

**HIGHER A-NUMBERS IN \mathbb{Z}_p -TOWERS VIA COUNTING LATTICE POINTS****Jeremy Booher***Department of Mathematics, University of Florida, Gainesville, Florida*
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cwu88@ubc.ca*Received: 11/21/24, Revised: 11/13/25, Accepted: 5/28/26, Published: 6/26/26***Abstract**

Booher, Cais, Kramer-Miller, and Upton study a class of \mathbb{Z}_p -towers of curves in characteristic p with ramification controlled by an integer d . In the special case that d divides $p - 1$, they prove a formula for the higher a -numbers of these curves involving the number of lattice points in a complicated region of the plane. Booher and Cais had previously conjectured that for n sufficiently large, the higher a -numbers of the n th curve are given by formulae of the form $\alpha(n)p^{2n} + \beta(n)p^n + \lambda_r(n)n + \nu(n)$, where $\alpha, \beta, \nu, \lambda_r$ are periodic functions of n . This is an example of a new kind of Iwasawa theory. We establish this conjecture by carefully studying these lattice points.

1. Introduction

In this paper, we solve a combinatorial problem involving counting lattice points in order to prove a result in arithmetic geometry. We begin by discussing the motivation and broader context, which requires some background in algebraic geometry.¹ The remainder of the paper does not require this background.

Let k be a perfect field of characteristic p . We consider a smooth projective curve X over k : a variety of dimension one over k that is non-singular. We furthermore assume that X is geometrically connected, which means that X remains connected when extending scalars to the algebraic closure of k . The Jacobian of X is a variety $\text{Jac}(X)$ of dimension equal to the genus of X , which is also an abelian group (i.e., it is an abelian variety). For a positive integer n , the n -torsion in the Jacobian, denoted $\text{Jac}(X)[n]$, is the kernel of the multiplication by n map. If n is prime to p , we may view the n -torsion points defined over k as an abelian group: it is isomorphic to the n -torsion in the class group of the function field $k(X)$. However, if $n = p$, we must view $\text{Jac}(X)[p]$ as a group scheme. As the characteristic of k is p , the group scheme $\text{Jac}(X)[p]$ is not reduced, so it is not determined by its points. The k -points of $\text{Jac}(X)[p]$ are still isomorphic to the p -torsion in the class group of $k(X)$, but this misses some information. Thus, it is natural to investigate whether some version of Iwasawa theory holds for the group scheme $\text{Jac}(X)[p]$ in \mathbb{Z}_p -towers of curves.

A concrete way to investigate the structure of $\text{Jac}(X)[p]$ is to study higher a -numbers (defined below). These are defined using the *Cartier operator*, which is a function V_X from the space of meromorphic differentials on X to itself. It is characterized by the fact that for all differentials ω, ω' on X and all rational functions f on X , we have

- $V_X(f^p\omega + \omega') = fV_X(\omega) + V_X(\omega')$;
- $V_X(df) = 0$ and $V_X(df/f) = 0$;
- $V_X(H^0(X, \Omega_X^1)) \subset H^0(X, \Omega_X^1)$.

A formal definition can be found in [5]. Recall that $H^0(X, \Omega_X^1)$ denotes the k -vector space of regular differentials on X , whose dimension is the genus g_X of X . Following [2], we define the higher a -numbers of X as follows.

Definition 1. For a positive integer r , we define the r th *higher a -number* of X as

$$a^r(X) := \dim_k \ker (V_X^r : H^0(X, \Omega_X^1) \rightarrow H^0(X, \Omega_X^1)).$$

The higher a -numbers reflect part of the structure of $\text{Jac}(X)[p]$. The Dieudonné module of this group scheme, $\mathbb{D}(\text{Jac}(X)[p])$, is naturally identified with the de Rham

¹The interested reader can find additional background material in [7].

cohomology of X [10]. It is a k -vector space of dimension $2g_X$ with a pair of operators, the Frobenius F and Verschiebung V . The subspace $H^0(X, \Omega_X^1) \subset H_{\text{dR}}^1(X)$ is the kernel of F on the Dieudonné module, and the Cartier operator is the restriction of the Verschiebung V . This is most familiar for the first a -number of X , where

$$\begin{aligned} a^1(X) &= \dim_{\mathbb{F}_p} \text{Hom}_{\bar{k}}(\alpha_p, \text{Jac}(X)[p]) = \dim_k (\ker F \cap \ker V) \\ &= \dim_k \ker(V_X|_{H^0(X, \Omega_X^1)}). \end{aligned}$$

Inspired by Iwasawa theory and guided by computation, Booher and Cais formulated conjectures that the higher a -numbers of “reasonable” \mathbb{Z}_p -towers of curves over k “behave regularly” in the tower [1]. Booher, Cais, Kramer-Miller, and Upton proved a formula for the higher a -numbers in a specific class of reasonable \mathbb{Z}_p -towers [2]. The goal of this paper is to connect the two and establish the conjecture for this class of towers.

We first state the conjecture in this special case. Let $\{X_n\}_{n \geq 0}$ be a \mathbb{Z}_p -tower of curves over the projective line, i.e., a sequence of smooth projective curves over k with morphisms

$$\dots \rightarrow X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0$$

such that X_n is a branched $\mathbb{Z}/p^n\mathbb{Z}$ -cover of $X_0 \simeq \mathbb{P}_k^1$. We assume that each X_n is totally ramified over $\infty \in \mathbb{P}_k^1 \simeq X_0$ and unramified elsewhere. Finally, we assume that the tower has minimal break ratios [2, Definition 3.2]: if s_n are the breaks in the upper-numbering ramification filtration for the tower above infinity, then there is an integer d such that $s_n = dp^{n-1}$ for $n \geq 1$. The integer d is the *ramification invariant* of the tower.

Conjecture 1 (Booher-Cais [1, Conjecture 3.8]). *Let $\{X_n\}$ be a \mathbb{Z}_p -tower totally ramified over one point of $X_0 \simeq \mathbb{P}^1$ with minimal break ratios and ramification invariant d . For each $r \geq 1$, there exists an integer m_r and functions $b_r, \nu_r, \lambda_r : \mathbb{Z}/m_r\mathbb{Z} \rightarrow \mathbb{Q}$ such that*

$$a^r(X_n) = d \cdot \frac{r(p-1)}{2(p+1)((p-1)r + (p+1))} \cdot p^{2n + b_r(n)p^n + \lambda_r(n)n + \nu_r(n)} \quad \text{for } n \gg 0.$$

Remark 1. For any d prime to p , there are many \mathbb{Z}_p -towers over \mathbb{P}_k^1 which are totally ramified at ∞ with ramification invariant d . A simple class of examples is provided by the *basic \mathbb{Z}_p -towers* of [1, §2.3]. Conjecture 1 was inspired by explicit calculations for these basic towers. In these calculations it appeared that $b_r(n) = 0$ (cf. Conjecture 1.2 of [1]). It was not clear to what extent this generalized to other \mathbb{Z}_p -towers; see Theorem 2 below.

In this paper, we focus on the case when $d \mid p - 1$. This restriction on the ramification is natural as it guarantees that the a -number and Newton polygon of X_1 are determined by d [6, 11]. Furthermore, the main theorem in [2] is most precise

when $d \mid p - 1$. In particular, we will be able to prove an exact formula establishing Conjecture 1 in this case, as opposed to a weaker asymptotic statement (see [1, Conjecture 3.4] and [2, Corollary 1.2]).

More precisely, when $d \mid p - 1$, [2, Theorem 1.1] gives a formula for $a^r(X_n)$ in terms of the number of lattice points in a complicated region in the plane. To state it, we need to set up a bit of notation. It relies on ordering integer sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ using the lexicographic ordering:

$$(a_n)_{n \geq 0} > (b_n)_{n \geq 0} \text{ if there exists } m \geq 0 \text{ such that} \\ a_i = b_i \text{ for } 0 \leq i < m \text{ and } a_m > b_m. \tag{1}$$

Definition 2. Fix a prime p and $d \mid p - 1$. For $i \in \mathbb{Z}_{\geq 0}$, let $\frac{p+1}{d}i = \sum_{n \in \mathbb{Z}} i_n p^n$ be the (Archimedean) base- p expansion of $\frac{p+1}{d}i$. Then define

$$\mu(i) := \begin{cases} \lfloor \frac{p+1}{d}i \rfloor & \text{if } (i_n)_{n \geq 0} > (i_{-1-n})_{n \geq 0} \\ \lceil \frac{p+1}{d}i \rceil & \text{otherwise.} \end{cases}$$

Definition 3. Fix p, r , and $d \mid p - 1$. Let $\gamma := \frac{(p-1)r+(p+1)}{d}$. For a positive integer n , let $t_n := \lfloor \gamma^{-1}p^n \rfloor = \lfloor \frac{dp^n}{(r+1)p - (r-1)} \rfloor$ and define

$$\Delta_n := \{ (i, j) \in \mathbb{Z}_{>0}^2 : i > t_n \text{ and } \mu(i) \leq j \leq p^n - 1 \}.$$

Theorem 1 (Booher, Cais, Kramer-Miller, and Upton [2]). *Let $\{X_n\}$ be a \mathbb{Z}_p -tower totally ramified over one point of $X_0 \simeq \mathbb{P}^1$ with minimal break ratios and ramification invariant d . When $d \mid p - 1$, the r th higher a -number of X_n is given by*

$$a^r(X_n) = \frac{r(p-1)t_n(t_n+1)}{2d} + \#\Delta_n.$$

In this paper, we connect Conjecture 1 with Theorem 1 by carefully analyzing $\#\Delta_n$.

Theorem 2. *Let $\{X_n\}$ be a \mathbb{Z}_p -tower totally ramified over one point of $X_0 \simeq \mathbb{P}^1$ with minimal break ratios and ramification invariant d . If $d \mid p - 1$, then there exist positive $N_r \in \mathbb{Z}$, $\lambda_r \in \mathbb{Q}$, and a periodic function $\nu_r(n) : \mathbb{Z}_{>0} \rightarrow \mathbb{Q}$ such that for $n \geq N_r$*

$$a^r(X_n) = \frac{r}{r + \frac{p+1}{p-1}} \frac{d}{2(p+1)} p^{2n} + \lambda_r n + \nu_r(n). \tag{2}$$

Furthermore, writing D_r for the prime-to- p part of the denominator of $d/(r(p-1) + (p+1))$ in lowest terms, the minimal period of ν_r divides the least common multiple of 2 and the order of p modulo D_r .

