all components of A_α are images of J_ℓ. Moreover, J_r and $T_\alpha(J_r) = [\gamma_r', \beta)$ are components on the boundary of Δ_1. This means that there is hardly any interference between images of J_ℓ and J_r. If the CP of an arrangement is $F^n - F^{n-2} - 1$, the interference is still modest. However, if the difference between the exponents of F in the CP is 3 or more, the interference demands more space for the components of A_α, implying smaller components, i.e., smaller expansion factors and therefore larger values of N. The two arrangements of Figure 9 illustrate this nicely. In both cases we have $N = 900$.

In the left arrangement of Figure 9 there are 10 gaps, in a similar arrangement as the left one in Figure 7. The periodic point with shortest orbit length in A_α is $[\overline{abb}]$, and similar arrangements have $F^6 - F^4 - 1$ as a CP, as we have already seen. However, if we exclude $[\overline{abb}]$, the periodic point with shortest orbit length, 13, is $\mathcal{X} = [\overline{ab, a, ab}]$. Arrangements with gaps around the orbit points of $[\overline{1}]$, $[\overline{2}]$, $[\overline{a}]$, $[\overline{ab}] = [\overline{2, 1, 1, 2, 2}]$ and $\mathcal{X}$ only, have $F^7 - F^4 - 1$ as a CP; see the right arrangement of Figure 9. There the orbit of $\mathcal{X}$ is dashed, but it seems to coincide with the endpoints of the gaps. The dotted orbit of $[\overline{a}]$ is still visible. A close inspection of $\mathcal{X}$ is enlightening: its first 9 digits are the same as those of $[\overline{ab}]$, resulting in an orbit that is close to that of $[\overline{ab}]$. This is especially notable since $\mathcal{X}$ and $[\overline{ab}]$ are the only periodic points in $I_\alpha \subset A_\alpha$, except $[\overline{1}]$, $[\overline{2}]$ and $[\overline{a}]$. This explains why such an

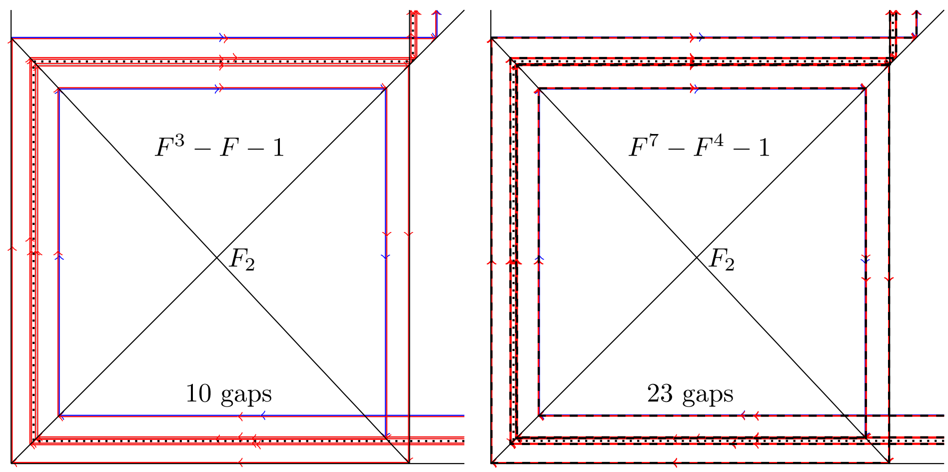


Figure 9: The connection between F^3-F-1 and F^7-F^4-1

arrangement is only possible for relatively large N: the opening of $\mathcal{X}$ is the root of $\beta_{17}=\beta_{43}$, and the closure is the root of $\alpha_5=\alpha_{31}$, so the difference between the index numbers in the equations for α_c and α_o is twice the orbit length of $\mathcal{X}$. For $\alpha_c<\alpha<\alpha_o$ we have an arrangement with $G=23$. In the next section we will see that this is 1 *less* than the 24 gaps of the arrangement with $F^{10}-F^9-1$ as a CP. Moreover, the smallest N associated with 23 gaps is 739, and for this N even more than 24 gaps are possible, as we will see in the next section.

In the following example, the interference between the images of J_ℓ and J_r is exceptionally high and requires particularly large values of N. This time, we construct an arrangement with F^8-2 as a CP, based on F^6-F^4-1; see the left arrangement of Figure 10. Both arrangements of Figure 10 are rendered as if all components and gaps have equal length. Especially for large values of G, this way of rendering a (complete) arrangement has visual advantages.

In the left arrangement, $\mathcal{Z}=[\overline{abb,ab,a,ab}]$ is the periodic point with smallest orbit length in A_α. Note that $|\mathcal{Z}|=20$. The orbit of $\mathcal{Z}$ has two points in P_2 and two points in the pre-image of P_2. The sequence of images of $J_\ell=[\beta_{19},p_2]$ and $J_r=(p_2,\alpha_{19}]$ commences as follows:

$$\begin{aligned}
&[\beta_{19},p_2]\overbrace{\xrightarrow{T_\alpha}[\alpha,\beta_{20}]\xrightarrow{T_\alpha}\cdots\xrightarrow{T_\alpha}[\beta_{37},\alpha_{17}]\xrightarrow{T_\alpha}[\alpha_{18},\beta_{38}]\xrightarrow{T_\alpha}}^{2,2,1,1,2,2,2,2,2,1,1,2,2,2,1,1,2,1,1,2}[\beta_{39},\alpha_{19}];\\
&(p_2,\alpha_{19}]\underbrace{\xrightarrow{T_\alpha}[\alpha_{20},\beta)\xrightarrow{T_\alpha}\cdots\xrightarrow{T_\alpha}(\beta_{17},\alpha_{37}]\xrightarrow{T_\alpha}[\alpha_{38},\beta_{18})\xrightarrow{T_\alpha}}_{1,1,2,1,1,2,2,2,1,1,2,2,2,2,2,1,1,2,2,2}(\beta_{19},\alpha_{39}],
\end{aligned}$$

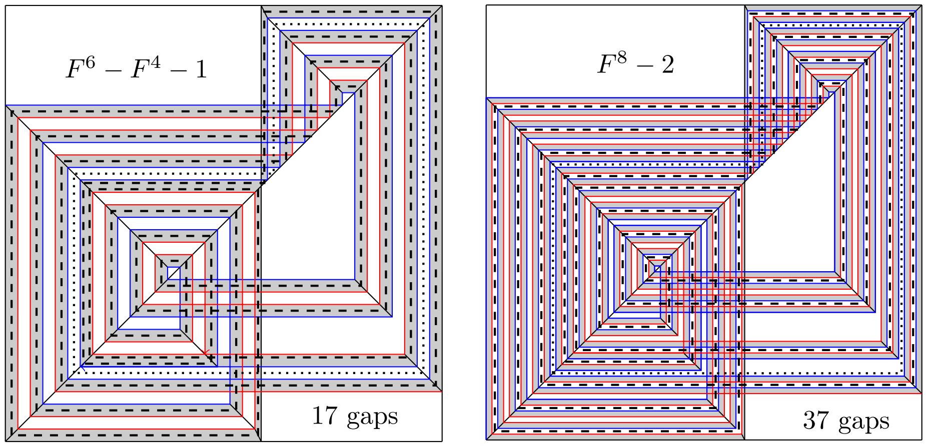


Figure 10: From $F^6 - F^4 - 1$ to $F^8 - 2$

$$\begin{aligned} T^i_\alpha[\beta_{19}, p_2] \cap T^i_\alpha(p_2, \alpha_{19}] &= \emptyset \text{ for } i = 0, \dots, 18; \\ T^{19}_\alpha[\beta_{19}, p_2] \cup T^{19}_\alpha(p_2, \alpha_{19}] &= [\alpha_{18}, \beta_{18}); \\ T^{19}_\alpha[\beta_{19}, p_2] \cap T^{19}_\alpha(p_2, \alpha_{19}] &= [\alpha_{38}, \beta_{38}); \\ T^{20}_\alpha[\beta_{19}, p_2] \cup T^{20}_\alpha(p_2, \alpha_{19}] &= P_2; \\ p_2 \in T^{20}_\alpha[\beta_{19}, p_2] \cap T^{20}_\alpha(p_2, \alpha_{19}] &= [\beta_{39}, \alpha_{39}]. \end{aligned}$$

We find $F^8|J_\ell| \approx F^8|J_r| \approx |P_2|$ and $F^8(J_\ell) \cup F^8(J_r) = P_2$, yielding $F^8 - 2$ as a CP.

In the arrangement with $F^8 - 2$ as a CP, we have the root of $\beta_{19} = \beta_{39}$ as the opening and $\alpha_{19} = \alpha_{39}$ as the closure, and find 2131 to be the smallest possible N with this arrangement, which contains 39 gaps. We find $\alpha_o = 45.16234959009\cdots$ and $\alpha_c = 45.16234959007\cdots$; the positive real root of $F^8 - 2$ is $2131.9\cdots$. Although 37 is the largest value of G we have seen so far, the maximum of $G(2131)$ is substantially larger, as we will shortly see.

