SUPER-m-ZUMKELLER NUMBERS

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Abstract

The study of divisor functions has inspired several generalizations of perfect numbers, among which Zumkeller numbers and m-Zumkeller numbers play a significant role. A Zumkeller number is a positive integer whose set of positive divisors can be partitioned into two disjoint subsets of equal sum, whereas an m-Zumkeller number requires a partition of divisors with equal product. In this paper, we introduce and investigate a new generalization, termed a super-m-Zumkeller number. A positive integer n is called a super-m-Zumkeller number if the set of all of the positive divisors of $T(n)$ (the product of all of the positive divisors of n) can be partitioned into two disjoint subsets with equal products. We establish necessary and sufficient conditions for an integer to be super-m-Zumkeller, derive infinite families of such numbers, and characterize their behavior under prime powers and products of primes. Furthermore, we examine their relationship with k-m-perfect numbers. These results extend the existing framework of Zumkeller-type numbers and reveal new structural links between additive and multiplicative divisor functions.

1. Introduction

The study of perfect numbers has a long and distinguished history in number theory, dating back to Euclid and later to Euler. A *perfect number* is defined as a positive integer equal to the sum of its proper positive divisors, and it represents one of the earliest instances of an arithmetic function inspiring rich structural investigation. Over time, various generalizations of perfect numbers have been introduced, not only to extend their scope but also to reveal deeper properties of divisor functions. Among these, Zumkeller numbers and their variants have attracted significant attention.

A *Zumkeller number* [7] is defined as a positive integer n whose set of positive divisors can be partitioned into two disjoint subsets of equal sum. This concept, introduced as a natural additive analogue of perfectness, broadens the classical notion and has been the subject of a number of recent investigations. The numbers

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$6, 12, 20, 24, 28, 30, 40,$ and 42 are the first few Zumkeller numbers. Building on this, the concept of m-Zumkeller numbers [6] was introduced. A positive integer n is said to be an *m-Zumkeller number* if its set of positive divisors can be partitioned into two disjoint subsets of equal product. The shift from additive to multiplicative partitioning opens up new directions in the interplay between divisor sums and divisor products, blending additive and multiplicative number theory. The numbers $6, 8, 10, 14, 15, 16, 21,$ and 22 are the first few m-Zumkeller numbers. Extensive research has been conducted on generalizations of Zumkeller and m-Zumkeller numbers, which can be found in [2, 3, 4, 5].

In this paper, we propose a further generalization, termed the super-m-Zumkeller number. For a positive integer n, let $T(n)$ denote the product of all of its positive divisors. We call n a *super-m-Zumkeller number* if the set of positive divisors of $T(n)$ can be partitioned into two disjoint subsets with equal products. This definition extends the m-Zumkeller property to a higher-order structure by transferring the partition condition from divisors of n to divisors of $T(n)$. In this way, the study of super-m-Zumkeller numbers enriches the existing framework by introducing an iterative dimension to the divisor product function.

The motivation for this work lies in both the structural elegance of such generalizations and their unexpected connections to other well-studied classes of numbers. In particular, we explore the relationship between super-m-Zumkeller numbers and *k-m-perfect numbers* [8], defined as those integers n satisfying $T(n) = n^k$, where $k \in \mathbb{Z}$ and $k \geq 2$.

The contributions of this paper are threefold. First, we formalize the definition of super-m-Zumkeller numbers and establish fundamental necessary and sufficient conditions for their existence. Second, we derive structural results and infinite families of such numbers by examining prime power cases, products of distinct primes, and certain congruence classes. Third, we study their relationship with k-m-perfect numbers, identifying conditions under which these two families intersect. These results not only extend the body of knowledge on Zumkeller-type numbers but also suggest new avenues for the exploration of divisor functions and multiplicative partitions.

