COUNTING SYMMETRIC AND NON-SYMMETRIC PEAKS IN A SET PARTITION

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Abstract

The aim of this paper is to derive explicit formulas for two statistics. The first one is the total number of symmetric peaks in a set partition of $[1,n]$ with exactly k blocks, and the second one is the total number of non-symmetric peaks in a set partition of $[1,n]$ with exactly k blocks. We represent these results in two ways: first, by using the theory of generating functions, and second, by using combinatorial tools.

1. Introduction

A *partition* Π of the set $[1,n]=\{1,2,\ldots,n\}$ of size k (*a partition of* $[1,n]$ *with exactly* k *blocks*) is a collection $\{B_1,B_2,\ldots,B_k\}$ that satisfies: $\emptyset \neq B_i \subseteq [1,n]$ for all i, $B_i \bigcap B_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{k} B_i = [1,n]$. The elements B_i are called *blocks*. We use the assumption that blocks are listed in an increasing order of their minimal elements, that is, $\min B_1 < \min B_2 < \cdots < \min B_k$. We let $P_{n,k}$ denote the set of all partitions of $[1,n]$ with exactly k blocks (see [19]). Note that $|P_{n,k}| = S_{n,k}$, which are known as the *Stirling numbers of the second kind* (see A008277 in [18]). A partition Π can be written as $\pi_1\pi_2\cdots\pi_n$, where $i \in B_{\pi_i}$ for all i. This form is called the *canonical sequential form*. For example, $\Pi = \{\{12\},\{3\}\}$ is a partition of $[1,3]$ and the canonical sequential form is $\pi = 112$. Then we can see that any partition Π of the set $[1,n]$ with exactly k blocks can be represented as a word of length n over the alphabet $[1,k]$. A *word* ω *of length* n *over the alphabet* $[1,k]$ is an element of $[1,k]^n$ (see [7]) of the form $1(1)^*2(2)^*\cdots k(k)^*$, where $(m)^*$ is a word over the alphabet $[1,m]$. These words are called *restricted growth functions*. For other properties of set partitions, see [16].

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The study of patterns in combinatorial structures is very popular. For example, Kitaev [8] researched patterns in permutations and words. For more examples of patterns in various combinatorial structures, refer to [9, 10, 12], and [17]. Given $\pi = \pi_1\pi_2\cdots\pi_n \in [1,k]^n$, a pattern $\pi_i\pi_{i+1}\pi_{i+2}$ is called a *peak* at i if it satisfies $\pi_i < \pi_{i+1} > \pi_{i+2}$ for $1 \le i \le n-2$. For instance, let $\pi = 1322141251$ be a word in $[1,5]^{10}$. It has 3 peaks, which are 132, 141, and 251. Mansour and Shattuck [14] determined the number of peaks in all words of length n over the alphabet $[1,k]$, and derived the number of peaks over $P_{n,k}$.

We say that $\pi = \pi_1\pi_2\cdots\pi_n \in [1,k]^n$ contains *a symmetric peak*, if there exists $1 \le i \le n-2$ such that $\pi_i\pi_{i+1}\pi_{i+2}$ is a peak and $\pi_i = \pi_{i+2}$. Similarly, we say that π contains *a non-symmetric peak*, if there exists $1 \le i \le n-2$ such that $\pi_i\pi_{i+1}\pi_{i+2}$ is a peak and $\pi_i \ne \pi_{i+2}$. In the case of the example $\pi = 1322141251$, there is one symmetric peak 141, and two non-symmetric peaks, namely 132, and 251. Asakly [1] determined the number of symmetric and non-symmetric peaks in all words of length n over the alphabet $[1,k]$. Since then, researchers have investigated these statistics in various combinatorial structures, and over the past decade they have been extended to multiple contexts. In 2021, Elizalde, Flórez, and Ramírez [4] studied symmetric peaks in non-decreasing Dyck paths. In 2022, Mansour, Moreno, and Ramírez [13] examined symmetric and asymmetric peaks in compositions. More recently, Baril, Flórez, and Ramírez [2, 3] have considered Motzkin and Dyck paths with air pockets. For further related results, see also [6, 20]. Beyond symmetric and non-symmetric peaks, other approaches to symmetry have been considered. For instance, Elizalde [5] studied the degree of symmetry of lattice paths.

