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DIRICHLET CONVOLUTIONS AND DIVISOR PROBLEMS OVER B-FREE POLYNOMIALS OVER FUNCTION FIELDS

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Abstract

We extend the concept of B-free numbers, originally due to Erdős, to the function field case $\mathbb{F}_q[T]$, where q is a prime power. We consider distributions of the divisor function d_h for $h \geq 1$ over B-free polynomials, where each member in the sequence B is a power of a monic irreducible polynomial, extending some results of Camargo for averages over arithmetic progressions of the divisor function d, and we also explore the distribution in intervals.

1. Introduction

The idea of B-free numbers was introduced by Erdős [11] as follows. Let $b_1 < b_2 < \cdots$ be a sequence of positive integers satisfying

$$\sum_{\ell=1}^{\infty} \frac{1}{b_\ell} < \infty, \quad \text{and} \quad (b_\ell, b_m) = 1, \quad \ell \neq m.$$

A positive integer n is called *B-free* if it is not divisible by any b_ℓ. In the case when the b_ℓ's are the squares of the prime numbers, the B-free numbers are then the square-free numbers. Erdős [11] proved that for some $c < 1$ and all large enough N, the interval $[N, N + N^c]$ contains B-free numbers and conjectured that for any $\varepsilon > 0$ there exists $N_{B,\varepsilon}$ such that for any $N \geq N_{B,\varepsilon}$, the interval $[N, N + N^\varepsilon]$ contains at least one B-free number. Szemerédi [26] proved Erdős' result for $c > \frac{1}{2}$. Further improvements were obtained in [7, 28, 29], with the best result due to

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Sargos and Wu [23] for $c > \frac{40}{97}$. Plaksin [21] proved that Erdős' conjecture is true except for a very small proportion of intervals. In the case of square-free numbers, much more can be proven. For example, Granville [13] proved Erdős' conjecture conditionally on the ABC-conjecture, while Filaseta and Trifonov [12] proved unconditionally the square-free case with $c > \frac{1}{5}$. In [2], Alkan and Zaharescu considered the problem of finding B-free numbers in short arithmetic progressions and formulated a generalization of Erdős' conjecture to this setting. A survey of these results can be found in [4].

B-free numbers have implications to non-vanishing Fourier coefficients of cusp forms, as first observed by Balog and Ono [6]. In fact, one can control the non-vanishing by applying B-free numbers distribution results over short intervals to certain strategic choices of B. This was further explored in several works including [1, 3, 17, 19].

In recent work, Camargo [9] developed results on weighted sums of Dirichlet convolutions of completely multiplicative functions and used them to make estimates for the average of the divisor function over B-free integers over arithmetic progressions, extending previous work over square-free integers [8, 14]. More precisely, Camargo considered, for a sequence of positive integers $\{\kappa_\ell\}_{j\geq 1}$ such that

$$\kappa_{\min} := \min_{j\geq 1}\{\kappa_\ell\} \geq 2,$$

the B_κ-free integers given by

$$b_\ell = p_\ell^{\kappa_\ell},$$

where $p_1, p_2, \ldots$ is the sequence of prime numbers given in ascending order. Define

$$\widetilde{D}_{B_\kappa}(x,a,m) := \sum_{\substack{n\leq x \\ n\equiv a \pmod m}} \mathbb{1}_{B_\kappa}(n) d(n),$$

where $\mathbb{1}_{B_\kappa}$ is the indicator function for the B_κ-integers, d is the number of positive divisors function, and a, m are integers such that $0 \leq a < m$ and $(a,m) = 1$. It is proven in [9, Corollary 3] that there is a certain constant $\widetilde{c}_{\kappa,a,m}$ such that

$$\widetilde{D}_{B_\kappa}(x,a,m) = \frac{\varphi(m)}{m^2} \prod_{\substack{\ell=1 \\ p_\ell\nmid m}}^{\infty} \left[1 - \frac{\kappa_\ell + 1}{\kappa_\ell} + \frac{\kappa_\ell}{p_\ell^{\kappa_\ell+1}}\right] x\log x + \widetilde{c}_{\kappa,a,m}x + O\left(x^{\eta_\kappa+\varepsilon}\right), \quad (1)$$

where φ is Euler's totient function and

$$\eta_\kappa = \begin{cases} 1/2, & \text{if } \kappa_{\min} = 2, \\ 1/3, & \text{if } \kappa_{\min} \geq 3. \end{cases}$$