2. Super-m-Zumkeller Numbers

Definition 1. Let $T(n)$ denote the product of all of the positive divisors of the positive integer n. Then n is a *super-m-Zumkeller number* if the set of all of the positive divisors of the number $T(n)$ can be partitioned into two disjoint subsets of equal product.

A *super-m-Zumkeller partition* for a super-m-Zumkeller number n is a partition $\{A, B\}$ of the set of positive divisors of $T(n)$ so that the product of the elements in

A yields the same value as the product of the elements in B.

In what follows, $\tau(n)$ denotes the number of positive divisors of n and $T(n)$ denotes the product of all of the positive divisors of n. We will need the following well-known lemma (see for example, [1]).

Lemma 1 ([1]). *Let the prime factorization of n be $\prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are distinct primes and $\alpha_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, r$. Then we have*

$$T(n) = n^{\frac{\tau(n)}{2}},$$

where $\tau(n) = \prod_{j=1}^{r} (\alpha_j + 1)$.

Theorem 1. *If n is a super-m-Zumkeller number, then $\tau(n) \geq 3$.*

Proof. Since $n > 1$, if $\tau(n) = 2$, then n must be a prime number and $T(T(n)) = T(n) = n$, which is also a prime number. In this case, n is not a super-m-Zumkeller number. Therefore, for an integer to be a super-m-Zumkeller number, we must have $\tau(n) \geq 3$. □

Theorem 2 provides a necessary condition for a positive integer n to be a super-m-Zumkeller number. The following theorem establishes a necessary and sufficient condition.

Theorem 2. *An integer $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are distinct primes and $\alpha_i \in \mathbb{N}$, is a super-m-Zumkeller number if and only if 8 divides $\alpha_i \tau(n)\tau(T(n))$ for all $i = 1, 2, \ldots, r$.*

Proof. Assume that $n = \prod_{i=1}^{r} p_i^{\alpha_i}$. Then we have

$$\begin{aligned} T(n) &= \left(\prod_{i=1}^{r} p_i^{\alpha_i} \right)^{\frac{\tau(n)}{2}} \\ &= \prod_{i=1}^{r} p_i^{\frac{\alpha_i \tau(n)}{2}}. \end{aligned}$$

It follows that $T(T(n)) = \left(\prod_{i=1}^{r} p_i^{\frac{\alpha_i \tau(n)}{2}} \right)^{\frac{\tau(T(n))}{2}}$. Thus, n is a super-m-Zumkeller number if and only if 8 divides $\alpha_i \tau(n)\tau(T(n))$ for all $i = 1, 2, \ldots, r$. □

Remark 1. Let p, q, and r denote distinct primes. Then the integers of the form $p^2, p^5, p^7, p^8, p^{10}, pqr$, and p^3q are always super-m-Zumkeller numbers.

Corollary 1. *The product of distinct prime numbers* $\prod_{i=1}^{r} p_i$*, where* $r \geq 3$*, is always a super-m-Zumkeller number.*

Proof. Let $n = \prod_{i=1}^{r} p_i$, where $r \geq 3$. Then $\tau(n) = 2^r$. Since $r \geq 3$, it follows that 8 divides $\tau(n)$. Hence, n is a super-m-Zumkeller number. □

We can conclude that there are infinitely many super-m-Zumkeller numbers as a result of Corollary 1.

Corollary 2. *The integers of the form* $2^k \prod_{i=1}^{r} p_i$*, where* $p_1, p_2, \ldots, p_r$ *are distinct primes, are super-m-Zumkeller numbers if either of the following hold: (i)* $r = 2$ *and* k *is an odd positive integer, or (ii)* $r \geq 3$ *and* $k \in \mathbb{N}$*.*

Proof. Let $n = 2^k \prod_{i=1}^{r} p_i$, which implies $\tau(n) = (k+1)2^r$. If (i) holds, where k is an odd integer and $r = 2$, then $k+1$ is even, making $(k+1)2^2$ a multiple of 8; thus, $8 \mid \tau(n)$ and n is a super-m-Zumkeller number. Similarly, if (ii) holds such that $r \geq 3$, then 2^r is at least 2^3, ensuring that $8 \mid \tau(n)$ and n is a super-m-Zumkeller number. □