The previous case shows that, for N large enough, gaps around periodic points with long consecutive strings of $1, 1, 2$ exist. Examples (with CP and the opening and closure equations) are:

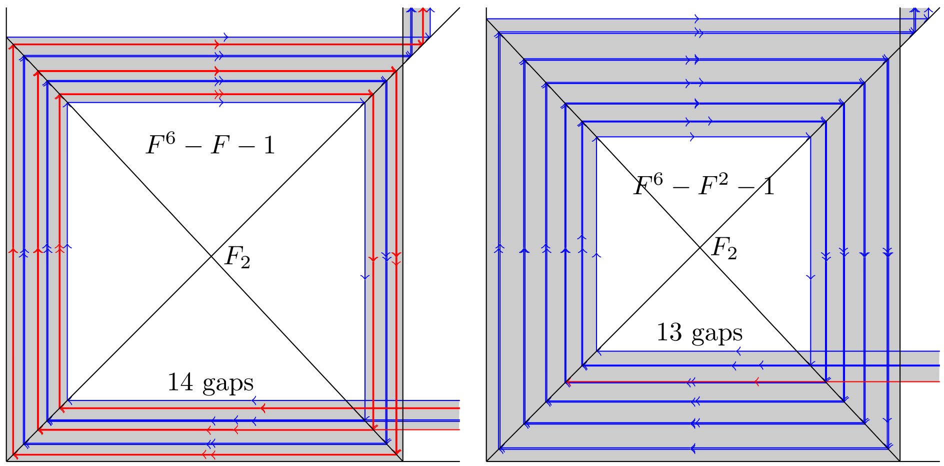
$$N = 3330, \quad G = 43, \quad F^8 - 2, \quad \beta_{19} = \beta_{45}, \quad \alpha_{19} = \alpha_{45}, \quad [\overline{abb, ab, \underbrace{a, \dots, a}_{4 \text{ times } a}, b}];$$

$$N = 4796, \quad G = 49, \quad F^{10} - 2, \quad \beta_{19} = \beta_{51}, \quad \alpha_{19} = \alpha_{51}, \quad [\overline{abb, ab, \underbrace{a, \dots, a}_{6 \text{ times } a}, b}];$$

$$N = 6527, \quad G = 55, \quad F^{12} - 2, \quad \beta_{19} = \beta_{57}, \quad \alpha_{19} = \alpha_{57}, \quad [\overline{abb, ab, \underbrace{a, \dots, a}_{8 \text{ times } a}, b}].$$

In these cases, $[\overline{abb}]$, $[\overline{ab}]$, $[\overline{a}]$, $[\overline{2}]$, and $[\overline{1}]$ are also contained in gaps.

The last examples of arrangements in Phase I with relatively few gaps are those with $\beta_2 \le p_2$. In that case, the minimal m such that $T_\alpha^m(J_r) \cap P_2 \ne \emptyset$ is 10; for J_ℓ this number of iterations is at least 7. It follows that if $\beta_2 \le p_2$, all arrangements have $F^n - F^{n-m} - 1$ as a CP, where $3 \le m \le n$. There is much interference between the images of J_ℓ and J_r, resulting in a relatively small G. Figure 11 shows two arrangements for $N = 900$. In the left one, the smallest periodic point in A_α is $[\overline{abb, ab, ab, ab, b}]$; in the right one, it is $[\overline{abb, b, b, b}]$.



$N = 900; \alpha = 28.9986$ $N = 900; \alpha = 28.99618$

Figure 11: Two arrangements for $N = 900$ and $\beta_2 \le p_2$

We conclude that for fixed N, the maximal number of gaps is attained for $\alpha \in (\mathcal{A}_1, \mathcal{A}_2)$ (see Section 3: $\mathcal{A}_1$ is the root of $\beta_2 = p_2$ and $\mathcal{A}_1$ is the root of $\beta_2 = p_2$). More specifically, in the rest of this paper we will confine ourselves to arrangements with CP either $F^n - F^{n-1} - 1$ or $F^n - F^{n-2} - 2$.

6. Optimal and Ideal Arrangements

In this section, we will explicitly look for arrangements with the largest possible number of gaps, for N fixed. We use the following definition.

Definition 6. An arrangement $\Upsilon_{N,\alpha}$ is called *optimal* if $G(N, \alpha)$ is maximal.

It is convenient to introduce some periodic points that will be important for the determination of optimal arrangements. The way they are defined, as well as the role they play in the classification of optimal arrangements, is very similar to the definition of the numbers c_n in Theorem 1.

Definition 7. Let $ab := 2,1,1,2,2$ and $abb := 2,1,1,2,2,2,2$. Let $d_1 := [\overline{ab}]$, $d_2 := [\overline{abb}]$, $d_3 := [\overline{abb, ab, ab}]$, and $d_4 := [\overline{abb, ab, ab, abb, abb}]$. Let $n \geq 5$ and let j be a string of partial quotients. Then we define d_n recursively as follows: if $d_{n-1} = [\overline{j, ab, ab}]$, then $d_n = [\overline{j, ab, ab, j, abb}]$; if $d_{n-1} = [\overline{j, abb}]$, then $d_n = [\overline{j, abb, j, ab, ab}]$.

The following lemmas are similar to Corollaries 1 and 2 in [1]; the proofs are similarly straightforward.

Lemma 4. *Let $d_1, d_2, d_3 \ldots$ be the sequence of periodic points in Definition 7. Then $k > \ell \geq 1$ if and only if $d_k < d_\ell$.*

Lemma 5. *Let d_n be a periodic point as introduced in Definition 7. Then for $n \geq 1$ the number d_n is the smallest of all points in the orbit of d_n.*

Note that if $d_m < d_n$, then $T_\alpha^{-1}(d_n) < T_\alpha^{-1}(d_m) < p_2$. From this the next lemma follows immediately, since $d_n \in A_\alpha$ if and only if $T_\alpha^{-1}(d_n) \in A_\alpha$.

Lemma 6. *Let $d_1, d_2, d_3 \ldots$ be the sequence of periodic points in Definition 7. Let $m, n \in \mathbb{N}$, with $m > n$. If $d_n \in A_\alpha$, then $d_m \in A_\alpha$.*

Now recall the development from the left arrangement to the middle one in Figure 8. In the left one the 7 orbit points of $[\overline{abb}]$ are still in A_α, rendered with dashes. In the middle arrangement, these orbit points are contained in $I_\alpha \setminus A_\alpha$, adding 7 gaps to those containing the orbit points of $[\overline{1}]$, $[\overline{2}]$ and $[\overline{ab}]$. We want to do something similar based on the arrangement with 14 gaps, with $F^6 - F^5 - 1$ as a CP. In that case, we have $P_2 = [\beta_5, \beta_2]$. The periodic point with smallest orbit length in A_α, the point $[\overline{a}]$ excluded, has an orbit point x in $K = [\alpha, \beta_6]$, the leftmost component of A_α. Following the road map of the arrangement (in grey), we find that $T_\alpha^{13}(x) \in P_2$, and $x = [\overline{abb, ab, \ldots}]$. For all $y = [abb, ab, \mathbf{2}, \ldots]$, we have $T_\alpha(y) \in K$, implying $y = [abb, ab, abb, ab, \ldots]$. For all $y = [abb, ab, \mathbf{1}, \ldots]$, we have $T_\alpha(y) \in (\beta_3, \beta)$, $T_\alpha^2(y) \in (\beta_2, \beta_4)$, and $T_\alpha^3(y) \in P_2$. As a consequence, there are $y = [abb, ab, 1, \ldots]$ such that $T_\alpha^3(y) \in J_\ell$ and $T_\alpha^4(y) \in K$. More specifically, it follows that $x = d_3$.

Remark 11. We identified d_3 in an arrangement in which the periodic points $[\overline{1}]$, $[\overline{2}]$, d_1, and d_2 were contained in $|[\overline{1}]| + |[\overline{2}]| + |d_1| + |d_2|$ separate gaps. This approach can easily be applied to identify d_n in arrangements in which the periodic points $[\overline{1}]$, $[\overline{2}]$, $d_1, \ldots, d_{n-1}$ are contained in $|[\overline{1}]| + |[\overline{2}]| + |d_1| + \ldots + |d_{n-1}|$ separate gaps, with $n \geq 4$. It is very similar to the construction of the orbit points c_n in [1].