A natural question is whether Camargo's results extend to function fields, that is, to elements of $\mathbb{F}_q[T]$. The function field setting is particularly appealing in many problems, in part because one can often prove unconditional results there, since the Riemann Hypothesis holds in this context. This setting also allows one to consider the limit $q \to \infty$, which is typically more accessible and often leads to conjectures in the number field setting.

The distribution of the divisor function and its generalization d_h, which counts the number of ways of writing a positive integer as a product of h positive integer factors, has been extensively studied, as it arises in various contexts related to moments of the zeta function and L-functions. The distribution of its function field analogue has also been considered and has been related to integrals from random matrix theory. See, for example, [5, 15, 18].

In this article, we are interested in exploring the analogue of B-free elements over $\mathbb{F}_q[T]$, where q is a prime power, with the goal of recovering the results of [9]. The study of B-free objects provides a natural way to impose multiplicative constraints, generalizing classical notions such as square-free or k-free integers. In the number field setting, such conditions have proved fruitful in problems related to modular forms and correlations of multiplicative functions. The function field setting offers a natural framework to investigate analogous phenomena, often allowing for sharper and unconditional results. For a non-zero $f \in \mathbb{F}_q[T]$, its norm is given by $|f| = q^{\deg(f)}$ and we let $|0| = 0$. Let $\mathcal{M}$ denote the set of monic elements of $\mathbb{F}_q[T]$. Let $B = \{b_\ell\}_\ell$ be a sequence of elements in $\mathcal{M}$ such that

$$\sum_{\ell} \frac{1}{|b_\ell|} < \infty, \quad \text{and} \quad (b_\ell, b_m) = 1, \quad \ell \neq m.$$

An $f \in \mathcal{M}$ is said to be B-free if f is not divisible by any of the b_ℓ's.

Let $\mathcal{P}$ be the set of monic irreducible elements of $\mathbb{F}_q[T]$. We consider the sequences B indexed by elements of $\mathcal{P}$ such that $b_P = P^{\kappa(P)}$, where $\kappa(P)$ is a sequence of non-zero integers. We refer to such sequence as B_κ. Thus, B_κ is determined by the choice of $\{\kappa(P)\}_{P\in\mathcal{P}}$. A polynomial $f \in \mathcal{M}$ is B_κ-free if f is not divisible by any of the b_P. For instance, the k-free polynomials are those that are B_k-free, where $b_P = P^k$ for all $P \in \mathcal{P}$. Let

$$\kappa_{\min} := \min_{P\in\mathcal{P}} \kappa(P).$$

Given B_κ and $h \in \mathbb{Z}_{\geq 1}$, we define

$$\tilde{D}_{B_\kappa,h}(N) = \sum_{f\in\mathcal{M}_{\leq N}} \mathbb{1}_{B_\kappa}(f) d_h(f),$$

where $\mathbb{1}_{B_\kappa}$ is the indicator function for the B_κ-free polynomials and d_h is the h-divisor function given by

$$d_h(f) = \sum_{\substack{g_1,\ldots,g_h\in\mathcal{M}\\ g_1\cdots g_h=f}} 1.$$

We remark that the cases of $h=1$ and $h=2$, involve the standard counting function and the classical divisor function d respectively.

We prove the following result.