Theorem 3. *If* p *is a prime, then* p^α*, where* α *is a non-zero positive integer, is a super-m-Zumkeller number if and only if* $\alpha \equiv 0, 2, 5, 7, 8, 10, 13,$ *or* $15 \pmod{16}$*.*

Proof. The number p^α is a super-m-Zumkeller number if and only if 8 divides $\alpha(\alpha+1)\tau(p^{\frac{\alpha(\alpha+1)}{2}})$. This is equivalent to the condition that 8 divides $\alpha(\alpha+1)[\frac{\alpha(\alpha+1)}{2}+1]$, which can be written as the congruence

$$\alpha(\alpha+1)\left[\frac{\alpha(\alpha+1)}{2}+1\right] \equiv 0 \pmod{8}. \tag{1}$$

`Case 1:` Let $\alpha \equiv 0, 7, 8,$ or $15 \pmod{16}$. If $\alpha \equiv 0 \pmod{8}$, then α is a multiple of 8. Similarly, if $\alpha \equiv 7 \pmod{8}$, then $(\alpha+1)$ is also a multiple of 8. In both instances, the product in Congruence (1) is clearly congruent to $0 \pmod{8}$. Thus, p^α is a super-m-Zumkeller number for these values.

`Case 2:` Let $\alpha \equiv 2$ or $10 \pmod{16}$. This implies $\alpha \equiv 2 \pmod{8}$. Then $\alpha = 8k+2$ for some $k \in \mathbb{N}$. Substituting this into the term $\frac{\alpha(\alpha+1)}{2}+1$, we get:

$$\begin{aligned}\frac{(8k+2)(8k+3)}{2}+1 &= (4k+1)(8k+3)+1 \\ &= 32k^2+20k+4 = 4(8k^2+5k+1) \equiv 0 \pmod{4}.\end{aligned}$$

Since $\alpha = 8k+2$ is even, the product $\alpha(\alpha+1)$ is divisible by 2. It follows that the product of $\alpha(\alpha+1)$ and $\left[\frac{\alpha(\alpha+1)}{2}+1\right]$ is divisible by $2 \cdot 4 = 8$. Thus, Congruence (1) is satisfied for $\alpha \equiv 2$ or $10 \pmod{16}$.

`Case 3:` Let $\alpha \equiv 5$ or $13 \pmod{16}$. This implies $\alpha \equiv 5 \pmod 8$. Then $\alpha = 8k+5$ for some $k \in \mathbb{N}$. We evaluate the expression:

$$\alpha(\alpha+1)\left[\frac{\alpha(\alpha+1)}{2}+1\right] = (8k+5)(8k+6)\left[\frac{(8k+5)(8k+6)}{2}+1\right].$$

This simplifies to $(8k+5)\cdot 2(4k+3)\cdot[(8k+5)(4k+3)+1]$. The term in the square bracket is $(32k^2+44k+16)$, which is a multiple of 4. The total product then contains factors of 2 and 4, making it congruent to $0 \pmod 8$. Thus, Congruence (1) is satisfied for $\alpha \equiv 5$ or $13 \pmod{16}$.

`Case 4:` Let $\alpha \in \{1,3,4,6,9,11,12,14\} \pmod{16}$. By direct substitution into Congruence (1), it can be shown that the expression is not divisible by 8. For example, if $\alpha \equiv 1 \pmod{16}$, then $\alpha = 16k+1$ for some $k \in \mathbb{N}$, and the expression becomes:

$$(16k+1)(16k+2)\left[\frac{(16k+1)(16k+2)}{2}+1\right] \equiv 1(2)\left[\frac{2}{2}+1\right] \equiv 4 \not\equiv 0 \pmod 8.$$

Similarly, testing the other values in this set shows that the product is never congruent to $0 \pmod 8$, thus these values of α do not yield a super-m-Zumkeller number.