Theorem 2 is a less precise version of Theorem 4, which provides explicit expressions for λ_r , N_r , and the function $\nu_r(n)$ (and, to an extent, its period). Theorem 2 confirms Conjecture 1 when $d \mid p - 1$. This provides the first theoretical evidence for Conjecture 1 in odd characteristic, as opposed to the weaker asymptotic conjecture established in [2, Corollary 1.2]. Note that it is not obvious that the right side of Equation (2) is an integer. The main p^{2n} term, which also appears in the asymptotic conjecture, is manifestly not an integer but has a natural interpretation as the area of a triangle appearing in [2] and Section 1.1. The higher a -numbers are clearly integers. Therefore, to make the right side of Equation (2) an integer, the rational number $\nu_r(n)$ must be somewhat complicated.

Example 1. When $p = 3$ we can either have $d = 1$ or $d = 2$. When $d = 2$, Corollary 6 implies that there is a periodic function $\nu_r : \mathbb{Z} \rightarrow \mathbb{Q}$ such that

$$a^r(X_n) = \frac{r}{4(r+2)}3^{2n} + \nu_r(n).$$

for n sufficiently large, and predicts the minimal period. For some values of r (i.e., $r = 1, 2, 7$) the function ν_r is constant. For others (i.e., $r = 3, 6$) it has period exactly two. Other values of r give larger minimal periods.

Similarly, when $d = 1$, there is a periodic $\nu_r : \mathbb{Z} \rightarrow \mathbb{Q}$ such that for sufficiently large n

$$a^r(X_n) = \frac{r}{8(r+2)}3^{2n} + \nu_r(n).$$

Example 2. When $p = 5$, $d = 4$, and $r = 2$, we see that

$$a^2(X_n) = \frac{4}{21} \cdot 5^{2n} + \frac{1}{3}n + \nu_2(n),$$

where $\nu_2(n)$ is given in Table 1. Note that the values of $\nu_2(n)$ make $4 \cdot 5^{2n} + 7n + 21 \cdot \nu_2(n)$ a multiple of 21 (necessary since $a^2(X_n)$ is an integer). Note that this is a case where the period given by Theorem 4 (6) is a multiple of the observed minimal period (3).

$n \pmod{3}$	0	1	2
$\nu_2(n)$	$-4/21$	$-2/21$	$2/7$

Table 1: Value of periodic function $\nu_2(n)$ for $n \geq 0$.

These parameters will be used as a recurring example throughout the paper (see Examples 6 to 8). Explicit instances of towers of curves realizing these parameters can be constructed using Artin–Schreier–Witt theory from the Witt vector equation

$$(y_0^p, y_1^p, \dots, y_{n-1}^p) - (y_0, \dots, y_{n-1}) = \sum_{i=1}^d [c_i x^i, 0, \dots, 0],$$

where $c_i \in k$ and $c_d \neq 0$. The curve X_1 is easy to describe: it is the Artin–Schreier curve determined by $y_0^p - y_0 = \sum c_i x^i$. The others are more involved to describe explicitly; see [2, Remark 3.4 and Section 3.2] for a review.

Example 3. In contrast, when $p = 5$, $d = 4$, and $r = 61$,

$$a^r(X_n) = \frac{122}{375} 5^{2n} + \frac{2}{3}$$

for $n \geq 3$, but the values for $n = 1$ and $n = 2$ are not given by the formula. The more precise Theorem 4 predicts that $N_{61} = 3$ (as $v_5(2/125) = -3$), which matches the actual behavior.

Remark 2. The case when $r = 1$ is particularly simple, and [2, §6] shows (by directly computing $\#\Delta_n$) that if $d \mid p - 1$

$$a^1(X_n) = \frac{d(p-1)}{4(p+1)}(p^{2n-1} + 1) - \begin{cases} 0 & d \text{ even} \\ \frac{p-1}{4d} & d \text{ odd.} \end{cases}$$

The difficulty of generalizing that argument to other values of r led to a project for students at the PROMYS program, and then to this paper.

Remark 3. The period for $a^r(X_n)$ seen in the precise version of Theorem 2 (i.e., Theorem 4) is similar in spirit but slightly different than the conjectured period in [1, Conjecture 1.2]. In particular, the result we prove involves the order of p modulo D_r , rather than the order of p^2 , and also may differ by a factor of 2. In particular, the prediction for the period given in [1, Conjecture 1.2] is not correct.

Remark 4. For any \mathbb{Z}_p -tower that is totally ramified over one point of \mathbb{P}^1 and unramified elsewhere, the Deuring-Shafarevich formula shows that the p -rank of each curve in the tower is 0. Thus, the p -part of the class group of the function field is trivial. In contrast, Theorem 2 shows the higher a -numbers can grow regularly with n in a non-trivial fashion.

Remark 5. For a curve X of genus g_X , the higher a -numbers eventually stabilize: $a^r(X) = g_X - f_X$ for large enough r , where f_X is the p -rank of X . More precisely, the p -rank is the stable rank of the Cartier operator and the stabilization must happen at least for $r \geq g_X - f_X$ by a simple (semi-)linear algebra argument. Thus, only finitely many of the higher a -numbers provide information. (The smallest integer r for which $a^r(X) = a^{r+1}(X)$ would be an interesting quantity to study.) In contrast, for a \mathbb{Z}_p -tower, as the genus of X_n goes to infinity, the higher a -numbers $a^r(X_n)$ are eventually interesting for any r .

Remark 6. The behavior of étale \mathbb{Z}_p -towers is different, although $\text{Jac}(X_n)[p^\infty]$ still depends on n in a regular manner [4]. Such towers exist only over $\overline{\mathbb{F}}_p$, not over finite fields. The (necessarily wild) ramification in our setting causes the genus (and higher a -numbers) to grow with n at a much faster rate.

1.1. Idea of the Proof

An elementary argument reinterprets the formula for the higher a -numbers in Theorem 1 as the cardinality of

$$\tilde{\Delta}_n := \Delta_n \cup \left\{ (i, j) \in \mathbb{Z}_{>0}^2 : i \leq t_n \text{ and } p^n - ir \frac{p-1}{d} \leq j \leq p^n - 1 \right\}.$$

The underlying idea in the proof is that $\#\tilde{\Delta}_n$ is approximately the number of lattice points in the triangle \mathcal{P}_n bounded by $y = p^n$, $y = \frac{p+1}{d}x$, and $y = p^n - r\frac{p-1}{d}x$. See Figure 1 for an illustration. If we let $\tau = \frac{p+1}{d}$ and $\gamma = \frac{p-1}{d}r + \tau$, then $p^n\gamma^{-1}$ is the x -coordinate of the lower vertex of the triangle (explaining the quantity t_n) and $p^n\tau^{-1}$ is the x -coordinate of the right vertex.

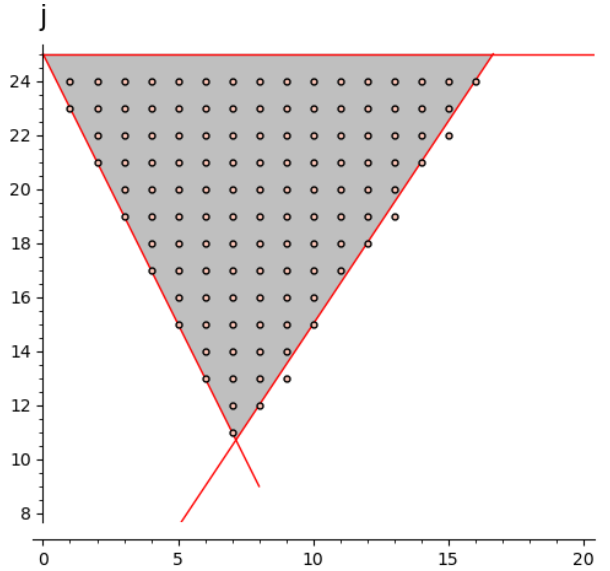


Figure 1: $\tilde{\Delta}_2$ and \mathcal{P}_2 (shaded) when $p = 5$, $r = 2$, and $d = 4$.

The key difficulty is that there is irregular behavior along the side of the triangle given by $y = \frac{p+1}{d}x$: this arises from the condition $\mu(i) \leq j$ in the definition of Δ_n . While $\mu(i)$ is approximately equal to $\frac{p+1}{d}i$, it is subtle to determine whether to round up or down in Definition 2. This controls whether the first lattice point with $x = i$ below the line $y = \frac{p+1}{d}x$ should be included in Δ_n or not. The blue line in Figure 2 shows the ragged lower right boundary of Δ_n given by $\mu(i)$.

Section 2.2 establishes properties of base- p rationals and periodic functions that are ubiquitous throughout the following sections. In Section 2.3, we define an indicator function $\delta(i)$ (taking on values 0 and 1) which reflects whether we round

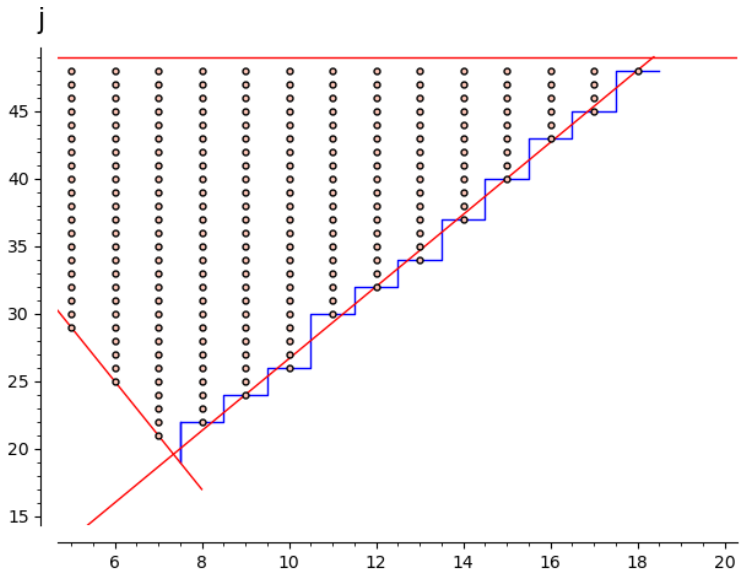


Figure 2: $\tilde{\Delta}_2$ and \mathcal{P}_2 , showing μ_i , when $p = 7$, $r = 2$, and $d = 3$.

up or down in Definition 2. The key insight is that δ is a periodic function and that we can compute its average value over a period. This gives control over sums of the form $\sum_{i=m}^n \delta(i)$, which reflect the number of lattice points in Figure 1 that lie outside the triangle. In Section 2.4, we essentially express higher a -numbers in terms of lattice points in \mathcal{P}_n and sums involving δ . A convenient expression for $a^r(X_n)$ is

$$\left(\sum_{i=1}^{\lfloor \tau^{-1}p^n \rfloor} (p^n - \lfloor \tau i \rfloor) - \sum_{i=1}^{\lfloor \gamma^{-1}p^n \rfloor} (p^n - \lfloor \gamma i \rfloor) \right) - \left(\sum_{i=1}^{\lfloor \tau^{-1}p^n \rfloor} \delta(i) - \sum_{i=1}^{\lfloor \gamma^{-1}p^n \rfloor} \delta(i) \right).$$

The sums involving floor functions count the number of lattice points in a variant of \mathcal{P}_n : see Remark 7.