Theorem 1. *Let $\{\kappa(P)\}_P$ be a sequence of integers greater than or equal to 2. Then, as $N\to\infty$, we have*

$$\tilde{D}_{B_\kappa,1}(N) = \frac{q^N}{1-\frac{1}{q}} \prod_{P\in\mathcal{P}} \left(1-\frac{1}{|P^{\kappa(P)}|}\right) + O\left(q^{(\frac{1}{\kappa_{\min}}+\varepsilon)N}\right)$$

and

$$\begin{aligned}\tilde{D}_{B_\kappa,2}(N) = &\frac{q^N}{1-\frac{1}{q}} \prod_{P\in\mathcal{P}} \left(1-\frac{\kappa(P)+1}{|P^{\kappa(P)}|}+\frac{\kappa(P)}{|P^{\kappa(P)+1}|}\right)\\ &\times\left[N+1-\frac{1}{q-1}+\sum_{P\in\mathcal{P}}\frac{\kappa(P)(\kappa(P)+1)\deg(P)(|P|-1)}{|P^{\kappa(P)+1}|-|P|(\kappa(P)+1)+\kappa(P)}\right]\\ &+O\left(q^{(\frac{1}{\kappa_{\min}}+\varepsilon)N}\right).\end{aligned}$$

We remark that the above result gives a smaller error term than (1) for $\kappa_{\min}>3$.

In the general case, we give an asymptotic result.

Theorem 2. *Let $\{\kappa(P)\}_P$ be a sequence of integers greater than or equal to 2 and $h\in\mathbb{Z}_{\geq 1}$. Then, as $N\to\infty$, we have*

$$\begin{aligned}\tilde{D}_{B_\kappa,h}(N) = &\frac{q^N N^{h-1}}{1-\frac{1}{q}} \prod_{P\in\mathcal{P}} \left[1-\frac{1}{|P^{\kappa(P)}|}\left(\binom{h+\kappa(P)-1}{\kappa(P)}\right.\right.\\ &\left.\left.+\frac{1}{|P|}\left[1-\binom{h+\kappa(P)-1}{\kappa(P)}\right]\right)\right]\left(1+O\left(\frac{1}{N}\right)\right).\end{aligned}$$

If the B-free condition is not imposed and we consider the full sum, the expected error term for $\sum_{n\leq x} d_h(n)$ in the number field case is $O\left(x^{\frac{h-1}{2h}}\right)$ (see [27, Chapter 12]). Meanwhile, in the function field setting one has the exact formula

$$\sum_{f\in\mathcal{M}_{\leq N}} d_h(f) = q^N\binom{N+h}{h}$$

(see [5]).

To shed some light on the B-free case, it is natural to consider the square-free setting. In this case, the generating function for d_h can be written as

$$\prod_{P\in\mathcal{P}}\left(1+h\,u^{\deg(P)}\right) = \mathcal{Z}(u)^h\mathcal{G}_{1,h}(u),$$

where $\mathcal{Z}(u) = \frac{1}{1-qu}$ and $\mathcal{G}_{1,h}(u)$ is analytic for $|u| < q^{-1/2}$. Applying Perron's formula as in Theorem 1 leads to an error term of size $O(q^{(\frac{1}{2}+\varepsilon)N})$.

This estimate can be refined by extracting additional factors from the generating function. For instance, one can factor further in terms of $\mathcal{Z}(u^m)$, which yields secondary main terms together with improved error terms, such as $O(q^{(\frac{1}{4}+\varepsilon)N})$ after extracting the first few contributions. More generally, this procedure can be iterated, leading to increasingly precise lower-order terms and an error of size $O(q^{(\frac{1}{m+1}+\varepsilon)N})$.

This behavior highlights the contrast with the number field setting and reflects the simpler analytic structure of the zeta function in the function field case. The general B-free case is more involved. Our results represent a first step toward understanding how the statistics of the divisor function interact with this type of multiplicative constraints in the function field setting.

At present, we are not aware of corresponding Ω-results in the general B_κ-free setting over function fields.