By combining the results of these cases, we conclude that p^α is a super-m-Zumkeller number if and only if $\alpha \equiv 0, 2, 5, 7, 8, 10, 13,$ or $15 \pmod{16}$. □

Theorem 4. *Let* $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ *be a super-m-Zumkeller number, where* $p_1 < p_2 < \cdots < p_r$ *are distinct primes and* $\alpha_i \in \mathbb{N}$ *for all* $i = 1, 2, \ldots, r$*. Let* p *be a prime such that* $\gcd(n,p) = 1$*. Then the integer* np *is also a super-m-Zumkeller number if and only if* $\alpha_i \not\equiv 0 \pmod 8$ *for all* $i = 1, 2, \ldots, r$*.*

Proof. Since $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ is a super-m-Zumkeller number, it follows that 8 divides $\alpha_i \tau(n)\tau(T(n))$ for all $i = 1, 2, \ldots, r$. This implies that if $\alpha_i \equiv 1, 3, 5,$ or $7 \pmod 8$, then $\tau(n)\tau(T(n))$ must be even. Similarly, if $\alpha_i \equiv 2, 4,$ or $6 \pmod 8$, then $\tau(n)\tau(T(n))$ must again be even. Now, if $\alpha_i \equiv 0 \pmod 8$, then α_i is of the form $8k_i$, where $k_i \in \mathbb{N}$. In this case, $\tau(n)$ must be of the form $8l+1$ for $l \in \mathbb{N}$. We have

$$\begin{aligned} T(n) &= T\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \\ &= \left(\prod_{i=1}^{r} p_i^{8k_i}\right)^{\frac{8l+1}{2}} \\ &= \prod_{i=1}^{r} p_i^{4k_i(8l+1)}. \end{aligned}$$

Therefore, $\tau(T(n))$ must be an odd number. Thus, $\tau(n)\tau(T(n))$ is odd if and only if

$$\alpha_i \equiv 0 \pmod{8}. \tag{2}$$

Now, consider $\tau(np) = \tau(n)\tau(p) = 2\tau(n)$. The number np is a super-m-Zumkeller number if and only if 8 divides $2\tau(n)\tau(T(n))\tau(T(p))$. This is equivalent to 4 dividing $\tau(n)\tau(T(n))\tau(p)$, which is equivalent to 2 dividing $\tau(n)\tau(T(n))$. By Congruence (2), this holds if and only if $\alpha_i \not\equiv 0 \pmod{8}$. □

Theorem 5. *Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ be a super-m-Zumkeller number, where $p_1 < p_2 < \cdots < p_r$ are distinct primes and $\alpha_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, r$. Let p be a prime such that $\gcd(n, p) = 1$. Then for $l \in \mathbb{N}$, the number np^l is also a super-m-Zumkeller number if and only if one of the following holds:*

(i) $l \equiv 0, 2, 5, 7 \pmod{8}$;

(ii) $l \equiv 1, 3, 4, 6 \pmod{8}$ *and* $\alpha_i \not\equiv 0 \pmod{8}$ *for all* $i = 1, 2, \ldots, r$.

Proof. First, assume that either (i) or (ii) holds. We prove that np^l is a super-m-Zumkeller number.

Since $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ is a super-m-Zumkeller number, it follows that 8 divides $\alpha_i\tau(n)\tau(T(n))$ for all $i = 1, 2, \ldots, r$. The integer np^l is a super-m-Zumkeller number if and only if 8 divides $l(l+1)\tau(n)\tau(T(n))\tau(T(p^l))$. This condition is equivalent to

$$8 \mid l(l+1)\tau(n)\tau(T(n))\left[\frac{l(l+1)}{2} + 1\right]. \tag{3}$$

We now consider the eight possible residue classes of $l \pmod{8}$.