As discussed in Section 2.5, the theory of Ehrhart polynomials gives a natural approach to counting the number of lattice points in \mathcal{P}_n , since \mathcal{P}_n is a triangle with rational vertices and is the triangle obtained from \mathcal{P}_0 by scaling by a factor of p^n . However, in Section 3.1 we give a direct proof using the same ideas as we use to study sums involving δ in Section 3.2. The basic idea is to exploit the periodicity of the function being summed and compare with the length of the sum. The latter depends on p^n , which explains the appearance of the order of p modulo D_r in Theorem 2. We put everything together in Section 3.3, and show that the p^n -terms in the various pieces cancel out in the resulting formula.

In Section 3.4, we prove additional properties about λ_r and the minimal possible integer N_r appearing in Theorem 2. We end with some further properties of $a^r(X_n)$ we have observed from empirical data, but have been unable to prove.

2. Preliminaries

2.1. Notation

Fix an odd prime p , a positive integer d dividing $p - 1$, and a positive integer r .

Notation 1. We define

- (i) $\tau = \frac{p+1}{d}$
- (ii) $\gamma = \frac{p-1}{d}r + \tau = \frac{(p-1)r+(p+1)}{d}$.

Note that, by hypothesis, $\gamma - \tau$ is integral and τ, γ are positive. Furthermore, recalling Definition 3, we have $t_n = \lfloor \gamma^{-1}p^n \rfloor$.

Notation 2. Let x be a positive rational number. We let x_N, x_D be the unique coprime positive integers such that

$$x = p^{v_p(x)} \frac{x_N}{x_D},$$

where as usual $v_p(x)$ is the p -adic valuation of x .

Since $d \mid p - 1$ and hence $\gcd(d, p + 1) = \gcd(d, 2)$, we see that $\tau_D = \frac{d}{\gcd(d, 2)}$ and $\tau_N = \frac{p+1}{\gcd(d, 2)}$.

For the convenience of the reader, we end with an index for other significant notation.

- The set of lattice points Δ_n is defined in Definition 3, and its variant $\tilde{\Delta}_n$ in Definition 8. The functions $\mu(i)$ and $\delta(i)$ describe the boundary of Δ_n , and are defined in Definition 2 and Definition 7. The latter also defines the variants $\delta_0(i)$ and $\tilde{\delta}(i)$.
- For a periodic function f , \bar{f} denotes the average value (Definition 4).
- For a positive rational number x , \bar{x} denotes the average value of the repeating part of its base- p expansion (Definition 6). The period of its repeating part is denoted L_x , and $[p^k](x)$ is the k th digit in the base- p expansion. The fractional part of x is $\{x\}$.
- For a positive rational number x , the functions A_{x-1} , B_{x-1} , and F_{x-1} appear over the course of Section 3 in Proposition 4, Proposition 5, and Lemma 9.
- The exact formula for λ_r appears in Theorem 4.

2.2. Base- p Expansions and Periodic Sequences

Definition 4. Let $(f_i)_{i \geq 1}$ be a sequence of real numbers. We say that f is *periodic* for $i \geq D$ if, for some $L \in \mathbb{Z}$ and all $i \geq D$, we have

$$f_{i+L} = f_i.$$

We will also sometimes less precisely call such functions *eventually periodic*. In this case we define the *average value* of f over its period to be

$$\bar{f} \text{ or } \bar{f}_i := \frac{1}{L} \sum_{i=D+1}^{D+L} f_i.$$

Note that the average value of a periodic function is independent of the choice of L and D . We may equivalently view the sequence as a function $f : \mathbb{N} \rightarrow \mathbb{R}$. It is harmless to start indexing sequences at 0 instead of 1 if convenient. We record the following well-known result; the proof is omitted.

Lemma 1 (Sum of a periodic sequence). *Let $f = (f_i)_{i \in \mathbb{N}}$ be a periodic sequence with period L . For any $N \geq 0$, we have*

$$\sum_{i=1}^N f_i = N\bar{f} + \sum_{k=1}^{N \bmod L} (f_k - \bar{f}).$$

As many of the sums we will deal with are only eventually periodic, it is useful to also have a formula for a sequence that is periodic after a delay.

Corollary 1 (Sum of an eventually periodic sequence). *Let $(f_i)_{i \in \mathbb{N}}$ be periodic for $i \geq D$ with period L . For $N \geq D - 1$,*

$$\sum_{i=1}^N f_i = \sum_{i=1}^{D-1} f_i + \left((N - D + 1)\bar{f} + \sum_{k=1}^{(N-D+1) \bmod L} (f_{k+D-1} - \bar{f}) \right).$$

Proof. Noting that

$$\sum_{i=1}^N f_i = \sum_{i=1}^{D-1} f_i + \sum_{i=D}^N f_i,$$

apply Lemma 1 to the sequence f_D, f_{D+1}, \dots , which by hypothesis is periodic with period L . □

Definition 5. Let x be a positive rational number. For any integer $k \in \mathbb{Z}$, we define $[p^k](x)$ to be the unique integer in $\{0, \dots, p - 1\}$ such that

$$[p^k](x) \equiv \left\lfloor \frac{x}{p^k} \right\rfloor \pmod{p}.$$

In other words, $[p^k](x)$ is the k th digit of the base- p expansion of x , so that

$$x = \sum_{k \in \mathbb{Z}} [p^k](x) p^k.$$

Since x is rational, the sequence $([p^{-k}](x))_{k \in \mathbb{N}}$ is eventually periodic [8, Theorem 136].

Definition 6. Let x be a positive rational number. We set L_x to be the minimal period of the eventually periodic sequence $[p^{-k}](x)$. We define the *delay* D_x to be the minimal D such that the sequence is periodic for $k \geq D + 1$. We also set $\bar{x} = \overline{[p^{-k}](x)}$, where the bar is used to indicate the average value as in Definition 4.

Example 4. In base-5

$$\frac{2}{3} = .3131\dots \quad \text{and} \quad \frac{2}{7} = .120324120324\dots$$

Thus, $L_{2/3} = 2$, $D_{2/3} = 0$, and $\overline{2/3} = (3 + 1)/2 = 2$. Similarly $L_{2/7} = 6$, $D_{2/7} = 0$, and $\overline{2/7} = 2$. On the other hand, the (base-10) rational number $x = 2803/672$ has base-7 expansion $4.1124612461\dots$ with delay $D_x = 1$ and period $L_x = 4$.

Recall that x_D is defined in Notation 2.

Lemma 2. Let x be a positive rational number. Then L_x is the order of p in $(\mathbb{Z}/x_D\mathbb{Z})^\times$ and $D_x = -v_p(x)$.

Proof. This is well-known, see for example [8, Theorem 136]. □

Corollary 2. For any positive integer m , $\frac{m}{d}$ has a periodic base- p expansion with period 1.

Proof. Let $x = \frac{m}{d}$, so x_D divides d which divides $p - 1$. Thus, $p \equiv 1 \pmod{x_D}$ has order one. □

2.3. Properties of δ

Our next goal is to study when the two cases in Definition 2 occur. We make use of the lexicographic ordering on sequences (recall Equation (1)).

Definition 7. We define a function $\delta : \mathbb{N} \rightarrow \{0, 1\}$ by

$$\delta(i) := \begin{cases} 0 & \text{if } ([p^n](\tau i))_{n \geq 0} > ([p^{-1-n}](\tau i))_{n \geq 0} \\ 1 & \text{otherwise.} \end{cases}$$

We define variants $\delta_0, \tilde{\delta} : \mathbb{N} \rightarrow \{0, 1\}$ by

$$\delta_0(i) := \begin{cases} 0 & \text{if } p \mid i \\ \delta(i) & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{\delta}(i) := \begin{cases} 1 & \text{if } \tau_D \mid i \\ \delta(i) & \text{otherwise.} \end{cases}$$

Recalling Definition 2, it is immediate that $\mu(i) = \lfloor \frac{p+1}{d}i \rfloor + \delta(i)$. Note that if $\frac{p+1}{d}i$ is an integer (i.e., $\tau_D \mid i$) we have $\delta(i) = 0$ while $\tilde{\delta}(i) = 1$, so we may similarly write $\mu(i) = \lceil \frac{p+1}{d}i \rceil - 1 + \tilde{\delta}(i)$.

Example 5. Table 2 shows $\delta(i)$ and $\delta_0(i)$ for small i when $p = 5$ and $d = 4$. In this case $\tau_D = 2$.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\delta(i)$	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	0
$\delta_0(i)$	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0
$\tilde{\delta}(i)$	1	1	0	1	1	1	1	1	0	1	1	1	0	1	0	1	1	1	0

Table 2: Values of $\delta(i)$ with $d = 4, p = 5$.

We now turn to studying $\delta(i)$.

Lemma 3. For a positive integer i , we have that $[p^{-1}](\tau i) = [p^0](\tau i)$ if and only if $p \mid i$.