In this paper, we aim to determine the total number of symmetric and non-symmetric peaks over $P_{n,k}$. Mansour, Shattuck, and Yan [15] found the total number of occurrences of the subword pattern 121 (a symmetric peak) in all the partitions of $[1,n]$ with exactly k blocks. Additionally, they found the number of occurrences of the subword patterns 231 and 321 (non-symmetric peaks) in all the partitions of $[1,n]$ with exactly k blocks. We present two alternative proofs that yield the same results: the first one by using the theory of generating functions and the results that Asakly [1] obtained, and the second one by using combinatorial tools.

2. Counting Symmetric Peaks

2.1. The Ordinary Generating Function for the Number of Partitions with Exactly k Blocks According to the Number of Symmetric Peaks

Let $\mathrm{sp}(\pi)$ denote the number of symmetric peaks in partition π. Let $\mathrm{SP}_k(x,q)$ be the ordinary generating function for the number of partitions of $[1,n]$ with exactly

k blocks according to the number of symmetric peaks, that is

$$\mathrm{SP}_k(x,q) = \sum_{n\geq k} x^n \sum_{\pi\in P_{n,k}} q^{\mathrm{sp}(\pi)}.$$

Theorem 1. *The generating function for the number of partitions of* $[1,n]$ *with exactly* k *blocks according to the number of symmetric peaks is given by*

$$\mathrm{SP}_k(x,q) = x^k(xq+1-x)^{k-1}\prod_{j=1}^{k} W_j(x,q),$$

where

$$W_j(x,q) = \frac{x(q-1)+(1-x(q-1))W_{j-1}(x,q)}{1-x(1-q)(1-(j-1)x)-xW_{j-1}(x,q)(x(j-1)+q(1-x(j-1)))},$$

with initial condition $W_0(x,q)=1$.

Proof. In this context we want to use the canonical sequential form of a partition π of $[1,n]$ with exactly k blocks, where $k\geq 2$. Any partition π of $[1,n]$ with exactly k blocks, can be decomposed as $\pi = \pi' k\omega$, where π' is a partition of $[1,n_1]$ with exactly $k-1$ blocks and ω is a word of length n_2 over the alphabet $[1,k]$, satisfying $n_1+1+n_2=n$. There are three possibilities: ω is empty, ω starts with a letter equal to the last letter in π', or ω starts with a letter different from the last letter of π'. Let $W_k(x,q)$ be the generating function for the number of words ω of length n over the alphabet $[1,k]$, according to the statistic of symmetric peaks. According to Lemma 2 in [1], we have

$$W_k(x,q) = \frac{x(q-1)+(1-x(q-1))W_{k-1}(x,q)}{1-x(1-q)(1-(k-1)x)-xW_{k-1}(x,q)(x(k-1)+q(1-x(k-1)))}.$$

The corresponding generating functions for the aforementioned cases are

$$x\,\mathrm{SP}_{k-1}(x,q),$$

$$x^2q\,\mathrm{SP}_{k-1}(x,q)W_k(x,q),$$

$$x\,\mathrm{SP}_{k-1}(x,q)(W_k(x,q)-xW_k(x,q)-1).$$

This leads to

$$\mathrm{SP}_k(x,q) = x\,\mathrm{SP}_{k-1}(x,q)W_k(x,q)(xq+1-x),$$

with initial conditions $\mathrm{SP}_0(x,q)=1$ and $\mathrm{SP}_1(x,q)=\frac{x}{1-x}$. By induction, we obtain the required result. □

Note that, by substituting $q = 1$ in Theorem 1, we obtain

$$\mathrm{SP}_k(x,1) = x^k \prod_{j=1}^{k} W_j(x,1) = x^k \prod_{j=1}^{k} \frac{1}{1-jx},$$

which is the generating function for the number of partitions of $[1,n]$ with k blocks. We aim to find the total number of symmetric peaks over all partitions of $[1,n]$ with exactly k blocks. We proceed as follows:

- First, differentiate the generating function $\mathrm{SP}_k(x,q)$ with respect to the variable q.
- Second, substitute $q = 1$ in $\frac{\partial}{\partial q}\mathrm{SP}_k(x,q)$.
- Finally, find the coefficients of x^n in $\frac{\partial}{\partial q}\mathrm{SP}_k(x,q)\mid_{q=1}$ to obtain the total number of symmetric peaks over all $P_{n,k}$.