However, an arrangement with $d_3 \in I_\alpha \setminus A_\alpha$ does not exist for $N = 324$. For $N = 529$ it exists, although the arrangement does not have $14 + 17 = 31$ gaps, but merely $7 + 17 = 24$ gaps; see Figure 12. It shows two graphic representations of the arrangement $\Upsilon_{529,21.99822}$, the "classic one" and the one rendered as if all gaps and components have equal length. The orbit of d_3 is drawn with dashed line segments,

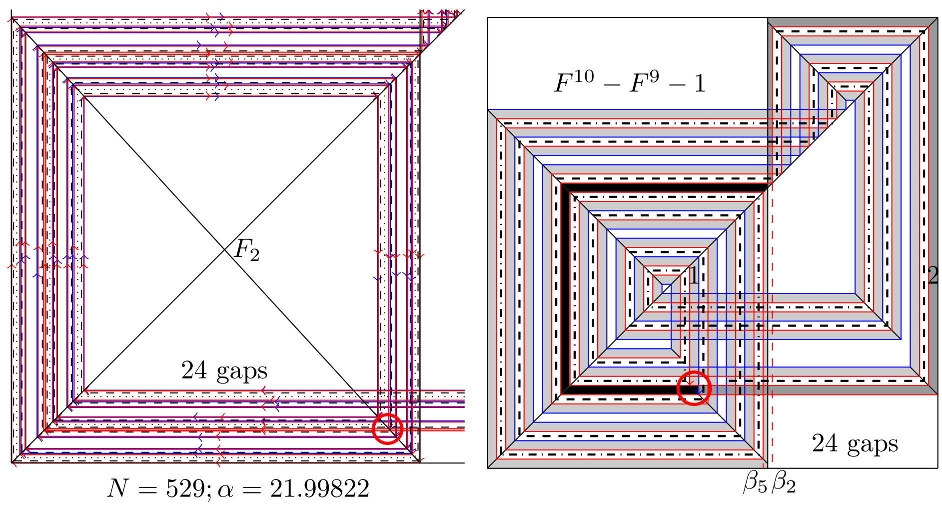


Figure 12: Two illustrations of the arrangement for $N = 529$ and $\alpha = 21.99822$

and the orbit of d_2 with dots. A drawback of the former representation is that, with so many gaps, the visual support is severely affected. In the latter, the orbit of d_3 is rendered with dashes and the orbit of d_2 with dashes and dots combined, showing more clearly that there are seven gaps that contain orbit points of both d_3 and d_2. The components of A_α around the orbit points of $[\overline{a}]$ are in dark grey, and the component containing both an orbit point from $[\overline{a}]$ and one of d_3, as well as the road to and from that component, are in black. In both representations of the arrangement, the red circles enclose a very specific part, i.e., where $\alpha_4 \approx \alpha_{28}$ and $\alpha_{22} \approx \beta_1$; we will discuss an important consequence of this shortly. In the left arrangement this is drawn for $N = 529$ and $\alpha = 21.99822$ particularly; the right arrangement is solely determined by the CP it has in common with the left one[12].

It is noteworthy that arrangements preceding those with 24 gaps have 7 gaps instead of 14 gaps; see Figure 13. We will elaborate on this by inspecting the endpoints of the gaps in the arrangements of Figures 12 and 13.

The gaps in $\Upsilon_{529,21.9984}$ are, from left to right,

$$\boldsymbol{(\alpha_{17}, \alpha_{24})}, (\alpha_7, \alpha_{12}), \boldsymbol{(\alpha_{15}, \alpha_{22})}, (\alpha_5, \alpha_{10}), \boldsymbol{(\alpha_{27}, \alpha_{20})}, (\alpha_3, \alpha_4), \boldsymbol{(\alpha_{21}, \alpha_{28})},$$
$$(\alpha_{11}, \alpha_6), \boldsymbol{(\alpha_{23}, \alpha_{16})}, (\alpha_{13}, \alpha_8), \boldsymbol{(\alpha_{25}, \alpha_8)}, (\alpha_1, \alpha_2), \boldsymbol{(\alpha_{19}, \alpha_{26})}, (\alpha_9, \alpha_{14}).$$

The gaps in boldface are related to the orbit of d_2; the difference of the indices of their endpoints is $|d_2| = 7$. These gaps close for $\alpha_c = 21.9983608\cdots$ (where

[12]A disadvantage of the representation on the right is that it suggests that components 1 and 2 (black digits) have the same image component (in black). This is because the representation does not allow for real subsets.

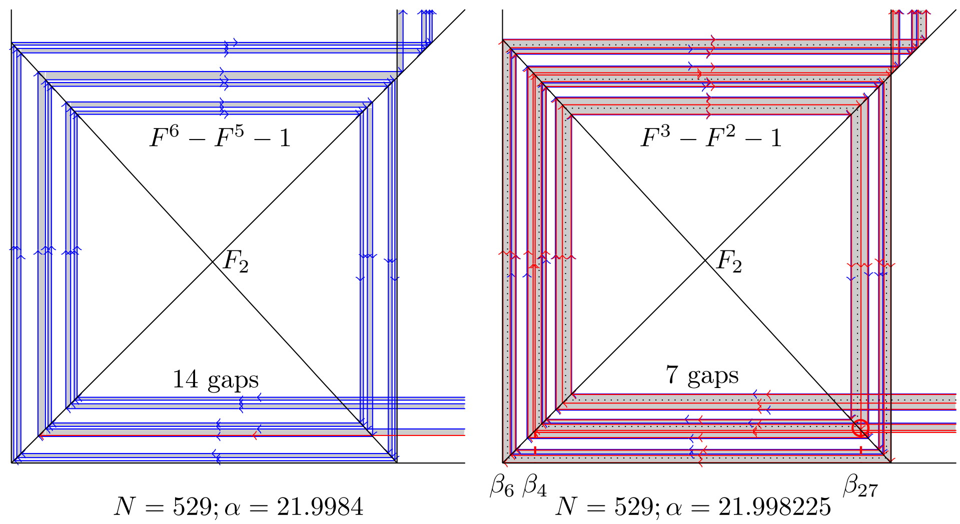


Figure 13: Preceding arrangements for one with 24 gaps for $N = 529$

$\alpha_{17} = \alpha_{24}$). Note that gaps around d_3 may open when α decreases beyond the root of $\beta_6 = d_3 = \beta_{6+|d_3|} = \beta_{23}$, which is 21.9983618. However, the associated closure of d_3 is $21.99846\cdots$. This root of the equation $\alpha_7 = \alpha_{24}$ is larger than the opening, preventing the gaps around d_3 from emerging. Moreover, due to Lemma 6, these gaps only exist if those around d_2 exist. As α decreases even more, though, something interesting happens. In $\Upsilon_{529,21.998225}$, we still have $d_2 \in A_\alpha$, the arrangement has still only 7 gaps, and all endpoints of gaps are still images of α. Then, for α decreasing beyond the root of $\beta_4 = \beta_{28}$, which is for $\alpha = \alpha_o = 21.99822\cdots$, the gaps around d_2 reopen, as do those around d_3. As a consequence, in the arrangement with 24 gaps, 7 of these contain an orbit point of both d_3 and d_2. While in many cases the opening equation involves β_6, in this situation it involves β_4. Later in this section we will come across similar, even more complex situations. If α is close to but smaller than α_o, the first three images of $J_r = (p_2, \beta_2]$ are $[\beta_3, \beta)$, $(\beta_1, \beta_4]$, and $[\beta_5, \beta_2)$, respectively. The first 23 images of $[\beta_5, p_2]$ are

$$[\alpha, \beta_6],\ [\beta_7, \alpha_1],\ \ldots\ , [\alpha_{21}, \beta_{27}],\ [\beta_{28}, \alpha_{22}],\ \text{ while }\ [\beta_{28}, \alpha_{22}] \subset [\beta_4, \beta_1].$$

In the right arrangement of Figure 12 this is illustrated with the red and the blue arrow in the red circle, indicating β_{28} and α_{22}. As soon as α is smaller than the root of $\alpha_{22} = \beta_1$ (which is $21.9982177\cdots$), all endpoints of gaps are images of α. This is for α larger than $\alpha_c = 21.998217\cdots$, which is the root of $\alpha_{25} = \alpha_{49}$; this last equation is related to α_o through the equation $\alpha_{22} = \beta_1$. For $\alpha \leq \alpha_c$ in Phase I, we have $d_2 \in A_\alpha$ and therefore $d_3 \in A_\alpha$ as well. The CP of the arrangement with 24 gaps is $F^{10} - F^9 - 1$, with real root $F \approx 1.1975$, associated with $N \approx 493$. The smallest N such that $\alpha_c > \alpha_0$ is indeed 493, with $(\alpha_c, \alpha_o) = (21.2017198\cdots, 21.2017199\cdots)$.

Being similar to the left arrangement of Figure 13, the left arrangement of Figure 14 has 14 gaps as well, but this time we have $N = 900$.