We consider the case of arithmetic progressions. Let $m, a \in \mathbb{F}_q[T]$ and let N be a non-negative integer. Given B_κ, we define

$$\tilde{D}_{B_\kappa,h}(N,a,m) = \sum_{\substack{f\in\mathcal{M}_{\leq N}\\ f\equiv a \pmod{m}}} \mathbb{1}_{B_\kappa}(f) d_h(f),$$

that is, the average of the h-divisor function on B_κ-free elements in the corresponding arithmetic progression.

We prove the following statement.

Theorem 3. *Let $\{\kappa(P)\}_P$ be a sequence of integers greater than or equal to 2. Let $m, a \in \mathbb{F}_q[T]$ such that $(a,m) = 1$ and let N be a non-negative integer. Then, as $N \to \infty$, we have*

$$\tilde{D}_{B_\kappa,1}(N,a,m) = \frac{q^N}{(1-\frac{1}{q})|m|} \prod_{\substack{P\in\mathcal{P}\\ P\nmid m}} \left(1 - \frac{1}{|P^{\kappa(P)}|}\right) + O\left(q^{(\frac{1}{\kappa_{\min}}+\varepsilon)N}\right)$$

and

$$\begin{aligned}\tilde{D}_{B_\kappa,2}(N,a,m) =& \frac{q^N\Phi(m)}{(1-\frac{1}{q})|m|^2} \prod_{\substack{P\in\mathcal{P}\\ P\nmid m}} \left(1 - \frac{\kappa(P)+1}{|P^{\kappa(P)}|} + \frac{\kappa(P)}{|P^{\kappa(P)+1}|}\right) \\ &\times \left[N + 1 - \frac{1}{q-1} + 2\sum_{\substack{P\in\mathcal{P}\\ P\mid m}} \frac{\deg P}{|P|-1}\right. \\ &\left.+ \sum_{\substack{P\in\mathcal{P}\\ P\nmid m}} \frac{\kappa(P)(\kappa(P)+1)\deg(P)\,(|P|-1)}{|P^{\kappa(P)+1}| - |P|(\kappa(P)+1) + \kappa(P)}\right] + O\left(q^{(\frac{1}{\kappa_{\min}}+\varepsilon)N}\right),\end{aligned}$$

where Φ *denotes Euler's totient function, given by* $\Phi(f) = |f| \prod_{P|f} \left(1 - \frac{1}{|P|}\right)$.

In the general case we have the following result.

Theorem 4. *Let* $\{\kappa(P)\}_P$ *be a sequence of integers greater than or equal to* 2 *and* $h \in \mathbb{Z}_{\geq 1}$. *Let* $m, a \in \mathbb{F}_q[T]$ *such that* $(a, m) = 1$ *and let* N *be a non-negative integer. Then we have*

$$\begin{aligned}\tilde{D}_{B_\kappa,h}(N, a, m) =& \frac{q^N N^{h-1}}{(h-1)!} \frac{\Phi(m)^{h-1}}{|m|^h} \left(1 - \frac{1}{q}\right)^{-1} \\ &\times \prod_{\substack{P \in \mathcal{P} \\ P \nmid m}} \left[1 - \frac{1}{|P^{\kappa(P)}|} \left(\binom{h + \kappa(P) - 1}{\kappa(P)} \right.\right. \\ &\left.\left. + \frac{1}{|P|} \left[1 - \binom{h + \kappa(P) - 1}{\kappa(P)}\right] \right) \right] \left(1 + O\left(\frac{1}{N}\right)\right).\end{aligned}$$

We also consider the case of intervals. Let $A \in \mathbb{F}_q[T]$ and $0 \leq \Delta \leq \deg(A) - 2$. Define the interval centered at A of width Δ:

$$I(A; \Delta) := \{f \in \mathbb{F}_q[T] \,:\, |f - A| \leq q^\Delta\}.$$

We are interested in studying

$$\tilde{N}_{B_\kappa,h}(A, \Delta) = \sum_{\substack{f \in \mathbb{F}_q[T] \\ f \in I(A;\Delta)}} \mathbb{1}_{B_\kappa}(f) d_h(f),$$

that is, the average of the h-divisor function on B_κ-free elements of $I(A; \Delta)$.