Lemma 1. *The partial derivative* $\frac{\partial}{\partial q}\mathrm{SP}_k(x,q)\mid_{q=1}$ *is given by*

$$\frac{\partial}{\partial q}\mathrm{SP}_k(x,q)\mid_{q=1} = (k-1)x\,\mathrm{SP}_k(x,1) + \mathrm{SP}_k(x,1)\sum_{m=1}^{k}\frac{x^3\binom{m}{2}}{(1-mx)}. \tag{1}$$

Proof. We need to differentiate the generating function $\mathrm{SP}_k(x,q)$ with respect to the variable q and then evaluate the result at $q = 1$:

$$\begin{aligned}\frac{\partial}{\partial q}\mathrm{SP}_k(x,q)\mid_{q=1} &= x^{k+1}(k-1)(xq+1-x)^{k-2}\prod_{j=1}^{k} W_j(x,q)\mid_{q=1}\\ &\quad + x^k(xq+1-x)^{k-1}\left(\sum_{m=1}^{k}\frac{\partial}{\partial q}W_m(x,q)\prod_{j=1,j\neq m}^{k} W_j(x,q)\right)\mid_{q=1}\\ &= x^{k+1}(k-1)\prod_{j=1}^{k} W_j(x,1)\\ &\quad + x^k\left(\sum_{m=1}^{k}\frac{\frac{\partial}{\partial q}W_m(x,q)}{W_m(x,q)}\prod_{j=1}^{k} W_j(x,q)\right)\mid_{q=1}.\end{aligned}$$

This leads to

$$\frac{\partial}{\partial q}\mathrm{SP}_k(x,q)\mid_{q=1} = x^{k+1}(k-1)\prod_{j=1}^{k} W_j(x,1) + x^k\left(\sum_{m=1}^{k}\frac{\frac{\partial}{\partial q}W_m(x,1)}{W_m(x,1)}\prod_{j=1}^{k} W_j(x,1)\right). \tag{2}$$

Using the facts that

$$\mathrm{SP}_k(x,1) = x^k \prod_{j=1}^{k} W_j(x,1) = x^k \prod_{j=1}^{k} \frac{1}{1-jx},$$

and Theorem 3 in [1],

$$\frac{\partial}{\partial q} W_k(x,q) \mid_{q=1} = \frac{x^3\binom{k}{2}}{(1-kx)^2}.$$

Together with Equation (2), we obtain the required result. □

Corollary 1. *The total number of symmetric peaks in all partitions of* $[1,n]$ *with exactly* k *blocks is*

$$(k-1)S_{n-1,k} + \sum_{j=2}^{k} \binom{j}{2} \sum_{i=3}^{n-k} j^{i-3} S_{n-i,k}.$$

Proof. In order to enumerate the total number of partitions of $[1,n]$ with exactly k blocks according to the symmetric peaks, we need to find the coefficients of x^n in Equation (1). Due to the fact that

$$\mathrm{SP}_k(x,1) = \frac{x^k}{\prod_{j=1}^{k}(1-jx)} = \sum_{r\geq k} S_{r,k} x^r$$

we get the required result. □

2.2. Combinatorial Proof

In this subsection, we present a combinatorial proof for Corollary 1. For that, we need the following definitions. Consider any set partition $\pi = \pi_1\pi_2\cdots\pi_n$, represented by its canonical sequence. We say that π contains a *rise (descent)* at i if $\pi_i < \pi_{i+1}$ ($\pi_i > \pi_{i+1}$, *respectively*). For instance, for $\pi = 1121324323 \in P_{10,4}$, we have four rises at $i = 2,4,6,9$ and four descents at $i = 3,5,7,8$. Furthermore, we say that π_i is a *record* if $\pi_i > \pi_j$ for all $j = 1,2,\ldots,i-1$, and i is called the *index* of the record π_i (see [11]).