Proof. Since $d \mid p - 1$, we know from Corollary 2 that the base- p expansion of τi consists of a single repeating digit h . Then we see the fractional part of τi is $\{\tau i\} = 0.\bar{h} = \frac{h}{p-1}$ and hence $[p^0](\tau i) \equiv \left(\tau i - \frac{h}{p-1}\right) \pmod{p}$. We then see

$$\begin{aligned}
 [p^0](\tau i) = [p^{-1}](\tau i) &\iff \tau i - \frac{h}{p-1} \equiv h \pmod{p} \\
 &\iff \tau i + h \equiv h \pmod{p} \\
 &\iff \tau i \equiv 0 \pmod{p} \\
 &\iff i \equiv 0 \pmod{p},
 \end{aligned}$$

where the last step uses that $p + 1$ and d are units modulo p . □

Lemma 4. Suppose $p \nmid i$. Then $[p^{-1}](\tau i) > [p^0](\tau i) \iff \delta(i) = 1$.

Proof. As $[p^{-1}](\tau i) \neq [p^0](\tau i)$ by Lemma 3, we see that

$$\begin{aligned}
 [p^{-1}](\tau i) > [p^0](\tau i) &\text{ if and only if } (i_n)_{n \geq 0} \leq (i_{-1-n})_{n \geq 0} \\
 &\text{ if and only if } \delta(i) = 1.
 \end{aligned}$$
□

Proposition 1. For an integer i with $0 \leq i < \tau_D p$ and $p \nmid i$:

- (i) if $\tau_D \nmid i$ then $\delta(i) + \delta(\tau_D p - i) = 1$.
- (ii) if $\tau_D \mid i$ then $\delta(i) = \delta(\tau_D p - i) = 0$.

For an integer j with $0 \leq j < d$ and $p \nmid j$:

(iii) if $\tau_D \nmid j$ then $\delta(j) + \delta(d - j) = 1$.

(iv) if $\tau_D \mid j$ then $\delta(j) = 0$.

Proof. We begin with the first statement. Note that $\tau(\tau_D p - i) = \tau_N p - \tau i$. Because $\tau_N p$ is integral and $\tau i \notin \mathbb{Z}$, when computing $\tau_N p - \tau i$ we must borrow from the ones place and so

$$[p^{-1}] (\tau_N p - \tau i) = (p - 1) - [p^{-1}] (\tau i).$$

Similarly, as $\tau_N p$ is a multiple of p we must borrow when computing $\tau_N p - \tau i$ so

$$[p^0] (\tau_N p - \tau i) = (p - 1) - [p^0] (\tau i).$$

As $p \nmid i$ and $p \nmid \tau_D p - i$, we may apply Lemma 4 to see that $\delta(i)$ and $\delta(\tau_D p - i)$ depend only on these two calculated digits. Hence, $\delta(\tau_D p - i) = 1$ if and only if $(p - 1) - [p^{-1}] (\tau i) > (p - 1) - [p^{-1}] (\tau i)$ if and only if $\delta(i) = 0$.

The third statement is similar. Note that $\tau j + \tau(d - j) = (p + 1)$. If $\tau_D \nmid j$, then τj and $\tau(d - j)$ are not integers and so consist of a single non-zero repeating digit after the radix point. Thus, adding digits and keeping track of carries, we see that

$$[p^0] (\tau j) + [p^0] (\tau(d - j)) = p \quad \text{and} \quad [p^{-1}] (\tau j) + [p^{-1}] (\tau(d - j)) = p.$$

We conclude using Lemma 4.

Finally, if $\tau_D \mid i$ and $i > 0$ then $\delta(i) = 0$ since $\tau i = \frac{\tau_N}{\tau_D} i$ is an integer, and hence has $[p^{-1}] (\tau i) = 0$, while $[p^0] (\tau i) > 0$. □

Theorem 3. *Let i and n be positive integers, and τ_D be the denominator of $\frac{p+1}{d}$ when written in lowest terms. Then we have:*

(i) $\delta(p^n i) = \delta(i)$.

(ii) $\delta_0(i) = \delta_0(i + \tau_D p)$.

(iii) if $\tau_D \nmid i$ and $p \nmid i$ then $\delta_0(i) + \delta_0(\tau_D p - i) = 1$.

(iv) if $\tau_D \mid i$ or $p \mid i$, then $\delta_0(i) = \delta_0(\tau_D p - i) = 0$.

Proof. (i) By Corollary 2, the base- p expansions of τi and $\tau p^n i$ consist of a single repeating digit after the radix point, and that is the same digit h for both. Thus, $\delta(i) = 1$ if and only if $[p^\ell] (\tau i) \leq h$ for all $\ell \geq 0$. Similarly, $\delta(p^n i) = 1$ if and only if $[p^\ell] (\tau i p^n) = [p^{\ell-n}] (\tau i) \leq h$ for $\ell \geq 0$. These two conditions are easily seen to be equivalent as $[p^{\ell-n}] (\tau i) = h$ for $\ell < n$.

(ii) As τi and $\tau(i + \tau_D p) = \tau i + \tau_N p$ have the same digits to the right of the radix point, we see $[p^{-1}] (\tau(i + \tau_D p)) = [p^{-1}] (\tau i)$. Similarly, we see that $[p^0] (\tau(i + \tau_D p)) = [p^0] (\tau i)$ as adding $\tau_N p$ (a multiple of p) does not change the

digit in the ones place in the base- p expansion of τi . We may then apply Lemma 4 to complete the proof, as $p \nmid i$ and $p \nmid \tau(i + \tau_D p) = \tau i + \tau_N p$.

Recall that $\delta(i) = \delta_0(i)$ unless i is a multiple of p , in which case $\delta_0(i) = 0$. Then (iii) and (iv) are restatements of Proposition 1. \square

In particular, δ_0 is periodic with period $\tau_D p$ so we may apply the ideas of Section 2.2.

Proposition 2. *We have $\overline{\delta_0} = \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right)$.*

Proof. The proposition is a consequence of Theorem 3. As δ_0 is immediately periodic — meaning the delay D_{δ_0} is zero — with period $\tau_D p$, it suffices to sum over the interval from 1 to $\tau_D p$. If $\tau_D \mid i$, then $\delta_0(i) = 0$. Thus, it suffices to consider those i in $[1, \tau_D p]$ such that $\tau_D \nmid i$ and $p \nmid i$. There are $(p - 1)(\tau_D - 1)$ such i , and exactly one of $i, \tau_D p - i$ makes δ_0 equal to 1. Thus, we have $\sum_{i=1}^{\tau_D p} \delta_0(i) = \frac{(p-1)(\tau_D-1)}{2}$. Dividing by the period $\tau_D p$ gives the proposition. \square

We will return to these properties to compute more involved sums in Section 3.2.

2.4. Simplifying Higher a -Numbers

We now turn our attention to obtaining an alternate version of the formula for higher a -numbers given by Theorem 1 and expressing it using $\delta(i)$.

Definition 8. For fixed p, r , and $d \mid p - 1$ as usual, for a positive integer n define

$$\tilde{\Delta}_n := \Delta_n \cup \left\{ (i, j) \in \mathbb{Z}_{>0}^2 : i \leq t_n \text{ and } p^n - ir \frac{p-1}{d} \leq j \leq p^n - 1 \right\}.$$

Lemma 5. *Continuing the hypotheses and notation of Theorem 1, $a^r(X_n) = \#\tilde{\Delta}_n$.*

Proof. For $i \leq t_n$, the number of elements of $\tilde{\Delta}_n$ with first coordinate i is $ir \frac{p-1}{d}$. The claim is then a restatement of Theorem 1 as

$$\sum_{i=1}^{t_n} ir \frac{p-1}{d} = \frac{(p-1)rt_n(t_n+1)}{2d}. \quad \square$$

Proposition 3. *With notation as before,*

$$\begin{aligned} a^r(X_n) &= \sum_{i=1}^{\lfloor \gamma^{-1} p^n \rfloor} (\gamma - \tau)i + \sum_{i=\lfloor \gamma^{-1} p^n \rfloor + 1}^{\lfloor \tau^{-1} p^n \rfloor} (p^n - \lfloor \tau i \rfloor) - \sum_{i=\lfloor \gamma^{-1} p^n \rfloor + 1}^{\lfloor \tau^{-1} p^n \rfloor} \delta(i) \\ &= \left(\sum_{i=1}^{\lfloor \tau^{-1} p^n \rfloor} (p^n - \lfloor \tau i \rfloor) - \sum_{i=1}^{\lfloor \gamma^{-1} p^n \rfloor} (p^n - \lfloor \gamma i \rfloor) \right) \end{aligned}$$

$$- \left(\sum_{i=1}^{\lfloor \tau^{-1}p^n \rfloor} \delta(i) - \sum_{i=1}^{\lfloor \gamma^{-1}p^n \rfloor} \delta(i) \right).$$

Proof. Recall that by definition $t_n = \lfloor \gamma^{-1}p^n \rfloor$, and notice that the largest potential first coordinate of a point in Δ_n is $\lfloor \tau^{-1}p^n \rfloor$. Recalling Definitions 3 and 8, we see that for $1 \leq i \leq t_n$, the number of elements of $\tilde{\Delta}_n$ with first coordinate i is $ir^{\frac{p-1}{d}} = (\gamma - \tau)i$. Similarly, for $t_n < i$, the number of elements of Δ_n with first coordinate i is $p^n - \mu(i)$. As $\mu(i) = \lfloor \tau i \rfloor + \delta(i)$, we conclude that

$$\#\tilde{\Delta}_n = \sum_{i=1}^{\lfloor \gamma^{-1}p^n \rfloor} (\gamma - \tau)i + \sum_{i=\lfloor \gamma^{-1}p^n \rfloor+1}^{\lfloor \tau^{-1}p^n \rfloor} (p^n - \lfloor \tau i \rfloor) - \sum_{i=\lfloor \gamma^{-1}p^n \rfloor+1}^{\lfloor \tau^{-1}p^n \rfloor} \delta(i).$$

The first equality follows using Lemma 5. The second equality is elementary, noticing that $\gamma - \tau \in \mathbb{Z}_{>0}$ so $(p^n - \lfloor \tau i \rfloor) - (p^n - \lfloor \gamma i \rfloor) = (\gamma - \tau)i$. \square

2.5. Connection with Ehrhart Quasi-polynomials

We end by explaining a slightly different perspective using the theory of Ehrhart quasi-polynomials for rational polytopes, which is not needed for our approach. The following triangle is closely related to the region Δ_n :

Definition 9. Let $\mathcal{P}_n := p^n\mathcal{P}_0$ be the triangle with vertices $(0, p^n)$, $(\tau^{-1}p^n, p^n)$, and $(\gamma^{-1}p^n, \tau\gamma^{-1}p^n)$. In particular, \mathcal{P}_0 is the triangle with vertices $(0, 1)$, $(\tau^{-1}, 1)$, and $(\gamma^{-1}, \tau\gamma^{-1})$.