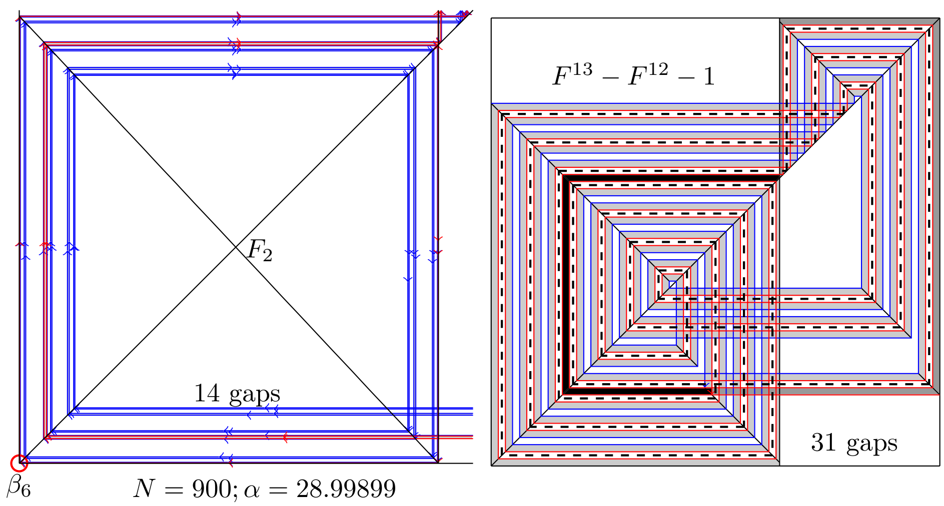


Figure 14: From 14 gaps to 31 gaps

In this case the closure of d_2 (the root of $\alpha_{17} = \alpha_{24}$, which is $28.99894\cdots$) is *smaller* than the opening of d_3, the root of $\beta_6 = \beta_{23}$, which is $28.99898\cdots$. As a consequence, less images of β are needed for opening d_3 and less images of α are needed for closing it. The closure of d_3 is $\alpha_c = 28.9989734\cdots$, which is the root of $\alpha_{31} = \alpha_{48}$. The smallest N with an arrangement similar to the one for $N = 900$ with 31 gaps is for 701, with $(\alpha_c, \alpha_o) = (25.4751303\cdots, 25.4751304\cdots)$.

For arrangements such as $\Upsilon_{900,28.99898}$, with $G = 31$, we use the following definition.

Definition 8. For each d_n, $n \geq 1$, arrangements are called *ideal up to* d_n if $[\overline{1,1,2}] \in P_2$ and all gaps contain exactly one point from the orbits of $[\overline{1}], [\overline{2}], d_1, \ldots, d_n$.

From Remark 7 it follows immediately that ideal arrangements have characteristic polynomial $F^m - F^{m-1} - 1$, for some $m \in \mathbb{N}$. We have already discussed the ideal arrangements up to d_1, d_2, and d_3, with $F^3 - F^2 - 1$, $F^6 - F^5 - 1$, and $F^{13} - F^{12} - 1$, respectively, as a CP.

Our approach for finding optimal arrangements is quite similar to the one we took in the case $\alpha = \alpha_{\max}$. It also reflects the construction of arrangements with

$F^{10} - F^9 - 1$ as a CP based on those with $F^6 - F^5 - 1$ as a CP. Where the sequences $2, 1, 1$ and $2, 2$ played a central role in the case $\alpha = \alpha_{\max}$, in Phase I the numbers d_n play this role. We will shortly explain this, but first we turn back to the arrangement with 31 gaps for $N = 900$. In Table 3 the opening and the closure of the gaps around the orbit points of d_3 for $N = 900$ is visualized numerically. The gaps around the orbit points of d_3 exist only if both $\beta_6 < \beta_{23}$ and $\alpha_{31} < \alpha_{48}$, as we will shortly explain. We use the color blue for the values $\alpha_{31} - \alpha$ and $\alpha_{48} - \alpha$ to show their volatility; similarly, we use the color red for the values $\beta_6 - \alpha$ and $\beta_{23} - \alpha$.

α	$\alpha_{31} - \alpha$	$\beta_6 - \alpha$	$d_3 - \alpha$	$\beta_{23} - \alpha$	$\alpha_{48} - \alpha$	$p_2 - \alpha$	$\beta_2 - \alpha$
28.999100	0.03402	0.00006	NA	0.0428	0.00407	0.03400	0.03442
28.999000	0.99986	0.00003	0.00001	0.99997	0.03435	0.03420	0.03441
28.998990	0.99993	0.00003	0.00002	0.00001	0.03424	0.03421	0.03441
28.998980	1.00000	0.00003	0.00003	0.00005	0.03235	0.03423	0.03441
28.998977	0.00002	0.00003	0.00004	0.00007	0.00009	0.03424	0.03441
28.998974	0.00004	0.00003	0.00004	0.00008	0.00005	0.03424	0.03441
28.998971	0.00006	0.00002	0.00004	0.00009	0.00001	0.03425	0.03441

Table 3: Opening and closure of d_3 for $N = 900$

As far as the opening is concerned: for α in $(\mathcal{A}_1, \alpha_{\max})$[13] the first six digits of the N-expansion of β are $1, 2, 1, 1, 2, 1$. In Phase I, for $\alpha < \mathcal{A}_1$, the sixth digit is 2 instead of 1, yielding the small value of β_6 that is needed to open what will be the periodic point with largest orbit length in $I_\alpha \setminus A_\alpha$. In the case of the opening of d_3, which has an orbit length of 17, we have $\beta_6 = d_3 = \beta_{6+17} = \beta_{23}$. Generally, for all ideal arrangements up to d_n we have $\beta_6 = \beta_{|d_n+6|}$ as the opening equation. As for the closure images of α: for α in $(\mathcal{A}_2, \alpha_{\max})$[14] the first seven digits of the N-expansion of α are $2, 1, 1, 2, 2, 2, 1$. In Phase I, for $\alpha < \mathcal{A}_2$, the seventh digit is 2 instead of 1. As a consequence, the first seven digits of the N-expansion of α are equal to those of d_2, which has orbit length 7. For the closure of d_1, with orbit length 5, we then find $\alpha_7 = \alpha_{7+5} = \alpha_{12}$; see Section 3. It follows that if the orbit points of d_1 and those of d_2 are contained in gaps (and those of $[\overline{a}]$ are not), the first $12 + 5 = 17$ digits of the N-expansion of α are equal to those of d_3. For the closure of d_2, with orbit length 7, we then find $\alpha_{17} = \alpha_{17+7} = \alpha_{24}$; see Section 4. Generally, for all ideal arrangements up to d_n the closure equation is $\alpha_{|d_{n+1}|} = \alpha_{|d_n|+|d_{n+1}|}$.

Remark 12. Table 3 illustrates why for most N it is hard to even find the right digits in the N-expansions of the images of α and β in the opening and closure equations. In the case of $N = 900$, for instance, we have $d(\alpha_{31}) = 1$ for $\alpha =$

[13]See Section 3: $\mathcal{A}_1$ is the root of $\beta_2 = p_2$.
[14]See Section 3: $\mathcal{A}_2$ is the root of $\alpha_6 = p_2$.

$28.99898\cdots$ and $d(\alpha_{31}) = 2$ for $\alpha = 28.99897\cdots$, while $d(\alpha_{48}) = 2$ for both $\alpha = 28.99898\cdots$ and $\alpha = 28.99897\cdots$. For the author it was often a matter of making "educated trial and error" computations before finding an α with the right digits for both sides of the closure or opening equations.

Since $|d_n| = 2\cdot|d_{n-1}| - 3\cdot(-1)^n$, we obtain the next lemma inductively.

Lemma 7. *Let d_n be a periodic point as defined above, with $n \geq 1$. Then $|d_n| = 2^{n+1} - (-1)^n$.*

Applying Lemma 7, we find the following formula for G.

Corollary 1. *Let $\Upsilon_{N,\alpha}$ be ideal up to d_n. Then*

$$G(N,\alpha) = 2 + \sum_{k=1}^{n} |d_k| = \begin{cases} 2^{n+2} - 1, \text{ if } n \text{ is odd;} \\ 2^{n+2} - 2, \text{ if } n \text{ is even.} \end{cases}$$

From Lemma 7 it follows that for ideal arrangements up to d_n, the number of components of A_α has the same parity as n. For ideal arrangements up to d_n, with n odd, the periodic point with smallest orbit length, $[\overline{a}]$ excluded, is d_{n+1}, along which J_ℓ expands until its image intersects P_2; see for instance the arrangement in the middle of Figure 7. For ideal arrangements up to d_n, with n even, the periodic point with smallest orbit length, $[\overline{a}]$ excluded, is d_n, along which J_ℓ expands *twice*; see for instance the right arrangement of Figure 7.

Now recall that the factor F is related to a sequence $a = 2, 2$ or $b = 2, 1, 1$. Let $a(n)$ and $b(n)$ be the number of a's and b's, respectively, in d_n, for $n \geq 1$. From the definition of the d_n, it is not hard to find the following lemma.

Lemma 8. *Let $n \in \mathbb{N}$. Then* $\begin{cases} a(n) = 2^{n-1}; \\ b(n) = \begin{cases} \frac{2^n+1}{3} \text{ if } n \text{ is odd;} \\ \frac{2^n-1}{3} \text{ if } n \text{ is even.} \end{cases} \end{cases}$

It follows that the degree of the CP of an arrangement that is ideal up to d_n is $a(n+1) + b(n+1)$ if n odd, and is $2(a(n) + b(n))$ if n is even. From this we induce the following corollary.

Corollary 2. *If $\Upsilon_{N,\alpha}$ is ideal up to d_n, with $n \geq 1$, its characteristic polynomial has degree* $\begin{cases} \frac{5\cdot 2^n - 1}{3}, \text{ if } n \text{ is odd;} \\ \frac{5\cdot 2^n - 2}{3}, \text{ if } n \text{ is even.} \end{cases}$

If we combine Corollary 1 and Corollary 2, we obtain the following result.