Proof. Let us divide the proof into two parts. In the first part, our focus is on symmetric peaks $\pi_\ell\pi_{\ell+1}\pi_{\ell+2}$ where $\pi_{\ell+1}$ is a record. In the second part, our attention will turn to a symmetric peak where $\pi_{\ell+1}$ is not a record.

Let $\pi = \pi_1\pi_2\cdots\pi_{n-1}$ be a canonical sequential form for $\Pi \in P_{n-1,k}$. Choose any record $\pi_{\ell+1}$ in $\Pi \in P_{n-1,k}$, where $1 \leq \ell \leq n-1$, and let $a = \pi_\ell$. It is obvious that we have a rise at ℓ. Add $\ell+2$ to block a and increase by one any member in their block that is greater than or equal to $\ell+2$. As a result, we obtain the canonical sequential form $\pi' = \pi_1\pi_2\cdots\pi_\ell\pi_{\ell+1}a\pi_{\ell+2}\cdots\pi_{n-1}$, where $\pi_{\ell+1}a\pi_{\ell+2}$ forms a symmetric peak.

With $k-1$ options for choosing a record, we have a total of $(k-1)S_{n-1,k}$ symmetric peaks.

Considering i and j, where $3 \leq i \leq n-k$ and $2 \leq j \leq k$, the total number of symmetric peaks at ℓ where $\pi_{\ell+1}$ is not a record can be determined by summing the possible partitions in $P_{n,k}$ that can be decomposed as $\pi = \pi'' j\alpha\beta$ for each i and j. Here, π'' is a partition with $j-1$ blocks, α is a word of length i over the alphabet $[1,j]$ whose last three letters form a symmetric peak, and β may be an empty word. There are $\binom{j}{2} j^{i-3} S_{n-i,k}$ members of $P_{n,k}$ with this form. This is because there are j^{i-3} choices for the first $i-3$ letters of α, $\binom{j}{2}$ for the final three letters in α (representing a symmetric peak), and $S_{n-i,k}$ choices for the remaining letters $\pi = \pi'' j\beta$, ensuring that they form a partition of an $(n-i)$-set into k blocks. □

3. Counting Non-Symmetric Peaks

3.1. The Ordinary Generating Function for the Number of Partitions with Exactly k Blocks According to the Number of Non-Symmetric Peaks

Let $\text{nsp}(\pi)$ denote the number of non-symmetric peaks in partition π. Let $\text{NSP}_k(x,q)$ be the ordinary generating function for the number of partitions of $[1,n]$ with exactly k blocks according to the number of non-symmetric peaks. That is,

$$\text{NSP}_k(x,q) = \sum_{n \geq k} x^n \sum_{\pi \in P_{n,k}} q^{\text{nsp}(\pi)}.$$

Theorem 2. *The generating function for the number of partitions of $[1,n]$ with exactly k blocks according to the number of non-symmetric peaks is given by,*

$$\text{NSP}_k(x,q) = x^k \prod_{i=1}^{k} \widetilde{W}_i(x,q) \prod_{j=3}^{k} \left((j-2)xq + 1 - (j-2)x\right),$$

where

$$\widetilde{W}_j(x,q) = \frac{x(q-1) + (1 - x(q-1))\widetilde{W}_{j-1}(x,q)}{1 - x(1-q)(1-2x) - x\widetilde{W}_{j-1}(x,q)(2x + q(1-2x))},$$

with initial condition $\widetilde{W}_0(x,q) = 1$.