It is an easy exercise to check that \mathcal{P}_n can equivalently be described as the triangle bounded by the lines $y = p^n$, $y = \tau x = \frac{p+1}{d}x$, and $y = p^n - r\frac{p-1}{d}x = p^n - (\gamma - \tau)x$. An example is shown in Figure 1: the points on or below $y = \tau x$ are explained by those i where $\delta(i) = 0$ as shown in Table 2.

Remark 7. Letting $L(\mathcal{P}_n)$ denote the number of lattice points in \mathcal{P}_n , we will verify that

$$a^r(X_n) = \#\tilde{\Delta}_n = L(\mathcal{P}_n) - \lfloor \tau^{-1}p^n \rfloor - 1 + \sum_{i=t_n+1}^{\lfloor \tau^{-1}p^n \rfloor} (1 - \tilde{\delta}(i)). \tag{3}$$

The set of lattice points in \mathcal{P}_n (including points on the boundary) is almost $\tilde{\Delta}_n$, except that:

- the $\lfloor \tau^{-1}p^n \rfloor + 1$ points $(0, p^n), (1, p^n), \dots, (\lfloor \tau^{-1}p^n \rfloor, p^n)$ are lattice points of \mathcal{P}_n but not included in $\tilde{\Delta}_n$,
- for $i > t_n$, the lower bound on the possible second coordinates is $\frac{p+1}{d}i$ versus μ_i . The point $(i, \mu(i)) \in \tilde{\Delta}_n$ is not in \mathcal{P}_n if and only if $\mu(i) < \frac{p+1}{d}i$. Note this

happens if and only if $\delta(i) = 0$ and $\frac{p+1}{d}i$ is not an integer (i.e., $\tau_D \nmid i$). We may use $\tilde{\delta}$ to encapsulate this as the condition $\tilde{\delta}(i) = 0$,

- for points $(i, j) \in \Delta_n$ with $i > t_n$ and $j > \mu(i)$, it is automatic that $j \geq \frac{p+1}{d}i$.

Thus, \mathcal{P}_n contains $\lfloor \tau^{-1}p^n \rfloor + 1$ lattice points not appearing in $\tilde{\Delta}_n$ and misses $\lfloor \tau^{-1}p^n \rfloor$

$$\sum_{i=t_n+1}^{\lfloor \tau^{-1}p^n \rfloor} (1 - \tilde{\delta}(i)) \text{ elements of } \Delta_n.$$

Now, the triangle \mathcal{P}_n is the dilation of \mathcal{P}_0 by p^n , and the theory of Ehrhart quasi-polynomials describes the number of lattice points in the dilation. In particular, the main result of [9] implies that the Ehrhart quasi-polynomial of the triangle \mathcal{P}_0 is

$$\frac{1}{2}(\tau^{-1} - \gamma^{-1})t^2 + \frac{1}{2} \left(\gamma^{-1} + \tau^{-1} + \frac{\gcd(d, 2)}{d}(\tau^{-1} - \gamma^{-1}) \right) t + c(t)$$

for some periodic function $c(t)$. Furthermore, we know that the minimal period of $c(t)$ is a divisor of the least integer D such that $D\tau^{-1}$, $D\gamma^{-1}$, and $D\tau\gamma^{-1}$ are integral. The key observation is that the affine spans of each side of the triangle contain lattice points, so that the periods of the linear and quadratic terms are one. As discussed in [3], determining the minimal period is subtle. Taking $t = p^n$ makes the main term in Theorem 2 transparent.

Equation (3) shows that (just as in Proposition 3) the heart of the proof must be to understand sums involving $\delta(i)$. This is the subject of Section 3.2. We choose to directly study the sums involving floor functions appearing in Proposition 3, instead of this approach using the Ehrhart quasi-polynomial, as the easy arguments we give in Section 3.1 introduce the techniques we use for studying the sums involving $\delta(i)$ in Section 3.2.

3. Computing Higher a -Numbers

In this section we analyze the sums appearing in Proposition 3. In particular, for a positive rational number x we study sums of the form

$$\sum_{i=1}^{\lfloor x^{-1}p^n \rfloor} (p^n - \lfloor xi \rfloor) \quad \text{and} \quad \sum_{i=1}^{\lfloor x^{-1}p^n \rfloor} \delta(i).$$

3.1. Sums Involving Floor Functions

For a rational number x , recall that $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x and that L_x , D_x , and \bar{x} are defined in Definition 6 in terms of the base- p decimal expansion of x .

Lemma 6. For any $x \in \mathbb{Q}_{>0}$, $\{xp^n\}$ is periodic in n for $n \geq D_x$ with period L_x and average value $\overline{\{xp^n\}} = \bar{x}/(p-1)$.

Proof. Let $x = \sum x_i p^i$ be the base- p expansion of x . Then

$$\{xp^n\} = \sum_{i \leq -1} x_{i-n} p^i.$$

We know $x_{i-n} = x_{i-n-L_x}$ for $-(i-n) \geq D_x + 1$ as the base- p expansion of x is eventually periodic. Thus, if $n \geq D_x$, summing over $i \leq -1$ we see that $\{xp^n\} = \{xp^{n+L_x}\}$. Furthermore,

$$\overline{\{xp^n\}} = \frac{1}{L_x} \sum_{n=D_x}^{D_x+L_x-1} \{xp^n\} = \frac{1}{L_x} \sum_{i=D_x+1}^{D_x+L_x} x_{-i} (p^{-1} + p^{-2} + \dots) = \frac{\bar{x}}{p-1}. \quad \square$$

Lemma 7. Let x be a positive rational number. Then

- (i) $\lfloor xp^n \rfloor \equiv \lfloor xp^{n+L_x} \rfloor \pmod{x_N}$ for $n \geq D_x$.
- (ii) $\lfloor xp^n \rfloor \equiv \lfloor xp^{n+L_x} \rfloor \pmod{p}$ for $n \geq D_x + 1$.

Proof. By Lemma 6, $\{xp^e\} = \{xp^{e+L_x}\}$ for $e \geq D_x$, and hence

$$\lfloor xp^{e+L_x} \rfloor - \lfloor xp^e \rfloor = xp^{e+L_x} - xp^e.$$

Thus, we see that

$$\lfloor xp^{n+L_x} \rfloor - \lfloor xp^n \rfloor = p^{n-D_x} x (p^{D_x+L_x} - p^{D_x})$$

which implies that $\lfloor xp^{n+L_x} \rfloor \equiv \lfloor xp^n \rfloor \pmod{p}$ for $n \geq D_x + 1$. Furthermore, for $n \geq D_x$, the left side shows this quantity is an integer, while the right side shows that it is a multiple of x_N since the denominator x_D is coprime with x_N . Thus, $\lfloor xp^{n+L_x} \rfloor \equiv \lfloor xp^n \rfloor \pmod{x_N}$ for $n \geq D_x$ as well. \square

Proposition 4. Let x be a positive rational number with $v_p(x) \geq 0$. Then

$$\sum_{i=1}^{\lfloor x^{-1}p^n \rfloor} (p^n - \lfloor xi \rfloor) = \frac{1}{2} x^{-1} p^{2n} + \frac{1}{2} \left(x^{-1} \left(1 - \frac{1}{x_D} \right) - 1 \right) p^n + A_{x^{-1}}(n),$$

where the function

$$A_{x^{-1}}(n) := \frac{1}{2} \left(- \left(1 - \frac{1}{x_D} \right) + x(1 - \{x^{-1}p^n\}) \right) \{x^{-1}p^n\} + \sum_{k=1}^{\lfloor x^{-1}p^n \rfloor \bmod x_D} \left(\{xk\} - \frac{1}{2} \left(1 - \frac{1}{x_D} \right) \right)$$

is periodic for $n \geq D_{x^{-1}}$ with period $L_{x^{-1}}$.

Proof. We split the summand using $p^n - \lfloor xi \rfloor = (p^n - xi) + \{xi\}$. Note that

$$\sum_{i=1}^{\lfloor x^{-1}p^n \rfloor} (p^n - xi) = \lfloor x^{-1}p^n \rfloor p^n - x \cdot \frac{\lfloor x^{-1}p^n \rfloor (\lfloor x^{-1}p^n \rfloor + 1)}{2}.$$

Rewriting using $\lfloor x^{-1}p^n \rfloor = x^{-1}p^n - \{x^{-1}p^n\}$, we see that

$$\sum_{i=1}^{\lfloor x^{-1}p^n \rfloor} (p^n - xi) = \frac{x^{-1}}{2} p^{2n} - \frac{1}{2} p^n + \frac{x}{2} (1 - \{x^{-1}p^n\}) \{x^{-1}p^n\}. \tag{4}$$

It remains to consider the sum of the periodic function $\{xi\}$. Since $v_p(x) \geq 0$, the denominator of x (in lowest terms) is x_D so $\{xi\}$ is immediately periodic with period x_D . The average value of the function is $m := \frac{1}{2} \left(1 - \frac{1}{x_D}\right)$ as in a block of length x_D the fractional part $\{xi\}$ ranges through $0, 1/x_D, 2/x_D, \dots, (x_D - 1)/x_D$ in some order. Applying Lemma 1, we obtain

$$\sum_{i=1}^{\lfloor x^{-1}p^n \rfloor} \{xi\} = m \lfloor x^{-1}p^n \rfloor + \sum_{k=1}^{\lfloor x^{-1}p^n \rfloor \bmod x_D} (\{xk\} - m)$$

Expanding using $\lfloor x^{-1}p^n \rfloor = x^{-1}p^n - \{x^{-1}p^n\}$ gives

$$\sum_{i=1}^{\lfloor x^{-1}p^n \rfloor} \{xi\} = mx^{-1}p^n - m \{x^{-1}p^n\} + \sum_{k=1}^{\lfloor x^{-1}p^n \rfloor \bmod x_D} (\{xk\} - m) \tag{5}$$

The desired formula follows from combining Equations Equations (4) and (5). By Lemma 6, $\{x^{-1}p^n\}$ is periodic for $n \geq D_{x^{-1}}$ with period $L_{x^{-1}}$. By Lemma 7, $(\lfloor x^{-1}p^n \rfloor \bmod x_D)$ is periodic for $n \geq D_{x^{-1}}$ with period $L_{x^{-1}}$. This gives the desired periodicity for $A_{x^{-1}}(n)$. \square