`Case 1:` Let $l \equiv 0 \pmod{8}$. Then clearly from (3), np^l is a super-m-Zumkeller number.

`Case 2:` Let $l \equiv 1 \pmod 8$, so that $l = 8k+1$ for some $k \in \mathbb{N}$. Then from (3), np^l is a super-m-Zumkeller number if and only if 2 divides $\tau(n)\tau(T(n))$, which is equivalent to $\alpha_i \not\equiv 0 \pmod 8$.
`Case 3:` Let $l \equiv 2 \pmod 8$, so that $l = 8k+2$ for some $k \in \mathbb{N}$. Then from (3), np^l is a super-m-Zumkeller number if and only if 4 divides $(4k+1)(8k+3)\tau(n)\tau(T(n))(32k^2+20k+4)$. This holds for all $k \in \mathbb{N}$. Therefore, np^l is a super-m-Zumkeller number.
`Case 4:` Let $l \equiv 3 \pmod 8$, so that $l = 8k+3$ for some $k \in \mathbb{N}$. Then from (3), np^l is a super-m-Zumkeller number if and only if

$$2 \text{ divides } \tau(n)\tau(T(n))(32k^2+28k+7),$$

which holds if and only if $\alpha_i \not\equiv 0 \pmod 8$.
`Case 5:` Let $l \equiv 4 \pmod 8$. Then from (3), np^l is a super-m-Zumkeller number if and only if 2 divides $\tau(n)\tau(T(n))$, which is equivalent to $\alpha_i \not\equiv 0 \pmod 8$.
`Case 6:` Let $l \equiv 5 \pmod 8$, so that $l = 8k+5$ for some $k \in \mathbb{N}$. Then from (3), np^l is a super-m-Zumkeller number if and only if

$$8 \text{ divides } (8k+5)(8k+6)\tau(n)\tau(T(n))\left[(8k+5)(4k+3)+1\right].$$

This holds for all $k \in \mathbb{N}$. Therefore, np^l is a super-m-Zumkeller number.
`Case 7:` Let $l \equiv 6 \pmod 8$, so that $l = 8k+6$ for some $k \in \mathbb{N}$. Then from (3), np^l is a super-m-Zumkeller number if and only if 2 divides $\tau(n)\tau(T(n))$, which holds if and only if $\alpha_i \not\equiv 0 \pmod 8$.
`Case 8:` Let $l \equiv 7 \pmod 8$, so that $l = 8k+7$ for some $k \in \mathbb{N}$. Then from (3), np^l is a super-m-Zumkeller number if and only if

$$8 \text{ divides } (8k+7)(8k+8)\tau(n)\tau(T(n))\left[(8k+7)(4k+4)+1\right].$$

This holds for all $k \in \mathbb{N}$. Therefore, np^l is a super-m-Zumkeller number.

Conversely, assume that np^l is a super-m-Zumkeller number. Then from (3), we have

$$8 \mid l(l+1)\tau(n)\tau(T(n))\left[\frac{l(l+1)}{2}+1\right].$$

Analyzing this divisibility condition modulo 8, it follows that l must satisfy one of the congruences listed in (i) or (ii), and in the latter case the condition $\alpha_i \not\equiv 0 \pmod 8$ is necessary. Hence, the result follows. □

As previously defined in Section 1, a positive integer n is called a *k-m-perfect number* if $T(n) = n^k$, where $k \in \mathbb{Z}$ and $k \geq 2$.