Corollary 3. *If $\Upsilon_{N,\alpha}$ is ideal up to d_n, with $n \geq 1$, then the ratio between its number of gaps and the degree of its characteristic polynomial is* $\begin{cases} \frac{12}{5} - \frac{3}{25\cdot 2^n - 5} \text{ if } n \text{ is odd;} \\ \frac{12}{5} - \frac{6}{25\cdot 2^n - 10} \text{ if } n \text{ is even.} \end{cases}$

From Corollary 3, we induce that for ideal arrangements, the ratio between the number of gaps and the degree of the CP is approximately $\frac{12}{5}$. Generally, if $\Upsilon_{N,\alpha}$ has a CP of the form $F^n - F^{n-1} - 1$, all periodic points except $[\overline{1}]$ and $[\overline{2}]$ with orbit points in gaps consist of an almost equal amount of sequences ab and abb. Since one sequence ab and one sequence abb together consist of two sequences $2, 1, 1$ and three sequences $2, 2$, associated with 12 gaps and 5 factors F, we can use $G \approx \frac{12n}{5}$ as a rule of thumb in case $\Upsilon_{N,\alpha}$ has a CP of the form $F^n - F^{n-1} - 1$.

Our approach to find optimal arrangements in Phase I is to relate them to ideal arrangements. For N up to about 700, however, this is not very useful. First, we found $\max(G) = 9$ for $154 \leq N \leq 202$, with arrangements quite dissimilar to ideal ones. Second, optimal arrangements with gaps around the orbit points of $[\overline{a}]$ are quite frequent for N up to 700, while this is not so for larger N, as we will shortly see. This is why we will first focus on 3-*optimal arrangements*, i.e., arrangements with $[\overline{a}] \in I_\alpha \setminus A_\alpha$, which have CPs of the form $F^n - F^{n-2} - 1$; see Remark 7.
First we note that if $[\overline{a}] \in I_\alpha \setminus A_\alpha$, we have

$$(p_2, \gamma_r] \overset{T_\alpha}{\to} [\gamma_{r,1}, \beta) \overset{T_\alpha}{\to} (\beta_1, \gamma_{r,2}] \overset{T_\alpha}{\to} [\gamma_{r,3}, \beta_2) \overset{T_\alpha}{\to} (\beta_3, \gamma_{r,4}] \overset{T_\alpha}{\to} [\gamma_{r,5}, \beta_4) \overset{T_\alpha}{\to} (\beta_5, \gamma_{r,6}] \subset P_2.$$

It follows that the gaps around the orbit points of $[\overline{a}]$ are $(\gamma_r, \gamma_{r,3}), (\gamma_{r,4}, \gamma_{r,1})$, and $(\gamma_{r,2}, \gamma_{r,5})$. These gaps open if $\beta_n = \beta_{n+3} = \gamma_r$ and close if $\alpha_m = \alpha_{m,3} = \gamma_r$ for certain $n, m \in \mathbb{N}$.

We distinguish two cases of evolving arrangements in which gaps around the orbit points of $[\overline{a}]$ emerge and later vanish as α decreases from $\alpha_{\max}$. The first is that both $\Upsilon_{N,\alpha_o([\overline{a}])}$ and $\Upsilon_{N,\alpha_c([\overline{a}])}$ have $F^n - F^{n-1} - 1$ as a CP, for some $n \in \mathbb{N}$. The second case is that for some $\alpha \in (\alpha_c([\overline{a}]), \alpha_o([\overline{a}]))$ more gaps emerge, i.e., containing gaps around the orbits of some periodic point d_n. Then $\Upsilon_{N,\alpha_o([\overline{a}])}$ has $F^n - F^{n-1} - 1$ as a CP and $\Upsilon_{N,\alpha_c([\overline{a}])}$ has $F^m - F^{m-1} - 1$ as a CP, for some $m > n$.

An example of the first case is $N = 324$ and $\alpha \in (16.998, 16.9985)$, that we came across previously. For $\Upsilon_{324,16.9985}$, the CP is $F^3 - F^2 - 1$. The components of A_α, the boldfaced ones of which contain an orbit point of $[\overline{a}]$, are

$$[\alpha, \beta_6],\ \boldsymbol{[\beta_1, \beta_4]},\ [\beta_9, \alpha_3],\ [\alpha_4, \beta_{10}],\ \boldsymbol{[\beta_5, \beta_2]},\ [\beta_7, \alpha_1],\ [\alpha_2, \beta_8],\ \boldsymbol{[\beta_3, \beta)},$$

where $P_2 = [\beta_5, \beta_2]$. When the gaps around the orbit points of $[\overline{a}]$ emerge, the boldfaced components of A_α will split so as to create three additional gaps. The endpoints of these gaps are the subsequent images of β_{10}. This is the image of β with largest index number that is the endpoint of a component in $\Upsilon_{324,16.9985}$, which has $F^3 - F - 1$ as a CP. The ten components of this arrangement are

$$[\alpha, \beta_6],\ \boldsymbol{[\beta_1, \beta_{14}]},\ \boldsymbol{[\beta_{11}, \beta_4]},\ [\beta_9, \alpha_3],\ [\alpha_4, \beta_{10}],\ \boldsymbol{[\beta_5, \beta_{12}]},\ \boldsymbol{[\beta_{15}, \beta_2]},$$
$$[\beta_7, \alpha_1],\ [\alpha_2, \beta_8],\ \boldsymbol{[\beta_3, \beta_{16}]},\ \boldsymbol{[\beta_{13}, \beta)},$$

where $P_2 = [\beta_5, \beta_{12}]$. Then, when α decreases beyond the root of $\alpha_5 = \beta_4$, the CP remains $F^3 - F - 1$. Now the components, such as in $\Upsilon_{324,16.9981}$, are

$$[\alpha, \beta_6],\ [\boldsymbol{\beta_1}, \boldsymbol{\alpha_{15}}],\ [\boldsymbol{\alpha_{12}}, \boldsymbol{\alpha_5}],\ [\alpha_{10}, \alpha_3],\ [\alpha_4, \alpha_{11}],\ [\boldsymbol{\alpha_6}, \boldsymbol{\alpha_{13}}],\ [\boldsymbol{\alpha_{16}}, \boldsymbol{\beta_2}],$$
$$[\alpha_8, \alpha_1],\ [\alpha_2, \alpha_9],\ [\boldsymbol{\beta_3}, \boldsymbol{\alpha_{17}}],\ [\boldsymbol{\alpha_{14}}, \boldsymbol{\beta}).$$

Finally, when α decreases beyond the root of $\alpha_{12} = \alpha_{15}$, these components, such as in $\Upsilon_{324,16.998}$, become

$$[\alpha, \beta_6],\ [\boldsymbol{\beta_1}, \boldsymbol{\alpha_5}],\ [\alpha_{10}, \alpha_3],\ [\alpha_4, \alpha_{11}],\ [\boldsymbol{\alpha_6}, \boldsymbol{\beta_2}],\ [\alpha_8, \alpha_1],\ [\alpha_2, \alpha_9],\ [\boldsymbol{\beta_3}, \boldsymbol{\beta}).$$

In this case the development of arrangements is

$$F^3 - F^2 - 1 \to F^3 - F - 1 \to F^3 - F^2 - 1.$$

An example of the second case is $N = 529$ and $\alpha \in (21.9984, 21.9989)$. For $\alpha \in [\alpha_o(d_2), 21.9989)$, where $\alpha_o(d_2) = 21.9986\cdots$ is the root of $\beta_6 = \beta_{13}$, the development of the arrangements and their CPs is similar to the one for $N = 324$ above, up to and including $\Upsilon_{324,16.9981}$. Now that we have $N = 529$, however, gaps around the orbit points of d_2 emerge before those around $[\overline{a}]$ close. The components of $\Upsilon_{529,21.9986}$, with 17 gaps and $F^6 - F^4 - 1$ as a CP, are

$$[\alpha, \beta_6],\ [\beta_{13}, \alpha_7],\ [\boldsymbol{\beta_1}, \boldsymbol{\alpha_{15}}],\ [\boldsymbol{\alpha_{12}}, \boldsymbol{\beta_4}],\ [\beta_{11}, \alpha_5],\ [\alpha_{10}, \beta_{16}],\ [\beta_9, \alpha_3],\ [\alpha_4, \beta_{10}],$$
$$[\beta_{17}, \alpha_{11}],\ [\alpha_6, \beta_{12}], [\boldsymbol{\beta_5}, \boldsymbol{\alpha_{13}}],\ [\boldsymbol{\alpha_{16}}, \boldsymbol{\beta_2}],\ [\alpha_8, \beta_{14}],\ [\beta_7, \alpha_1],\ [\alpha_2, \beta_8],\ [\beta_{15}, \alpha_9],$$
$$[\boldsymbol{\beta_3}, \boldsymbol{\alpha_{17}}],\ [\boldsymbol{\alpha_{14}}, \boldsymbol{\beta}).$$