Corollary 3. *With notation as in Notation 1,*

$$\begin{aligned} \sum_{i=1}^{\lfloor \tau^{-1}p^n \rfloor} (p^n - \lfloor \tau i \rfloor) - \sum_{i=1}^{\lfloor \gamma^{-1}p^n \rfloor} (p^n - \lfloor \gamma i \rfloor) &= \frac{\tau^{-1} - \gamma^{-1}}{2} p^{2n} + \frac{(\tau^{-1} - \gamma^{-1})(\tau_D - 1)}{2\tau_D} p^n \\ &\quad + (A_{\tau^{-1}}(n) - A_{\gamma^{-1}}(n)). \end{aligned}$$

Proof. Note that $v_p(\tau) = 0$ and $v_p(\gamma) \geq 0$ because $d \mid p - 1$ and therefore $p \nmid d$. We can then apply Proposition 4 and rewrite

$$\sum_{i=1}^{\lfloor \tau^{-1}p^n \rfloor} (p^n - \lfloor \tau i \rfloor) - \sum_{i=1}^{\lfloor \gamma^{-1}p^n \rfloor} (p^n - \lfloor \gamma i \rfloor)$$

$$\begin{aligned}
 &= \left[\frac{1}{2} \tau^{-1} p^{2n} + \frac{1}{2} \left(\tau^{-1} \left(1 - \frac{1}{\tau_D} \right) - 1 \right) p^n + A_{\tau^{-1}}(n) \right] \\
 &\quad - \left[\frac{1}{2} \gamma^{-1} p^{2n} + \frac{1}{2} \left(\gamma^{-1} \left(1 - \frac{1}{\gamma_D} \right) - 1 \right) p^n + A_{\gamma^{-1}}(n) \right] \quad (\text{Proposition 4}) \\
 &= \frac{1}{2} (\tau^{-1} - \gamma^{-1}) p^{2n} + \frac{1}{2} (\tau^{-1} - \gamma^{-1}) \left(1 - \frac{1}{\tau_D} \right) p^n + A_{\tau^{-1}}(n) - A_{\gamma^{-1}}(n). \quad \square
 \end{aligned}$$

Example 6. For $p = 5$, $d = 4$, and $r = 2$,

$$\sum_{i=1}^{\lfloor \gamma^{-1} p^n \rfloor} (\gamma - \tau) i + \sum_{i=\lfloor \gamma^{-1} p^n \rfloor + 1}^{\lfloor \tau^{-1} p^n \rfloor} (p^n - \lfloor \tau i \rfloor) = \frac{4}{21} \cdot 5^{2n} + \frac{2}{21} \cdot 5^n + c^{(1)}(n),$$

where $c^{(1)}(n)$ is given in Table 3.

$n \pmod{6}$	0	1	2	3	4	5
$c^{(1)}(n)$	$-2/7$	$-5/21$	$-3/7$	$-2/21$	$-2/7$	$1/3$

Table 3: Value of periodic function $c^{(1)}(n)$ for $n \geq 0$.

3.2. Sums Involving δ

We now turn to the sums of $\delta(i)$ appearing in Proposition 3. We first reduce to considering the case where i is prime to p .

Lemma 8. *Let $x \in \mathbb{Q}_{>0}$ with $x \geq 1$. Then*

$$\sum_{i=1}^{\lfloor x^{-1} p^n \rfloor} \delta(i) = \sum_{e=0}^n \sum_{\substack{i=1 \\ p \nmid i}}^{\lfloor x^{-1} p^e \rfloor} \delta(i) = \sum_{e=0}^n \sum_{i=1}^{\lfloor x^{-1} p^e \rfloor} \delta_0(i).$$

Proof. This is an immediate consequence of Theorem 3(i) and rewriting using Definition 7. □

As $\delta_0(i)$ is periodic, the inner sum is easy to evaluate.

Lemma 9. *Let $x \in \mathbb{Q}_{>0}$ and e be a non-negative integer. Then*

$$\sum_{i=1}^{\lfloor x^{-1} p^e \rfloor} \delta_0(i) = \frac{1}{2} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{\tau_D} \right) (x^{-1} p^e - \{x^{-1} p^e\}) + F_{x^{-1}}(e),$$

where

$$F_{x^{-1}}(e) := \sum_{k=1}^{\lfloor x^{-1} p^e \rfloor \bmod \tau_D p} \left(\delta_0(k) - \frac{1}{2} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{\tau_D} \right) \right).$$

Proof. Theorem 3(ii) shows that δ_0 is an immediately periodic function with period $\tau_D p$, so we may apply Lemma 1. Proposition 2 shows that the average value of δ_0 over a block of length $\tau_D p$ is $\frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right)$. \square

Corollary 4. *Let $x \in \mathbb{Q}_{>0}$ and n be a positive integer with $x \geq 1$. Then we have*

$$\begin{aligned} \sum_{i=1}^{\lfloor x^{-1}p^n \rfloor} \delta(i) &= \frac{1}{2} \left(1 - \frac{1}{\tau_D}\right) \left(p^n - \frac{1}{p}\right) x^{-1} \\ &\quad - \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right) \sum_{e=0}^n \{x^{-1}p^e\} + \sum_{e=0}^n F_{x^{-1}}(e). \end{aligned}$$

Proof. We combine Lemma 8 with Lemma 9 to obtain

$$\begin{aligned} \sum_{i=1}^{\lfloor x^{-1}p^n \rfloor} \delta(i) &= \sum_{e=0}^n \sum_{i=1}^{\lfloor x^{-1}p^e \rfloor} \delta_0(i) \\ &= \sum_{e=0}^n \left[\frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right) (x^{-1}p^e - \{x^{-1}p^e\}) + F_{x^{-1}}(e) \right] \\ &= \sum_{e=0}^n \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right) (x^{-1}p^e) \\ &\quad - \sum_{e=0}^n \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right) \{x^{-1}p^e\} + \sum_{e=0}^n F_{x^{-1}}(e) \\ &= \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right) x^{-1} \sum_{e=0}^n p^e \\ &\quad - \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right) \sum_{e=0}^n \{x^{-1}p^e\} + \sum_{e=0}^n F_{x^{-1}}(e). \end{aligned}$$

We simplify the geometric series:

$$\begin{aligned} \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right) x^{-1} \sum_{e=0}^n p^e &= \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right) x^{-1} \frac{p^{n+1} - 1}{p - 1} \\ &= \frac{1}{2} \left(1 - \frac{1}{\tau_D}\right) \left(p^n - \frac{1}{p}\right) x^{-1}. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} \sum_{i=1}^{\lfloor x^{-1}p^n \rfloor} \delta(i) &= \frac{1}{2} \left(1 - \frac{1}{\tau_D}\right) \left(p^n - \frac{1}{p}\right) x^{-1} - \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right) \sum_{e=0}^n \{x^{-1}p^e\} \\ &\quad + \sum_{e=0}^n F_{x^{-1}}(e). \end{aligned} \quad \square$$

Proposition 5. *Suppose that $x_D = \tau_D$. Then $F_{x-1}(e)$ is periodic for $e \geq D_{x-1} + 1$ with period L_{x-1} . Moreover, for $n \geq D_{x-1}$ we have*

$$\sum_{i=1}^{\lfloor x^{-1}p^n \rfloor} \delta(i) = \frac{1}{2}x^{-1} \left(1 - \frac{1}{\tau_D}\right) p^n + \left(\overline{F_{x-1}} - \frac{\overline{x^{-1}}}{2p} \left(1 - \frac{1}{\tau_D}\right)\right) n + B_{x-1}(n),$$

where $B_{x-1}(n)$ is a periodic function for $n \geq D_{x-1}$ with period L_{x-1} defined in Equation (6).

Proof. Note that $F_{x-1}(n)$ depends only on the value of $\lfloor x^{-1}p^n \rfloor$ modulo $\tau_D p$. By Lemma 7, $\lfloor x^{-1}p^n \rfloor$ is periodic modulo $\tau_D p$ for $n \geq D_{x-1} + 1$ with period L_{x-1} . This gives the first claim.

By Corollary 1, for all $n \geq D_{x-1}$ we have

$$\begin{aligned} \sum_{e=0}^n F_{x-1}(e) &= \sum_{e=0}^{D_{x-1}} F_{x-1}(e) + \overline{F_{x-1}}(n - D_{x-1}) \\ &\quad + \sum_{k=1}^{(n-D_{x-1}) \bmod L_x} (F_{x-1}(k + D_{x-1}) - \overline{F_{x-1}}) \\ &= \overline{F_{x-1}} \cdot n + B'_{x-1}(n), \end{aligned}$$

where $B'_{x-1}(n)$ is periodic for $n \geq D_{x-1}$ with period L_{x-1} . Note that the above application of Corollary 1 is slightly different in that the 0-indexed term $F_{x-1}(0)$ is included in the summation. However, this term is excluded from the repeating portion of the sequence and is subsumed by $B'_{x-1}(n)$.

On the other hand, by Lemma 6 we know that $\{x^{-1}p^e\}$ is periodic for $e \geq D_{x-1}$ with period L_{x-1} and average value $\overline{x^{-1}}/(p-1)$. By Corollary 1, for all $n \geq D_{x-1} - 1$ we have

$$\begin{aligned} \sum_{e=0}^n \{x^{-1}p^e\} &= \sum_{e=0}^{D_{x-1}-1} \{x^{-1}p^e\} + \frac{\overline{x^{-1}}}{p-1}(n - D_{x-1} + 1) \\ &\quad + \sum_{k=1}^{(n-D_{x-1}+1) \bmod L_{x-1}} \left(\{x^{-1}p^k\} - \frac{\overline{x^{-1}}}{p-1}\right) \\ &= \frac{\overline{x^{-1}}}{p-1} \cdot n + B''_{x-1}(n), \end{aligned}$$

where $B''_{x-1}(n)$ is periodic for $n \geq D_{x-1} - 1$ with period L_{x-1} . Once again, $B''_{x-1}(n)$ subsumes the 0-indexed term. To complete the proof we apply Corollary 4 and set

$$B_{x-1}(n) := B'_{x-1}(n) - \frac{1}{2} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\tau_D}\right) B''_{x-1}(n) - \frac{1}{2p} \left(1 - \frac{1}{\tau_D}\right) x^{-1}. \quad (6) \quad \square$$

Example 7. Let $p = 5$, $d = 4$, and $r = 2$. Then we have

$$\sum_{i=1}^{\lceil \tau^{-1}p^n \rceil} \delta(i) = \frac{1}{6} \cdot 5^n + \begin{cases} -\frac{1}{6} & \text{if } n \text{ is even} \\ \frac{1}{6} & \text{if } n \text{ is odd} \end{cases}$$

$$\sum_{i=1}^{\lceil \gamma^{-1}p^n \rceil} \delta(i) = \frac{1}{14} \cdot 5^n + \frac{1}{3}n + c^{(2)}(n),$$

where $c^{(2)}(n)$ is given in Table 4.