Theorem 6. *Every k-m-perfect number, where $k \in \mathbb{Z}$ and $k \geq 2$, is a super-m-Zumkeller number if $k \equiv 0 \pmod 4$. However, the converse is not true.*

Proof. Let n be a k-m-perfect number. Then $T(n) = n^k$, which implies $n^{\tau(n)/2} = n^k$, or $\tau(n) = 2k$. If $k \equiv 0 \pmod 4$, it follows that 8 divides $\tau(n)$, and thus n is a super-m-Zumkeller number. The converse is not true because 4 is a super-m-Zumkeller number but is not a k-m-perfect number. □

Remark 2. If p, q, r, and s are distinct primes, then the following statements hold:

1. all of the 4-m-perfect numbers of the form pq^3, pqr, and p^7 are super-m-Zumkeller numbers;
2. all of the 8-m-perfect numbers of the form $pqrs, pqr^3, p^3q^3, pq^7$, and p^{15} are super-m-Zumkeller numbers;
3. all of the 12-m-perfect numbers of the form $pqrs^2, pq^{11}, pqr^5, p^{23}, pq^2r^3, p^2q^7$, and p^3q^5 are super-m-Zumkeller numbers.

Theorem 7. *Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ be a k-m-perfect number, where $p_1 < p_2 < \cdots < p_r$ are distinct primes and $\alpha_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, r$. Then n is a super-m-Zumkeller number if and only if one of the following holds:*

1. $k \equiv 0 \pmod 4$ *for any choice of* α_i*;*
2. $k \equiv 3 \pmod 4$ *and* $\alpha_i \equiv 1 \pmod 4$ *for all* $i = 1, 2, \ldots, r$*.*

Proof. If $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ is a k-m-perfect number, then it follows that

$$\tau(n) = 2k. \tag{4}$$

We observe that the number of divisors of $T(n)$ satisfies

$$\begin{aligned}\tau(T(n)) &= \tau\left(n^{\frac{\tau(n)}{2}}\right) \\ &= \tau\left[\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right)^k\right] \\ &= \prod_{i=1}^{r}(\alpha_i k + 1).\end{aligned}$$

Therefore, n is a super-m-Zumkeller number if and only if 8 divides $\alpha_i(2k)(\alpha_i k + 1)$ for all i. This is equivalent to the condition

$$\alpha_i k(\alpha_i k + 1) \equiv 0 \pmod 4. \tag{5}$$

`Case 1:` Let $k \equiv 0 \pmod 4$. Then clearly from Congruence (5), n is a super-m-Zumkeller number.

Case 2: Let $k \equiv 1 \pmod 4$, so that $k = 4l + 1$ for some $l \in \mathbb{N}$. In this case, Congruence (5) holds if and only if $\alpha_i(\alpha_i + 1) \equiv 0 \pmod 4$, which implies $\alpha_i \equiv 0 \pmod 4$ or $\alpha_i \equiv 3 \pmod 4$. However, if $\alpha_i \equiv 0$ or $3 \pmod 4$, then $\tau(n)$ would be an odd number. This contradicts Equation (4) because $\tau(n)$ must be even for a k-m-perfect number. Thus, n cannot be a super-m-Zumkeller number in this case.
Case 3: Let $k \equiv 2 \pmod 4$. Then from Congruence (5), n is a super-m-Zumkeller number if and only if $2\alpha_i \equiv 0 \pmod 4$, which is equivalent to $\alpha_i \equiv 0 \pmod 2$. If α_i is even, then $\tau(n)$ is odd. This again contradicts Equation (4), meaning n cannot be a super-m-Zumkeller number.
Case 4: Let $k \equiv 3 \pmod 4$, so that $k = 4l + 3$ for some $l \in \mathbb{N}$. Then from Congruence (5), n is a super-m-Zumkeller number if and only if $\alpha_i(\alpha_i - 1) \equiv 0 \pmod 4$. This condition implies $\alpha_i \equiv 0 \pmod 4$ or $\alpha_i \equiv 1 \pmod 4$. Since $\alpha_i \equiv 0 \pmod 4$ results in an odd value for $\tau(n)$, it follows that n is a super-m-Zumkeller number in this case only when $\alpha_i \equiv 1 \pmod 4$. □

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