When α decreases beyond $\alpha_c([\overline{a}]) = 21.99856\cdots$, which is the root of $\alpha_{12} = \alpha_{15}$, the boldfaced intervals merge into $[\beta_1, \beta_4]$, $[\beta_5, \beta_2]$, and $[\beta_3, \beta)$, and we obtain

$$[\alpha, \beta_6],\ [\beta_{13}, \alpha_7],\ [\boldsymbol{\beta_1}, \boldsymbol{\beta_4}],\ [\beta_{11}, \alpha_5],\ [\alpha_{10}, \beta_{16}],\ [\beta_9, \alpha_3],\ [\alpha_4, \beta_{10}],\ [\beta_{17}, \alpha_{11}],$$
$$[\alpha_6, \beta_{12}], [\boldsymbol{\beta_5}, \boldsymbol{\beta_2}],\ [\alpha_8, \beta_{14}],\ [\beta_7, \alpha_1],\ [\alpha_2, \beta_8],\ [\beta_{15}, \alpha_9],\ [\boldsymbol{\beta_3}, \boldsymbol{\beta}).$$

If α decreases beyond the root $21.998434\cdots$ of $\beta_{18} = \beta_{21}$, gaps around the orbit points of $[\overline{a}]$ emerge once more. The components of $\Upsilon_{529,21.21.998434}$ are

$$[\alpha, \beta_6],\ [\beta_{13}, \alpha_7],\ [\boldsymbol{\beta_1}, \boldsymbol{\beta_{18}}],\ [\boldsymbol{\beta_{21}}, \boldsymbol{\beta_4}],\ [\beta_{11}, \alpha_5],\ [\alpha_{10}, \beta_{16}],\ [\beta_9, \alpha_3],\ [\alpha_4, \beta_{10}],$$
$$[\beta_{17}, \alpha_{11}],\ [\alpha_6, \beta_{12}], [\boldsymbol{\beta_5}, \boldsymbol{\beta_{22}}],\ [\boldsymbol{\beta_{19}}, \boldsymbol{\beta_2}],\ [\alpha_8, \beta_{14}],\ [\beta_7, \alpha_1],\ [\alpha_2, \beta_8],\ [\beta_{15}, \alpha_9],$$
$$[\boldsymbol{\beta_3}, \boldsymbol{\beta_{20}}],\ [\boldsymbol{\beta_{23}}, \boldsymbol{\beta}).$$

In this case, the CP is $F^7 - F^5 - 1$. In Figure 15 the arrangements of $\Upsilon_{529,21.21.9986}$ and $\Upsilon_{529,21.21.998434}$ are drawn. The overlapping images of J_ℓ and J_r are in black; the component $[\beta_{17}, \alpha_{11}]$ is marked with an $*$. In the left arrangement, $T_\alpha^{13}(J_\ell) = T^{13}([\beta_5, p_2]) = T([\beta_{17}, \alpha_{11}]) \subset [\alpha_{12}, \beta_4]$ and $T_\alpha^5(J_r) = T_\alpha^5((p_2, \alpha_{13}] \subset [\alpha_{12}, \beta_4]$, while $T_\alpha([\alpha_{12}, \beta_4]) = [\beta_5, \alpha_{13}] = P_2$; in the right one, $T_\alpha^{13}(J_\ell) = T^{13}([\beta_5, p_2]) =$

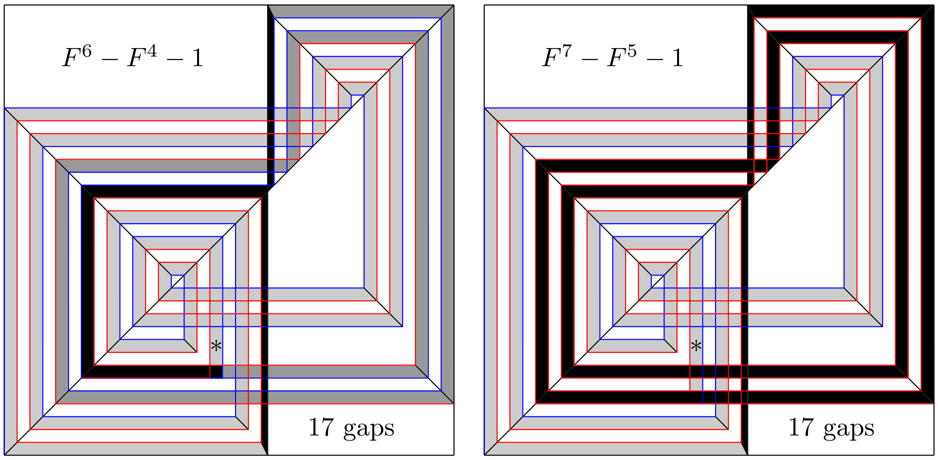


Figure 15: Arrangements with 17 gaps and different characteristic polynomials

$T_\alpha([\beta_{17}, \alpha_{11}]) \subset [\beta_1, \beta_{18}]$ and $T_\alpha^2(J_r) = T_\alpha^2((p_2, \beta_{22}]) \subset [\beta_1, \beta_{18}]$, while $T_\alpha^4([\beta_1, \beta_{18}]) = [\beta_5, \beta_{22}] = P_2$. The main difference between these two arrangements is that in the left one the smallest n such that $T_\alpha^n(J_\ell) \subset P_2$ is 14 and in the right one it is 17.

When α decreases beyond the root $29.998843\cdots$ of $\alpha_{12} = \beta_1$, we have an arrangement with 17 gaps and images of α for all endpoints of gaps:

$[\alpha, \beta_6]$, $[\alpha_{24}, \alpha_7]$, $\boldsymbol{[\alpha_{12}, \alpha_{29}]}$, $\boldsymbol{[\alpha_{32}, \alpha_{15}]}$, $[\alpha_{22}, \alpha_5]$, $[\alpha_{10}, \alpha_{27}]$, $[\alpha_{20}, \alpha_3]$, $[\alpha_4, \alpha_{21}]$, $[\alpha_{28}, \alpha_{11}]$, $[\alpha_6, \alpha_{23}]$, $\boldsymbol{[\alpha_{16}, \alpha_{33}]}$, $\boldsymbol{[\alpha_{30}, \alpha_{13}]}$, $[\alpha_8, \alpha_{25}]$, $[\alpha_{18}, \alpha_1]$, $[\alpha_2, \alpha_{19}]$, $[\alpha_{26}, \alpha_9]$, $\boldsymbol{[\alpha_{14}, \alpha_{31}]}$, $\boldsymbol{[\alpha_{34}, \beta)}$.

The closure equation of $[\overline{a}]$ in this case is $\alpha_{29} = \alpha_{32}$. The development of arrangements is

$$F^3 - F^2 - 1 \to F^3 - F - 1 \to F^6 - F^4 - 1 \to F^6 - F^5 - 1 \to F^7 - F^5 - 1.$$

Remark 13. The opening and closure equations of $[\overline{a}]$ are $\beta_{11} = \beta_{14}$ and $\alpha_{12} = \alpha_{15}$ only if the number of gaps for $\alpha = \alpha_o([\overline{a}])$ is 7, and the endpoints of the components are α through α_4 and β through β_{10}. If the number of gaps is 14 if $\alpha = \alpha_o([\overline{a}])$, then $\alpha_o([\overline{a}])$ is the root of $\beta_{18} = \beta_{21}$. In this case the CP is $F^7 - F^5 - 1$, with positive root $525.8\cdots$. Note that above we already saw that for $N \geq 493$ arrangements exist with 24 gaps. If we confine ourselves to optimal arrangements, those with a maximum of 17 gaps have $438 \leq N \leq 492$, so we take $\alpha_o([\overline{a}])$ to be the root of $\beta_{11} = \beta_{14}$ for $438 \leq N \leq 492$.

In the search for optimal arrangements for $N \geq 493$, the characteristic polynomial will be quite useful when it comes to the existence of 3-optimal arrangements. We already found that for $493 \leq N \leq 700$ arrangements exist with 24 gaps and that for $N \geq 701$ arrangements exist with 31 gaps. If there would be an optimal arrangement with 27 gaps, i.e., with $[\overline{a}] \in I_\alpha \backslash A_\alpha$ for $493 \leq N \leq 700$, the CP would be $F^{10} - F^8 - 1$ or $F^{11} - F^9 - 1$. Note that the latter CP would yield a larger minimum value of N. The positive root $F^{10} - F^8 - 1$ is $1.150963\cdots$, yielding a minimum of about 810 for N. From this it follows that no optimal arrangements with more than 24 gaps exist for $N \leq 700$. This can be understood more heuristically: in the arrangement with 24 gaps, we have $T_\alpha^{23}(J_\ell) = [\beta_{28}, \alpha_{22}] \subset [\beta_4, \beta_1]$, while a 3-optimal arrangement comes with splitting $[\beta_4, \beta_1]$. This is yet another example of interference in arrangements where $[\overline{a}] \in I_\alpha \setminus A_\alpha$.