$n \pmod{6}$	0	1	2	3	4	5
$c^{(2)}(n)$	$-1/14$	$13/42$	$23/42$	$1/14$	$1/42$	$5/42$

Table 4: Value of periodic function $c^{(2)}(n)$ for $n \geq 0$.

Remark 8. Note the formulas in this subsection appear quite similar to “classical” formulas in Iwasawa theory, i.e., of the form $\mu p^n + \lambda n + \nu$, with the only difference being that ν is a periodic function of n . In contrast, the formulas for the sums in Section 3.1 are of the form $\alpha p^{2n} + \beta p^n + \gamma$ with γ periodic: “classical” formulas in Iwasawa theory do not include a p^{2n} term.

3.3. Proof of the Main Theorem

We will now prove Theorem 2 by combining Proposition 3 with Corollaries 3 and 5 and attempting to simplify the resulting formula. The main simplification comes from the following.

Proposition 6. *The coefficient of n in Proposition 5 is zero when $x = \tau$. That is,*

$$\overline{F_{\tau^{-1}}} - \frac{\overline{\tau^{-1}}}{2p} \left(1 - \frac{1}{\tau_D} \right) = 0.$$

Proof. As $d < p$, we see that the base- p expansion of $\tau^{-1} = \frac{d}{p+1}$ is

$$\frac{d}{p+1} = (d-1)p^{-1} + (p-d)p^{-2} + (d-1)p^{-3} + (p-d)p^{-4} + \dots \tag{7}$$

Thus, $\overline{\tau^{-1}} = \frac{1}{2}((d-1) + (p-d)) = (p-1)/2$, so that

$$\frac{\overline{\tau^{-1}}}{2p} \left(1 - \frac{1}{\tau_D} \right) = \frac{1}{4} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{\tau_D} \right).$$

To evaluate $\overline{F_{\tau^{-1}}} = \frac{1}{2}(F_{\tau^{-1}}(1) + F_{\tau^{-1}}(2))$, note that the remainder when $\lceil \tau^{-1}p^e \rceil$ is divided by p is $d-1$ if e is a positive odd integer and is $p-d$ if e is a positive even

integer. Similarly, the remainder when $\lfloor \tau^{-1}p^e \rfloor$ is divided by τ_D is $\tau_D - 1$ if e is a positive odd integer and 0 if e is a positive even integer. The Chinese Remainder Theorem gives

$$\left(\lfloor \tau^{-1}p^e \rfloor \bmod \tau_D p\right) = \begin{cases} 0 & \text{if } e = 0 \\ d - 1 & \text{if } e \text{ is odd} \\ \tau_D p - d & \text{if } e \text{ is even and } e > 0. \end{cases}$$

Recalling the definition of $F_{x^{-1}}(n)$ from Lemma 9, we see that

$$\begin{aligned} F_{\tau^{-1}}(1) &= \sum_{i=1}^{d-1} \left(\delta_0(i) - \frac{1}{2} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{\tau_D} \right) \right) \\ F_{\tau^{-1}}(2) &= \sum_{i=1}^{\tau_D p - d} \left(\delta_0(i) - \frac{1}{2} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{\tau_D} \right) \right). \end{aligned}$$

Thus, we are led to consider the sum

$$\sum_{i=1}^{d-1} \delta_0(i) + \sum_{i=1}^{\tau_D p - d} \delta_0(i). \tag{8}$$

By Proposition 1, for $1 \leq i \leq d - 1$, we have $\delta_0(d - i) = \delta_0(\tau_D p - i)$. Thus,

$$\sum_{i=1}^{d-1} \delta_0(i) = \sum_{i=1}^{d-1} \delta_0(d - i) = \sum_{i=1}^{d-1} \delta_0(\tau_D p - i) = \sum_{i=\tau_D p - d + 1}^{\tau_D p - 1} \delta_0(i), \tag{9}$$

so by Proposition 2,

$$\begin{aligned} \sum_{i=1}^{d-1} \delta_0(i) + \sum_{i=1}^{\tau_D p - d} \delta_0(i) &= \sum_{i=\tau_D p - d + 1}^{\tau_D p - 1} \delta_0(i) + \sum_{i=1}^{\tau_D p - d} \delta_0(i) \\ &= \sum_{i=1}^{\tau_D p - 1} \delta_0(i) = \frac{(p - 1)(\tau_D - 1)}{2}. \end{aligned} \tag{10}$$

Thus, we see that

$$\begin{aligned} \frac{1}{2}(F_{\tau^{-1}}(1) + F_{\tau^{-1}}(2)) &= \frac{1}{2} \left(\sum_{i=1}^{d-1} \delta_0(i) + \sum_{i=1}^{\tau_D p - d} \delta_0(i) \right) \\ &\quad - \frac{1}{4}(\tau_D p - 1) \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{\tau_D} \right) \\ &= \frac{1}{4} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{\tau_D} \right) = \frac{\tau^{-1}}{2p} \left(1 - \frac{1}{\tau_D} \right). \quad \square \end{aligned}$$

Remark 9. We do not have a conceptual explanation for why the coefficient of n is zero in Proposition 6.

Corollary 5. *With $B_{x^{-1}}(n)$ as in Proposition 5,*

$$\sum_{i=\lfloor \gamma^{-1}p^n \rfloor + 1}^{\lfloor \tau^{-1}p^n \rfloor} \delta(i) = \frac{1}{2} (\tau^{-1} - \gamma^{-1}) \left(1 - \frac{1}{\tau_D}\right) p^n + \left(\frac{\overline{\gamma^{-1}}}{2p} \left(1 - \frac{1}{\tau_D}\right) - \overline{F_{\gamma^{-1}}}\right) n + (B_{\tau^{-1}}(n) - B_{\gamma^{-1}}(n)).$$

Proof. Combine Proposition 5 (as $\tau_D = \gamma_D$) and Proposition 6. □

We now synthesize all our previous results to arrive at a version of our main theorem.

Theorem 4. *Fix a positive integer r . With our standing assumptions that $\{X_n\}$ is a \mathbb{Z}_p -tower totally ramified over one point of $X_0 \simeq \mathbb{P}^1$ with minimal break ratios and ramification invariant d and that $d|p-1$, there exists a function $\nu_r : \mathbb{N} \rightarrow \mathbb{Q}$ such that*

$$a^r(X_n) = \frac{1}{2} (\tau^{-1} - \gamma^{-1}) p^{2n} + \lambda_r n + \nu_r(n),$$

where

$$\lambda_r = \overline{F_{\gamma^{-1}}} - \frac{\overline{\gamma^{-1}}}{2p} \left(1 - \frac{1}{\tau_D}\right)$$

and $\nu_r(n)$ is periodic for $n \geq D_{\gamma^{-1}}$ and has minimal period

$$\text{lcm}(L_{\gamma^{-1}}, 2) \quad \text{or} \quad \frac{1}{2} \cdot \text{lcm}(L_{\gamma^{-1}}, 2).$$

Recall that Lemma 2 shows that $D_{\gamma^{-1}} = -v_p(\gamma^{-1})$ and $L_{\gamma^{-1}}$ is the order of p modulo γ_N . Whether the minimal period of $\nu_r(n)$ is $\text{lcm}(L_{\gamma^{-1}}, 2)$ or $\frac{1}{2} \cdot \text{lcm}(L_{\gamma^{-1}}, 2)$ appears to be subtle: see Example 9.

Proof. Start with Proposition 3, and use Corollary 3 for an expression for the sums involving floor functions and Corollary 5 for the sums involving $\delta(i)$. Collecting terms gives the formula, with

$$\nu_r(n) := A_{\tau^{-1}}(n) - A_{\gamma^{-1}}(n) - B_{\tau^{-1}}(n) + B_{\gamma^{-1}}(n).$$

Recall that $A_{x^{-1}}(n)$ and $B_{x^{-1}}(n)$ are periodic for $n \geq D_{x^{-1}}$ with period $L_{x^{-1}}$ (Propositions 4 and 5). Thus, $\nu_r(n)$ is periodic for $n \geq \max(D_{\tau^{-1}}, D_{\gamma^{-1}})$ with period $\text{lcm}(L_{\tau^{-1}}, L_{\gamma^{-1}})$. Equation (7) shows that $L_{\tau^{-1}} \leq 2$ and $D_{\tau^{-1}} = 0$, which gives the claimed delay and $\text{lcm}(L_{\gamma^{-1}}, 2)$ as a bound on the minimal period.