Proof. Let π be any partition of $[1,n]$ with exactly k blocks, where $k \geq 3$. It can be decomposed as $\pi = \pi' k\omega$, where π' is a partition of $[1,n_1]$ with exactly $k-1$ blocks, and ω is a word of length n_2 over the alphabet $[1,k]$ satisfying $n_1 + 1 + n_2 = n$. There are three possibilities: ω is empty, ω starts with a letter different from the last letter in π', and ω starts with a letter equal to the last letter of π'. Let $\widetilde{W}_k(x,q)$

be the generating function for the number of words ω of length n over the alphabet $[1,k]$ according to the statistic of non-symmetric peaks. According to a result of Asakly [1], we have

$$\widetilde{W}_k(x,q)=\frac{x(q-1)+(1-x(q-1))\widetilde{W}_{k-1}(x,q)}{1-x(1-q)(1-2x)-x\widetilde{W}_{k-1}(x,q)(2x+q(1-2x))}.$$

Then the corresponding generating functions for the above cases respectively are

$$x\,\mathrm{NSP}_{k-1}(x,q),$$
$$x^2(j-2)q\,\mathrm{NSP}_{k-1}(x,q)\widetilde{W}_k(x,q),$$
$$x\,\mathrm{NSP}_{k-1}(x,q)(\widetilde{W}_k(x,q)-x(j-2)\widetilde{W}_k(x,q)-1).$$

This leads to

$$\mathrm{NSP}_k(x,q)=x\,\mathrm{NSP}_{k-1}(x,q)\widetilde{W}_k(x,q)(xq(j-2)+1-x(j-2)),$$

with initial conditions, $\mathrm{NSP}_0(x,q)=1$, $\mathrm{NSP}_1(x,q)=\frac{x}{1-x}$, and $\mathrm{NSP}_2(x,q)=\frac{x^2}{(1-x)(1-2x)}$. By induction we obtain the required result. □

By substituting $q=1$ in Theorem 2, we obtain $\mathrm{NSP}_k(x,1)=x^k\prod_{j=1}^k W_j(x,1)=x^k\prod_{j=1}^k\frac{1}{1-jx}$, which is the generating function for the number of partitions of $[1,n]$ with k blocks.

Our goal is to find the total number of the non-symmetric peaks over all partitions of $[1,n]$ with exactly k blocks. To achieve this, we repeat the same steps as presented in Lemma 1.

Lemma 2. *The partial derivative* $\frac{\partial}{\partial q}\mathrm{NSP}_k(x,q)\mid_{q=1}$ *is given by*

$$\frac{\partial}{\partial q}\mathrm{NSP}_k(x,q)\mid_{q=1}=\binom{k-1}{2}x\,\mathrm{NSP}_k(x,1)+\mathrm{NSP}_k(x,1)\sum_{m=3}^{k}\frac{2x^3\binom{m}{3}}{(1-mx)}.$$

Proof. By differentiating the generating function $\mathrm{NSP}_k(x,q)$ with respect to the variable q and then evaluating the result at $q=1$ we obtain

$$\begin{aligned}&\frac{\partial}{\partial q}\mathrm{NSP}_k(x,q)\mid_{q=1}\\&=\left(x^k\sum_{m=1}^{k}\frac{\partial}{\partial q}\widetilde{W}_m(x,q)\prod_{i=1,i\neq m}^{k}\widetilde{W}_i(x,q)\prod_{j=3}^{k}((j-2)xq+1-(j-2)x)\right)\mid_{q=1}\\&\quad+\left(x^k\prod_{i=1}^{k}\widetilde{W}_i(x,q)\sum_{m=3}^{k}(m-2)x\prod_{j=3,j\neq m}^{k}((j-2)xq+1-(j-2)x)\right)\mid_{q=1}.\end{aligned}$$

Evaluating at $q = 1$ gives

$$\begin{aligned}
&\frac{\partial}{\partial q}\,\mathrm{NSP}_k(x,q)\mid_{q=1}\\
&= x^k \sum_{m=1}^{k} \frac{\partial}{\partial q}\widetilde{W}_m(x,q)\mid_{q=1} \prod_{i=1, i\neq m}^{k} \widetilde{W}_i(x,1) + x^k \prod_{i=1}^{k} \widetilde{W}_i(x,1) \sum_{m=3}^{k} (m-2)x\\
&= x^k \sum_{m=1}^{k} \frac{\frac{\partial}{\partial q}\widetilde{W}_m(x,q)\mid_{q=1}}{\widetilde{W}_m(x,1)} \prod_{i=1}^{k} \widetilde{W}_i(x,1) + x\frac{(k-1)(k-2)}{2} x^k \prod_{i=1}^{k} \widetilde{W}_i(x,1).
\end{aligned}$$