We think that this is the best point to unfold our approach of determining optimal arrangements based on ideal arrangements. Let $N \geq 701$. A 3-optimal arrangement with $G = 34$ has $F^{13} - F^{11} - 1$ as a CP, the root of which is $1.1631\cdots$, yielding a minimal value of about 1125 for N to possibly have an arrangement with 34 gaps. We compute the smallest N such that the root α_o of $\beta_{35} = \beta_{38}$ is larger the root α_c of $\alpha_{60} = \alpha_{63}$ and find that this is indeed $N = 1125$. For $\alpha = 32.540182852$ the components of A_α are

$$[\alpha, \beta_6], [\beta_{23}, \alpha_{17}], [\alpha_{24}, \beta_{30}], [\beta_{13}, \alpha_7], [\alpha_{12}, \beta_{18}], \boldsymbol{[\beta_1, \beta_{38}]}, \boldsymbol{[\beta_{35}, \beta_4]}, [\beta_{21}, \alpha_{15}],$$
$$[\alpha_{22}, \beta_{28}], [\beta_{11}, \alpha_5], [\alpha_{10}, \beta_{16}], [\beta_{33}, \alpha_{27}], [\alpha_{20}, \beta_{26}], [\beta_9, \alpha_3], [\alpha_4, \beta_{10}], [\beta_{27}, \alpha_{21}],$$
$$[\alpha_{28}, \beta_{34}], [\beta_{17}, \alpha_{11}], [\alpha_6, \beta_{12}], [\beta_{29}, \alpha_{23}], [\alpha_{16}, \beta_{22}], \boldsymbol{[\beta_5, \beta_{36}]}, \boldsymbol{[\beta_{39}, \beta_2]}, [\beta_{19}, \alpha_{13}],$$
$$[\alpha_8, \beta_{14}], [\beta_{31}, \alpha_{25}], [\alpha_{18}, \beta_{24}], [\beta_7, \alpha_1], [\alpha_2, \beta_8], [\beta_{25}, \alpha_{19}], [\alpha_{26}, \beta_{32}], [\beta_{15}, \alpha_9],$$
$$[\alpha_{14}, \beta_{20}], \boldsymbol{[\beta_3, \beta_{40}]}, \boldsymbol{[\beta_{37}, \beta)}.$$

Note that the difference between the indices of neighbouring endpoints that are images of β is 17, except in the boldfaced ones where $[\beta_3, \beta)$, $[\beta_1, \beta_4]$, and $[\beta_5, \beta_2]$ are split in two. This means that the gaps around the orbit points of 17 have opened relatively recently. The endpoints of the gaps, except those around the orbit points of $[\overline{a}]$, are evenly divided among images of α and β. For $\alpha = 32.540182849$ (and not for $\alpha = 32.540182848$ or $\alpha = 32.54018285$) the components of A_α are

$$[\alpha, \alpha_{31}], [\alpha_{48}, \alpha_{17}], [\alpha_{24}, \alpha_{55}], [\alpha_{38}, \alpha_7], [\alpha_{12}, \alpha_{43}], \boldsymbol{[\beta_1, \alpha_{63}]}, \boldsymbol{[\alpha_{60}, \alpha_{29}]}, [\alpha_{46}, \alpha_{15}],$$
$$[\alpha_{22}, \alpha_{53}], [\alpha_{36}, \alpha_5], [\alpha_{10}, \alpha_{41}], [\alpha_{58}, \alpha_{27}], [\alpha_{20}, \alpha_{51}], [\alpha_{34}, \alpha_3], [\alpha_4, \alpha_{35}], [\alpha_{52}, \alpha_{21}],$$
$$[\alpha_{28}, \alpha_{59}], [\alpha_{42}, \alpha_{11}], [\alpha_6, \alpha_{37}], [\alpha_{54}, \alpha_{23}], [\alpha_{16}, \alpha_{47}], \boldsymbol{[\alpha_{30}, \alpha_{61}]}, \boldsymbol{[\alpha_{64}, \beta_2]}, [\alpha_{44}, \alpha_{13}],$$
$$[\alpha_8, \alpha_{39}], [\alpha_{56}, \alpha_{25}], [\alpha_{18}, \alpha_{49}], [\alpha_{32}, \alpha_1], [\alpha_2, \alpha_{33}], [\alpha_{50}, \alpha_{19}], [\alpha_{26}, \alpha_{57}], [\alpha_{40}, \alpha_9],$$
$$[\alpha_{14}, \alpha_{45}], \boldsymbol{[\beta_3, \alpha_{65}]}, \boldsymbol{[\alpha_{62}, \beta)}.$$

Now, the endpoints of almost all gaps are images of α, except β_1, β_2 and β_3. For these three endpoints to be images we would have to do with $F^{14} - F^{12} - 1$, corresponding with a minimal N of about 1237. However, for N as small as 1181 the optimal arrangement already has $G = 45$. For this, we computed the smallest

N such that the root α_o of $\beta_4 = \beta_{52}$ is larger the root α_c of $\alpha_{43} = \alpha_{91}$, and found that this is indeed $N = 1181$. For the indices of the images of α and β in the opening and closure equations, we followed the same reasoning as for arrangements for $N = 529$ with 24 gaps. In the case of optimal arrangements with 45 gaps, those around the 17 orbit points of d_3 are part of the 31 gaps containing d_4, yielding the opening equation $\beta_4 = \beta_{4+17+31} = \beta_{52}$; cf. the opening equation of the gaps around the orbit points of d_3 for $N = 529$; see Figure 13. The closure equation $\alpha_{43} = \alpha_{91}$ is also similar to the closure equation in the example with $N = 529$.

Remark 14. There are no optimal arrangements with 38 gaps. For these to exist, the orbit points of d_4 would have to be contained in gaps, while those of both d_2 and d_3 would be contained in the gaps around d_4; cf. arrangements with 24 gaps. However, the orbit points of d_2 are contained in their "own" gap for a large part of $(\mathcal{A}_1, \mathcal{A}_2)$, and for N large enough to possibly allow gaps around d_4 containing gaps around d_2 and d_3 as well, arrangements exist with more than 38 gaps, as we will shortly see.

For $N \in \{1841, \cdots, 1845\}$ only, the optimal arrangements have 48 gaps. For these N the root α_o of $\beta_{49} = \beta_{52}$ is larger than the root α_c of $\alpha_{88} = \alpha_{91}$. The indices of the images of α and β in these arrangements are found in a similar way as we found $\beta_{35} = \beta_{38}$ and $\alpha_{60} = \alpha_{63}$ for arrangements with 34 gaps on the previous page. The smallest N that is ideal up to d_4, with the opening equation $\beta_6 = \beta_{37}$ and the closure equation $\alpha_{65} = \alpha_{96}$, is 1846.

Proceeding this way, we first look at the smallest N which is ideal up to d_5, with the opening equation $\beta_6 = \beta_{71}$ and closure equation $\alpha_{127} = \alpha_{192}$. This appears to be $N = 5253$. Then we try and check whether there are optimal arrangements with $62 < G < 127$. This concerns arrangements with gaps around the orbit points of d_5 that contain orbit points of other periodic points as well. This way, we find that there are optimal arrangements with 79 gaps as well as with 96 gaps. In the case of 79 gaps, the gaps around d_5 not only contain the orbit points of d_4 but also those of d_3.

We will now give an exposition of the way we analyse these arrangements with 79 gaps. The computations are very intricate, but we have gained some experience in where to look and what to look for. As for "where", in Section 5 we concluded that optimal arrangements $\Upsilon_{N,\alpha}$ have $\alpha \in (\mathcal{A}_1, \mathcal{A}_2)$, which is but a small interval. For $N = 2916$, the root of $\beta_2 = p_2$ is $52.9996573\cdots$, and the root of $\alpha_6 = p_2$ is $52.9997097\cdots$. As for "what", we inspect arrangements for $N = 2916$, knowing that for N as small as 1846 ideal arrangements up to d_4 exist, with 62 gaps. Based on the previous results, we suspect that $N = 2916$ will be sufficiently large to attain 79 gaps. After a little trial and error with various α in $(\mathcal{A}_1, \mathcal{A}_2)$, we find that $\Upsilon_{2916,52.9996593}$ has indeed 79 gaps. This is quite interesting, since $\Upsilon_{2916,52.99965936}$ and $\Upsilon_{2916,52.99965924}$ have only 14 gaps. The development (as α decreases) from 14 to 79 and back to 14 gaps resembles the "jump" from 7 to 24 gaps for $N = 529$; see the discussion related to Figure 13. This time it is even more complex: the orbit points of both d_3 and d_4 are contained in the gaps around the orbit points of d_5, with $|d_5| = 65$. In this case, if α is close to but smaller than α_o, the first three images of $J_r = (p_2, \beta_2]$ are also $[\beta_3, \beta)$, $(\beta_1, \beta_4]$, and $[\beta_5, \beta_2)$, respectively. The first 78 images of $[\beta_5, p_2]$ are

$$[\alpha, \beta_6],\ [\beta_7, \alpha_1],\ \ldots,\ [\alpha_{76}, \beta_{82}],\ [\beta_{83}, \alpha_{77}], \quad \text{while} \quad [\beta_{83}, \alpha_{77}] \subset [\beta_1, \beta_4].$$

Together, $[\alpha, \beta_6]$, $[\beta_7, \alpha_1]$, $\ldots$, $[\alpha_{76}, \beta_{82}]$, P_2, $[\beta_3, \beta)$, and $(\beta_1, \beta_4]$ constitute A_α, the 80 components of which separate the 79 gaps of the arrangement. It appears that the set of gaps containing orbit points of d_3 has empty intersection with the set of gaps containing orbit points of d_4. What is more, for α smaller than but close to α_o all gaps around the orbit points of d_5 have both an image of α and one of β as endpoints, while the gaps containing orbit points of d_3 only have images of β as endpoints. As a consequence, the difference between the indices in the opening equation is $|d_3| + |d_5| = 82$. As soon as α is smaller than the root of $\alpha_{77} = \beta_4$ (which is $52.999659261\cdots$), all endpoints of gaps, except $\beta, \ldots, \beta_3$, are images of α. This is for α larger than $\alpha_c = 52.999659254\cdots$, which is the root of $\alpha_{77} = \alpha_{159}$; this last equation is related to α_o through the equation $\alpha_{77} = \beta_4$, yielding α_o to be the root of $\beta_4 = \beta_{86}$. As for determining the smallest N with a similar arrangement, we simply use the digits for the opening and closure equations, that are easily computed for $N = 2916$. This is how we find $N = 2598$ as the smallest with an arrangement of 79 gaps.