We now show the minimal period is either $\text{lcm}(L_{\gamma^{-1}}, 2)$ or half of it. We have that there is $\lambda_r \in \mathbb{Q}$ and $c : \mathbb{N} \rightarrow \mathbb{Q}$ such that for sufficiently large n ,

$$\begin{aligned} a^r(X_{n+L}) - a^r(X_n) &= \frac{1}{2}(\tau^{-1} - \gamma^{-1})p^{2(n+L)} + \lambda_r(n+L) + c(n+L) \\ &\quad - \left(\frac{1}{2}(\tau^{-1} - \gamma^{-1})p^{2n} + \lambda_r n + c(n) \right) \\ &= \frac{1}{2}(\tau^{-1} - \gamma^{-1})(p^{2L} - 1)p^{2n} + \lambda_r L \\ &= \frac{1}{2} \frac{d}{p+1} (p^{2L} - 1)p^{2n} - \frac{1}{2} \gamma^{-1} (p^{2L} - 1)p^{2n} + \lambda_r L \end{aligned}$$

The left side is clearly an integer. Note that $\frac{1}{2} \frac{d}{p+1} (p^{2L} - 1)p^{2n}$ is an integer as well: $p + 1$ must divide one of $p^L - 1$ or $p^L + 1$ and the other is a multiple of 2 (unless $p = 2$, in which case $2 \mid p^{2n}$). Similarly, we note that $(p^{2L} - 1)p^{2n}$ is even. Furthermore, note that if we clear denominators by multiplying both sides by γ_N , then $\frac{1}{2} \gamma_N \gamma^{-1} (p^{2L} - 1)p^{2n}$ is an integer. Thus, we can conclude that $\gamma_N \lambda_r L$ is an integer. Furthermore, reducing modulo γ_N shows that

$$0 \equiv -\frac{\gamma_D}{2} (p^{2L} - 1)p^{2n - v_p(\gamma)} + \gamma_N \lambda_r L \pmod{\gamma_N}$$

As this holds for all sufficiently large n , we see that $p^{2L} - 1 \equiv 0 \pmod{\gamma_N}$. In other words, by Lemma 2, $L_{\gamma^{-1}} \mid 2L$. As $L \mid \text{lcm}(L_{\gamma^{-1}}, 2)$ by Theorem 4, we see $L = \text{lcm}(L_{\gamma^{-1}}, 2)$ or $L = \frac{1}{2} \text{lcm}(L_{\gamma^{-1}}, 2)$. \square

Example 8. We can now establish Example 2, which has $p = 5$, $d = 4$, and $r = 2$. Combining Theorem 4 (or really just Proposition 3) with Examples 6 and 7 we see that

$$a^2(X_n) = \frac{4}{21} \cdot 5^{2n} + \frac{1}{3}n + \nu_2(n),$$

where $\nu_2(n)$ is given in Table 1. Note that the period given by the theorem (6) is not the minimal period (3), which we observe in this instance coincides with $L_{\gamma^{-1}}$.

Corollary 6. *When $d = 1$ or $d = 2$:*

(i) $\lambda_r = 0$.

(ii) *The minimal period of $a^r(X_n)$ is*

$$L = \begin{cases} L_{\gamma^{-1}} & \text{if } L_{\gamma^{-1}} \text{ is odd} \\ L_{\gamma^{-1}} \text{ or } \frac{L_{\gamma^{-1}}}{2} & \text{if } L_{\gamma^{-1}} \text{ is even.} \end{cases}$$

Proof. When $d = 1$ or $d = 2$, then $\tau_D = 1$ (recall Notation 2), so $\delta(i) = 0$ for all integers i . Hence, by Proposition 3 and Corollary 3,

$$a^r(X_n) = \sum_{i=1}^{\lfloor \tau^{-1} p^n \rfloor} (p^n - \lfloor \tau i \rfloor) - \sum_{i=1}^{\lfloor \gamma^{-1} p^n \rfloor} (p^n - \lfloor \gamma i \rfloor)$$

$$\begin{aligned} &= \frac{\tau^{-1} - \gamma^{-1}}{2} \cdot p^{2n} + \frac{(\tau^{-1} - \gamma^{-1})(\tau_D - 1)}{2\tau_D} \cdot p^n + (A_{\tau^{-1}}(n) - A_{\gamma^{-1}}(n)) \\ &= \frac{1}{2} (\tau^{-1} - \gamma^{-1}) p^{2n} + (A_{\tau^{-1}}(n) - A_{\gamma^{-1}}(n)), \end{aligned}$$

showing $\lambda_r = 0$. Furthermore, noting that $\lfloor \tau^{-1}p^n \rfloor \pmod{\tau_D}$ is 0 for $\tau_D = 1$, Proposition 4 gives

$$\begin{aligned} A_{\tau^{-1}}(n) &:= \frac{1}{2} \left(- \left(1 - \frac{1}{\tau_D} \right) + \tau(1 - \{\tau^{-1}p^n\}) \right) \{\tau^{-1}p^n\} + \\ &\quad \sum_{k=1}^{\lfloor \tau^{-1}p^n \rfloor \pmod{\tau_D}} \left(\{\tau k\} - \frac{1}{2} \left(1 - \frac{1}{\tau_D} \right) \right) \\ &= \frac{1}{2} \tau(1 - \{\tau^{-1}p^n\}) \{\tau^{-1}p^n\}. \end{aligned}$$

Notice that $1 - \{\tau^{-1}p^n\} = \{\tau^{-1}p^{n+1}\} = 1 - \{\tau^{-1}p^{n+2}\}$. Hence, $A_{\tau^{-1}}(n) = A_{\tau^{-1}}(n + 1)$, so $A_{\tau^{-1}}(n)$ is constant. Recalling that $A_{\gamma^{-1}}$ has period $L_{\gamma^{-1}}$, we conclude that $A_{\tau^{-1}}(n) - A_{\gamma^{-1}}(n)$ and $a^r(X_n)$ have period at most $L_{\gamma^{-1}}$. Theorem 4 then yields the desired period. \square

Example 9. We can observe the above corollary in Table 5, which shows the behavior of the period when $d = 2$ and $p = 5$, for $r = 1, \dots, 16$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
L	1	3	3	5	2	1	8	9	3	11	1	9	7	3	10	3
$L_{\gamma^{-1}}$	1	6	6	5	4	2	16	9	6	22	1	18	14	3	10	6
$\text{lcm}(L_{\gamma^{-1}}, 2)$	2	6	6	10	4	2	16	18	6	22	2	18	14	6	10	6

Table 5: Comparing L to $\text{lcm}(L_{\gamma^{-1}}, 2)$ with $d = 2, p = 5$.

We can also obtain an explicit formula in the following case.

Corollary 7. *When $r = p + 1$, we have $\lambda_r = 0$, the minimal period L is 1, and $\nu_r(n) = \frac{1}{2}(p - 1)(\tau^{-1} - 1)$ for $n \geq 1$.*

Proof. For $r = p + 1$, we have $\gamma^{-1} = \frac{1}{p}\tau^{-1}$, so $\overline{F_{\gamma^{-1}}} = \overline{F_{\tau^{-1}}}$ and $\overline{\gamma^{-1}} = \overline{\tau^{-1}}$. Thus, by Theorem 4 and Proposition 6, we obtain $\lambda_r = 0$. It is also immediate that $D_{\gamma^{-1}} = 1$.

To see the minimal period, we will analyze

$$\nu_r(n) = A_{\tau^{-1}}(n) - A_{\gamma^{-1}}(n) + B_{\gamma^{-1}}(n) - B_{\tau^{-1}}(n).$$

This is slightly tricky, as $A_{\tau^{-1}}(n) - A_{\gamma^{-1}}(n)$ and $B_{\gamma^{-1}}(n) - B_{\tau^{-1}}(n)$ both have period two, but the sum is constant. It is an elementary but involved exercise,

which we leave to the reader, to check

$$A_{\tau^{-1}}(n) - A_{\gamma^{-1}}(n) = \frac{1}{2}(p-1)(\tau^{-1} - 1) + (-1)^n \left(- \left(1 - \frac{1}{\tau_D} \right) \tau^{-1} \right) \quad \text{and}$$

$$\begin{aligned} B_{\gamma^{-1}}(n) - B_{\tau^{-1}}(n) &= B_{\tau^{-1}}(n-1) - B_{\tau^{-1}}(n) \\ &= (-1)^n \left(\frac{1}{2} \left(1 - \frac{1}{\tau_D} \right) \left(\frac{1}{p}d + \left(1 - \frac{1}{p} \right) \tau^{-1} \right) \right). \end{aligned}$$

There are two key observations:

- $\gamma^{-1} = \frac{1}{p}\tau^{-1}$, so for example $B_{\gamma^{-1}}(n) = B_{\tau^{-1}}(n-1)$.
- as $\lfloor \tau^{-1}p^n \rfloor \pmod{\tau_D}$ is 0 if n is even and $\tau_D - 1$ if n is odd, the sums appearing in the definitions of $A_{\tau^{-1}}(n)$ and $B_{\tau^{-1}}(n)$ are almost the same, with one summand depending on the parity of n .

It is then a long but elementary computation to work out the exact formula. □

3.4. Additional Information about Periods and λ_r

In this section, we tighten our predictions about the parameter λ_r .

Corollary 8. *The quantity $\lambda_r \cdot L$ is an integer.*

Proof. The proof of Theorem 4 shows $\gamma_N \lambda_r L \equiv 0 \pmod{\gamma_N}$, which is equivalent to $\lambda_r L \in \mathbb{Z}$. □

The conjecture is supported by a variety of examples we have investigated. The following proposition relates the formulas for $a^r(X_n)$ for certain pairs of r values.

Proposition 7. *Let r_0, r_1 be non-negative integers such that $r_1 = (r_0 + 1)p + 1$. Let λ_0, λ_1 (resp. D_0, D_1) be the values of λ_r (resp. $D_{\gamma^{-1}}$) for r_0, r_1 . Then $\lambda_1 = \lambda_0$ and $D_1 = D_0 + 1$.*

Proof. Let γ_1 and γ_0 be the values of $\gamma = \frac{p-1}{d}r + \tau = \frac{(p-1)r+(p+1)}{d}$ when $r = r_1$ and $r = r_0$ respectively. We see that

$$\gamma_1^{-1} = \frac{d}{(p-1)((r_0+1)p+1) + p+1} = \frac{d}{p((p-1)r_0 + (p-1)) + 2p} = \frac{1}{p}\gamma_0^{-1}$$

This implies that the repeating parts of the base- p expansions are the same and $D_{\gamma_1^{-1}} = D_{\gamma_0^{-1}} + 1$. Thus, $\overline{\gamma_1^{-1}} = \overline{\gamma_0^{-1}}$ and $F_{\gamma_1^{-1}}(e) = F_{\gamma_0^{-1}}(e+1)$. Since the latter implies $\overline{F_{\gamma_1^{-1}}} = \overline{F_{\gamma_0^{-1}}}$, the claim follows from the formula for λ_r in Theorem 4. □

In the above proof, $\gamma_1^{-1} = \frac{1}{p}\gamma_0^{-1}$ implies $L_{\gamma_1^{-1}} = L_{\gamma_0^{-1}}$, giving a relationship between the possible minimal periods of $a^{r_1}(X_n)$ and $a^{r_0}(X_n)$. We end with a conjecture.

Conjecture 2. *For two r -values, r_1, r_0 , where $r_1 = (r_0 + 1)p + 1$, the minimal periods for the $a^{r_1}(X_n)$ and $a^{r_0}(X_n)$ are the same.*

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