According to Theorem 5 in [1], we have

$$\frac{\partial}{\partial q}\widetilde{W}_k(x,q)\mid_{q=1} = \frac{2x^3\binom{k}{3}}{(1-kx)^2},$$

and the equalities

$$\widetilde{W}_k(x,q)\mid_{q=1} = \frac{1}{1-kx} \quad \text{and} \quad \mathrm{NSP}_k(x,1) = x^k \prod_{i=1}^{k} \widetilde{W}_i(x,1),$$

lead to the required result. □

Corollary 2. *The total number of non-symmetric peaks in all partitions of* $[1, n]$ *with exactly* k *blocks is*

$$\binom{k-1}{2} S_{n-1,k} + 2\sum_{j=3}^{k} \binom{j}{3} \sum_{i=3}^{n-k} j^{i-3} S_{n-i,k}.$$

Proof. By finding the coefficients of x in $\frac{\partial}{\partial q}\,\mathrm{NSP}_k(x,q)\mid_{q=1}$ we get the result. □

3.2. Combinatorial Proof

In this subsection, we present a combinatorial proof for Corollary 2.

Combinatorial Proof of Corollary 2. Let us focus on non-symmetric peaks $\pi_\ell\pi_{\ell+1}\pi_{\ell+2}$ where $\pi_{\ell+1}$ is a record. Consider $\pi = \pi_1\pi_2\cdots\pi_{n-1}$ as a canonical sequential form for $\Pi \in P_{n-1,k}$. Choose any record $\pi_{\ell+1}$ in $\Pi \in P_{n-1,k}$, where $1 \le \ell \le n-1$. It is obvious that we have a rise at ℓ. Add $\ell+2$ to any block a, where $a < \pi_{\ell+1}$ and $a \neq \pi_\ell$, by increasing any member in their block that is greater than or equal to $\ell+2$ by one. As a result, we obtain the canonical sequential form $\pi' = \pi_1\pi_2\cdots\pi_\ell\pi_{\ell+1}a\pi_{\ell+2}\cdots\pi_{n-1}$, where $\pi_{\ell+1}a\pi_{\ell+2}$ forms a non-symmetric peak. For each chosen record $\pi_{\ell+1}$, there are $\pi_{\ell+1}-2$ possible choices for the block a. The sum of these choices, considering all possible records from 3 to k (excluding 1 and 2 since they do not form non-symmetric peaks), gives $\frac{(k-1)(k-2)}{2}$ possibilities. This leads to $\binom{k-1}{2}S_{n-1,k}$ non-symmetric peaks.

Let us focus on non-symmetric peaks $\pi_\ell\pi_{\ell+1}\pi_{\ell+2}$, where $\pi_{\ell+1}$ is not a record. Considering i and j, where $3 \leq i \leq n-k$ and $2 \leq j \leq k$, the total number of non-symmetric peaks at ℓ where $\pi_{\ell+1}$ is not a record can be determined by summing the possible partitions in $P_{n,k}$ that can be decomposed as $\pi = \pi'' j\alpha\beta$ for each i and j. Here, π'' is a partition with $j-1$ blocks, α is a word of length i over the alphabet $[1, j]$ whose last three letters forming a non-symmetric peak, and β may be an empty word. There are $2\binom{j}{3}j^{i-3}S_{n-i,k}$ members of $P_{n,k}$ with this form. This is because there are j^{i-3} choices for the first $i-3$ letters of α, $2\binom{j}{3}$ for the final three letters in α (representing a non-symmetric peak), and $S_{n-i,k}$ choices for the remaining letters $\pi = \pi'' j\beta$, ensuring that they form a partition of an $(n-i)$-set into k blocks. □

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