For $f \in \mathbb{F}_q[T]$, $f \neq 0$, define

$$f^*(T) := T^{\deg(f)} f\left(\frac{1}{T}\right).$$

Thus, if

$$f(T) = f_n T^n + \cdots + f_0$$

then

$$f^*(T) = f_0 T^n + \cdots + f_n.$$

A sequence $\{\kappa(P)\}_P$ will be called symmetric if $\kappa(P) = \kappa(P^*)$.

Theorem 5. *Let* $\{\kappa(P)\}_P$ *be a symmetric sequence of integers greater than or equal to* 2. *Let* $A \in \mathcal{M}_N$ *and* $cN \leq \Delta \leq N - 2$ *such that*

$$c > \frac{h + \frac{2}{\kappa_{\min}}}{h + 2}.$$

Then, as $N \to \infty$, we have

$$\tilde{N}_{B_\kappa,h}(A,\Delta) = \frac{1}{q^{N-\Delta-1}}\left(\tilde{D}_{B_\kappa,h}(N) - \tilde{D}_{B_\kappa,h}(N-1)\right) + O\left(q^{\frac{h(N-\Delta)}{2}+(\frac{1}{\kappa_{\min}}+\varepsilon)N}\right).$$

In particular, for $h=1$, we have

$$\tilde{N}_{B_\kappa,1}(A,\Delta) = q^{\Delta+1}\prod_{P\in\mathcal{P}}\left(1-\frac{1}{|P^{\kappa(P)}|}\right) + O\left(q^{\frac{N-\Delta}{2}+(\frac{1}{\kappa_{\min}}+\varepsilon)N}\right).$$

For $h=2$,

$$\begin{aligned}\tilde{N}_{B_\kappa,2}(A,\Delta) =& q^{\Delta+1}\prod_{P\in\mathcal{P}}\left(1-\frac{\kappa(P)+1}{|P^{\kappa(P)}|}+\frac{\kappa(P)}{|P^{\kappa(P)+1}|}\right)\\ &\times\left[N+1+\sum_{P\in\mathcal{P}}\frac{\kappa(P)\deg(P)(\kappa(P)+1)(|P|-1)}{|P^{\kappa(P)+1}|-|P|(\kappa(P)+1)+\kappa(P)}\right]\\ &+O\left(q^{N-\Delta+(\frac{1}{\kappa_{\min}}+\varepsilon)N}\right).\end{aligned}$$

Both for arithmetic progressions and short intervals over function fields, there are results of Sawin for factorization functions [24, 25], with error terms of size $O_{q,h,N}\left(q^{(1-\delta)(N-\deg(m))}\right)$ (for some $\delta>0$) and $O_{q,h,N}\left(q^{\Delta/2}\right)$, respectively.

The B-free condition can be expressed as a factorization function when $\kappa(P)$ depends only on $\deg(P)$. Thus, these results provide particular instances of the B-free setting and suggest that stronger bounds should hold in certain cases. In particular, in the setting of Theorem 5, one expects the error term to depend only polynomially on N.

For Theorems 1, 2, 3, and 4, our methods of proof involve following the methods of Camargo [9] for weighted sums of Dirichlet convolutions and computations of generating functions and Perron's formula. The strong estimates for the error terms follow from the Riemann Hypothesis, which is true in this setting. The proof of Theorem 5 relies on more delicate estimates of Keating and Rudnick [16] to express sums of the h-divisor function in short intervals as sum over arithmetic progressions.

This article is organized as follows. Section 2 introduces some necessary background on function fields. Section 3 contains an adaptation of the results of [9] on weighted sums of Dirichlet convolutions of multiplicative functions, while Section 4 gathers other auxiliary results. The proofs of the main results are in Section 5.

2. Some Background on Function Fields

In this section, we recall a few results for function fields which will be very useful. Recall that $\mathcal{M}$ denotes the set of monic polynomials in $\mathbb{F}_q[T]$. We denote by $\mathcal{M}_n$

(respectively $\mathcal{M}_{\leq n}$) the set of elements in $\mathcal{M}$ of degree n (resp. degree less than or equal to n).