Remark 15. The previous exposition concerning arrangements with 79 gaps may appear unsatisfactory insofar hardly any explanation of results is given. However, it is the complexity of the subject that shows that so far hardly any general properties of arrangements with more than 2 gaps can be given. A lot of computing seems unavoidable, but has shown to sometimes be quite clarifying.

We conclude this paper with an overview of optimal gaps. In Table 4 the CPs of optimal arrangements are given for N up to 5252. The ideal arrangements are in boldface. We have not found 3-optimal arrangements for $N > 1845$. Figure 16 shows the optimal arrangement with $F^{26} - F^{25} - 1$ as a CP, which is not attainable for $N < 1846$. The even distribution of the gaps around the orbit points of the concerning periodic points is clearly visible. The gaps around d_1 through d_4 are rendered with four different patterns of dots and dashes.

7. Discussion

In comparison with the arrangements where $\alpha = \alpha_{\max}$, those in Phase I show a wide variety. To some extent, this paper aims to provide an introduction of a way to cope

$G_{\max}$	CP+1	α_o	α_c	optimal for N	ideal
2	$F^2 - F$	NA	NA	33 – 109	
7	$\boldsymbol{F^3 - F^2}$	$\beta_{\mathbf{1}} = \beta_{\mathbf{6}}$	$\alpha_{\mathbf{7}} = \alpha_{\mathbf{12}}$	**110 – 153**	up to $\boldsymbol{d_1}$
9	$F^4 - F^3$	$\beta_1 = \beta_{13}$	$\alpha_7 = \alpha_{19}$	154 – 202	
10	$F^3 - F$	$\beta_{11} = \beta_{14}$	$\alpha_{12} = \alpha_{15}$	203 – 253	
14	$\boldsymbol{F^6 - F^5}$	$\beta_{\mathbf{6}} = \beta_{\mathbf{13}}$	$\alpha_{\mathbf{17}} = \alpha_{\mathbf{24}}$	**254 – 437**	up to $\boldsymbol{d_2}$
17	$F^6 - F^4$	$\beta_{11} = \beta_{14}$	$\alpha_{12} = \alpha_{15}$	438 – 492	
24	$F^{10} - F^9$	$\beta_4 = \beta_{28}$	$\alpha_{25} = \alpha_{49}$	493 – 700	
31	$\boldsymbol{F^{13} - F^{12}}$	$\beta_{\mathbf{6}} = \beta_{\mathbf{23}}$	$\alpha_{\mathbf{31}} = \alpha_{\mathbf{48}}$	**701 – 1124**	up to $\boldsymbol{d_3}$
34	$F^{13} - F^{11}$	$\beta_{35} = \beta_{38}$	$\alpha_{60} = \alpha_{63}$	1125 – 1180	
45	$F^{19} - F^{18}$	$\beta_4 = \beta_{52}$	$\alpha_{43} = \alpha_{91}$	1181 – 1840	
48	$F^{19} - F^{17}$	$\beta_{49} = \beta_{52}$	$\alpha_{88} = \alpha_{91}$	1841 – 1845	
62	$\boldsymbol{F^{26} - F^{25}}$	$\beta_{\mathbf{6}} = \beta_{\mathbf{37}}$	$\alpha_{\mathbf{65}} = \alpha_{\mathbf{96}}$	**1846 – 2597**	up to $\boldsymbol{d_4}$
79	$F^{33} - F^{32}$	$\beta_4 = \beta_{86}$	$\alpha_{77} = \alpha_{159}$	2598 – 3453	
96	$F^{40} - F^{39}$	$\beta_4 = \beta_{100}$	$\alpha_{97} = \alpha_{193}$	3454 – 5252	
127	$\boldsymbol{F^{53} - F^{52}}$	$\beta_{\mathbf{6}} = \beta_{\mathbf{71}}$	$\alpha_{\mathbf{127}} = \alpha_{\mathbf{192}}$	**5253** – ...	up to $\boldsymbol{d_5}$

Table 4: Some maximal values of G in Phase I

with this. The combination of characteristic polynomials and opening and closure equations proved to be quite useful in determining the smallest N for which a given number of gaps are possible. We found arrangements with very small components with characteristic polynomials $F^n - F^3 - 1$, arrangements width periodic points of even length in gaps, and gaps containing multiple periodic points. As to the number of gaps $G(N, \alpha)$, we found that in Phase I this can be more than 6 times as large as when $\alpha = \alpha_{\max}$.

Still, the computations are very intricate, and even matters such as optimal arrangements with 38 gaps are often surprisingly hard to tackle. This is also the case with finding the digits to even generate opening and closure equations; see Remark 12. For $N \geq 2952$, we found arrangements with 65 gaps and $F^{27} - F^{25} - 1$ as a CP, and $\beta_{66} = \beta_{69}$ and $\alpha_{125} = \alpha_{128}$ as opening and closure equation. We did not found, however, arrangements with 65 gaps and $F^{26} - F^{24} - 1$ as a CP. It would be worthwhile to investigate this further. Of a more fundamental nature is a general formula for arrangements ideal up to d_n as a function of N. Interesting as this is, in a prospective paper we will discuss Phases II through IV, and we will show that in Phase IV optimal arrangements often have even more gaps than in Phase I.

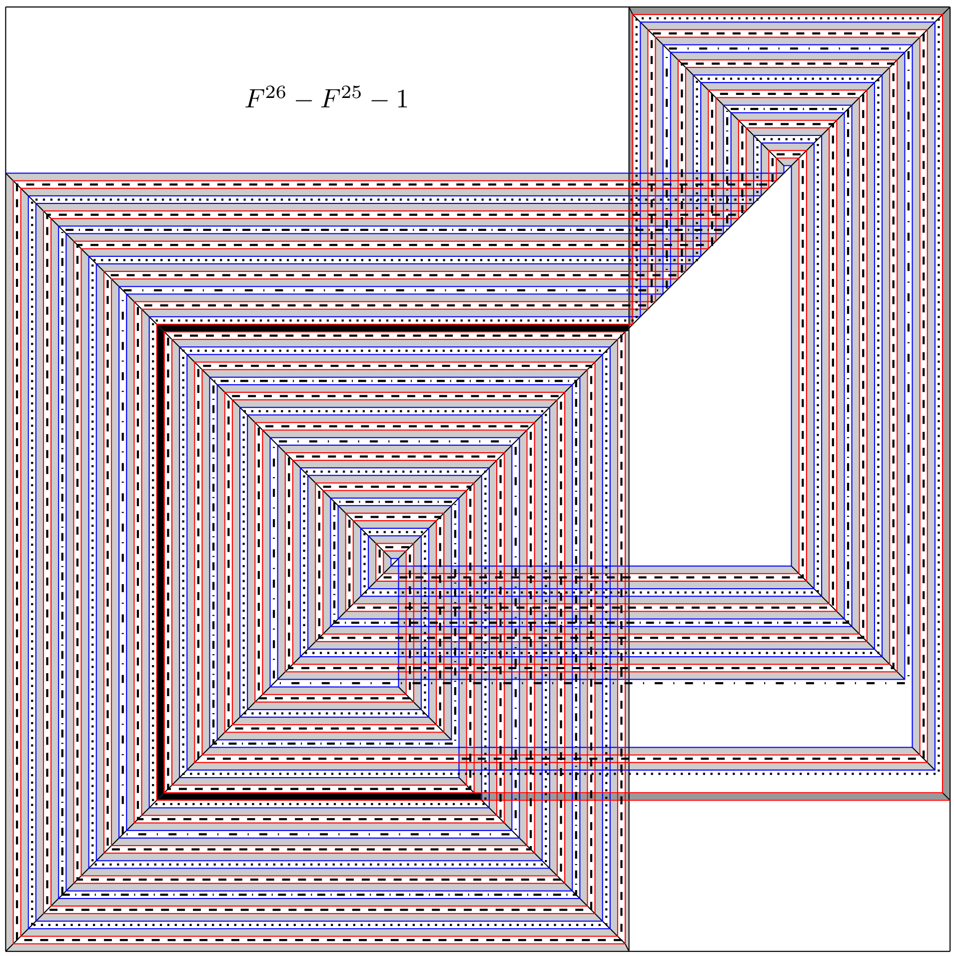


Figure 16: The maximum of $G(N)$ is 62 for $1846 \leq N \leq 2597$

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