The zeta function for $\mathbb{F}_q[T]$ is given by

$$\zeta_q(s) = \sum_{f \in \mathcal{M}} \frac{1}{|f|^s} = \prod_{P \in \mathcal{P}} \left(1 - \frac{1}{|P|^s}\right)^{-1} = \frac{1}{1 - q^{1-s}}.$$

While the initial sum and Euler product converge for $\mathrm{Re}(s) > 1$, the right-hand side identity provides a meromorphic continuation with a single pole at $s = 1$. By applying the change of variables $u = q^{-s}$, we can also write

$$\mathcal{Z}_q(u) = \sum_{f \in \mathcal{M}} u^{\deg(f)} = \prod_{P \in \mathcal{P}} \left(1 - u^{\deg(P)}\right)^{-1} = \frac{1}{1 - qu}.$$

For a polynomial $m(T) \in \mathbb{F}_q[T]$ of positive degree, the order of $(\mathbb{F}_q[T]/(m))^\times$ is given by the Euler's totient function $\Phi(m)$. A Dirichlet character modulo m is a homomorphism

$$\chi : (\mathbb{F}_q[T]/(m))^\times \to \mathbb{C}^\times,$$

which is extended to $\chi(f) = 0$ for $f \in \mathbb{F}_q[T]$ such that $(f, m) \neq 1$. The orthogonality relations give, for $a(T) \in \mathbb{F}_q[T]$,

$$\frac{1}{\Phi(m)} \sum_{\chi \pmod{m}} \overline{\chi}(a)\chi(f) = \begin{cases} 1 & f \equiv a \pmod{m}, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

(See [22, Proposition 4.2].)

Given a Dirichlet character χ, the corresponding Dirichlet L-series is given by

$$L_q(s, \chi) = \sum_{f \in \mathcal{M}} \frac{\chi(f)}{|f|^s}$$

for $\mathrm{Re}(s) > 1$. As in the zeta function case, we can make the change of variables $u = q^{-s}$ and write

$$\mathcal{L}_q(u, \chi) = \sum_{f \in \mathcal{M}} \chi(f) u^{\deg(f)} = \prod_{P \in \mathcal{P}} \left(1 - \chi(P) u^{\deg(P)}\right)^{-1}.$$

A Dirichlet character χ is called *even* if $\chi(c) = 1$ for any $c \in \mathbb{F}^\times$ and is called *odd* otherwise. By orthogonality, when χ is a nontrivial character, $\mathcal{L}_q(u, \chi)$ is a polynomial of degree $\Delta \leq \deg(m) - 1$ ([22, Proposition 4.3]). We may consider the reciprocals of the roots, i.e.,

$$\mathcal{L}_q(u, \chi) = \prod_{j=1}^{\Delta} (1 - \alpha_j u).$$

Then, for χ odd, the Riemann hypothesis implies that $|\alpha_j| = \sqrt{q}$. If χ is even, one root equals 1 and the others satisfy the Riemann hypothesis.

We recall Perron's formula over $\mathbb{F}_q[T]$. (See for example [20, 4.4.15] for the classical statement, and [10, Lemma 2.2] for the function field version.)

Lemma 1 (Perron's Formula)**.** *If the generating series* $\mathcal{A}(u) = \sum_{f\in\mathcal{M}} a(f)u^{\deg f}$ *is absolutely convergent in* $|u| \leq \rho < 1$*, then*

$$\sum_{f\in\mathcal{M}_{\leq n}} a(f) = \frac{1}{2\pi i}\int_{|u|=\rho} \frac{\mathcal{A}(u)}{u^n(1-u)}\frac{du}{u},$$

where the integral is taken over the circle oriented counterclockwise.

3. The Function Field Version of Camargo's Results

In this section, we state function field versions of the theory developed in Section 2 of [9] for multiplicative and completely multiplicative functions.

Let $\alpha, \beta : \mathbb{F}_q[T] \to \mathbb{C}$ be two multiplicative functions. We define the *Dirichlet convolution* of α and β by

$$(\alpha * \beta)(f) = \sum_{\substack{g\in\mathcal{M}\\ g|f}} \alpha(g)\beta\left(\frac{f}{g}\right) = \sum_{\substack{g\in\mathcal{M}, h\in\mathbb{F}_q[T]\\ gh=f}} \alpha(g)\beta(h).$$

In our applications, $f \in \mathcal{M}$, so that the last sum will be symmetric.

Recall that $\mu : \mathbb{F}_q[T] \to \mathbb{C}$ is the Möbius function given by $\mu(f) = (-1)^r$ if f is square-free and the product of r primes, and 0 otherwise.

Lemma 2. *Let* $\alpha, \beta : \mathbb{F}_q[T] \to \mathbb{C}$ *be completely multiplicative. For monic polynomials* f *and* g*, we have:*

$$(\alpha * \beta)(fg) = \sum_{\substack{\ell\in\mathcal{M}\\ \ell|(f,g)}} \mu(\ell)\alpha(\ell)\beta(\ell)(\alpha * \beta)\left(\frac{f}{\ell}\right)(\alpha * \beta)\left(\frac{g}{\ell}\right).$$

Proof. The proof follows directly from the fact that convolution is multiplicative (see also the argument from [9, Lemma 2]). □

For $\alpha, \beta : \mathbb{F}_q[T] \to \mathbb{C}$, $f \in \mathbb{F}_q[T]$ and $n \in \mathbb{Z}_{\geq 0}$, we define :

$$S_f[\alpha, \beta](n) := \sum_{g\in\mathcal{M}_{\leq n-\deg(f)}} (\alpha * \beta)(fg). \tag{3}$$

and

$$D[\alpha, \beta](n) := \sum_{g\in\mathcal{M}_{\leq n}} (\alpha * \beta)(g).$$

Lemma 3. *Let $\alpha, \beta : \mathbb{F}_q[T] \to \mathbb{C}$ be completely multiplicative. For $n \in \mathbb{Z}_{\geq 0}$, we have :*

$$S_f[\alpha,\beta](n) = \sum_{\substack{d\in\mathcal{M}\\ d|f}} \alpha(d)\beta\left(\frac{f}{d}\right) \sum_{\substack{\ell\in\mathcal{M}\\ \ell|d}} \mu(\ell)\beta(\ell) D[\alpha,\beta](n-\deg(f\ell)).$$

Proof. The argument follows the proof of [9, Lemma 3]. We apply Lemma 2 to (3) to obtain

$$\begin{aligned} S_f[\alpha,\beta](n) &= \sum_{g\in\mathcal{M}_{\leq n-\deg(f)}} \sum_{\substack{\ell\in\mathcal{M}\\ \ell|(f,g)}} \mu(\ell)\alpha(\ell)\beta(\ell)(\alpha*\beta)\left(\frac{f}{\ell}\right)(\alpha*\beta)\left(\frac{g}{\ell}\right) \\ &= \sum_{g\in\mathcal{M}_{\leq n-\deg(f)}} \sum_{\substack{\ell\in\mathcal{M}\\ \ell|(f,g)}} \mu(\ell)\alpha(\ell)\beta(\ell) \sum_{\substack{d\in\mathcal{M}\\ d|\frac{f}{\ell}}} \alpha(d)\beta\left(\frac{f}{d\ell}\right)(\alpha*\beta)\left(\frac{g}{\ell}\right). \end{aligned}$$

Making the change of variable $s = d\ell$ and interchanging the sums, we get

$$S_f[\alpha,\beta](n) = \sum_{\substack{s\in\mathcal{M}\\ s|f}} \alpha(s)\beta\left(\frac{f}{s}\right) \sum_{g\in\mathcal{M}_{\leq n-\deg(f)}} \sum_{\substack{\ell\in\mathcal{M}\\ \ell|(s,g)}} \mu(\ell)\beta(\ell)(\alpha*\beta)\left(\frac{g}{\ell}\right).$$

Setting $h = \dfrac{g}{\ell}$ and interchanging the last two sums, we obtain

$$S_f[\alpha,\beta](n) = \sum_{\substack{s\in\mathcal{M}\\ s|f}} \alpha(s)\beta\left(\frac{f}{s}\right) \sum_{\substack{\ell\in\mathcal{M}\\ \ell|s}} \mu(\ell)\beta(\ell) \sum_{\substack{h\in\mathcal{M}\\ \deg(h\ell)\leq n-\deg(f)}} (\alpha*\beta)(h),$$

which is the desired result, since the condition $\deg(h\ell) \leq n-\deg(f)$ can be rewritten as $\deg(h) \leq n - \deg(f\ell)$. □

Let $\alpha, \beta, \gamma : \mathbb{F}_q[T] \to \mathbb{C}$ be arithmetical functions defined over function fields. We define

$$D[\gamma,\alpha,\beta](n) := \sum_{f\in\mathcal{M}_{\leq n}} \gamma(f)(\alpha*\beta)(f).$$

Lemma 4. *Let $\alpha, \beta, \gamma : \mathbb{F}_q[T] \to \mathbb{C}$ be arithmetical functions defined over function fields. For $n \in \mathbb{Z}_{\geq 0}$,*

$$D[\gamma,\alpha,\beta](n) = \sum_{f\in\mathcal{M}_{\leq n}} \lambda(f) S_f[\alpha,\beta](n),$$

where $\lambda = \gamma * \mu$.

Proof. The argument follows the proof of [9, Lemma 4]. We have that

$$\begin{aligned} D[\gamma,\alpha,\beta](n) &= \sum_{g\in\mathcal{M}_{\le n}} \gamma(g)(\alpha*\beta)(g) = \sum_{g\in\mathcal{M}_{\le n}} (\lambda*1)(g)(\alpha*\beta)(g) \\ &= \sum_{g\in\mathcal{M}_{\le n}} \sum_{\substack{f\in\mathcal{M}\\ f\mid g}} \lambda(f)(\alpha*\beta)(g) = \sum_{g\in\mathcal{M}_{\le n}} \sum_{f\in\mathcal{M}_{\le n}} \lambda(f)(\alpha*\beta)(g)\delta_{f|g}, \end{aligned}$$

where $\delta_{f|g} = 1$ if $f \mid g$ and 0 otherwise. Making the change of variables $g = f\ell$, the last equality becomes

$$D[\gamma,\alpha,\beta](n) = \sum_{f\in\mathcal{M}_{\le n}} \lambda(f) \sum_{\ell\in\mathcal{M}_{\le n-\deg(f)}} (\alpha*\beta)(f\ell),$$

and the result follows. □

The next result corresponds to [9, Theorem 1].

Theorem 6. *Let $\alpha,\beta : \mathbb{F}_q[T] \to \mathbb{C}$ be completely multiplicative. For any arithmetical function γ and $n \in \mathbb{Z}_{\ge 0}$, we have*

$$D[\gamma,\alpha,\beta](n) = \sum_{\deg(f)\le n} (\gamma*\mu)(f) \sum_{d|f} \alpha(d)\beta\left(\frac{f}{d}\right) \sum_{\ell|d} \mu(\ell)\beta(\ell) D[\alpha,\beta](n-\deg(f\ell)).$$

Proof. The proof follows immediately from Lemmas 3 and 4. □

Let $\kappa = \{\kappa(P)\}_P$ be a sequence of non-zero integers and $f \in \mathcal{M}$. Let $f = \prod_{\mathcal{P}} P^{\nu_P(f)}$ be the decomposition of f into monic irreducible polynomials (here $\nu_P(f) = 0$ for almost all P, and $\prod_{\mathcal{P}}$ denotes the product with P going over all the elements of $\mathcal{P}$). We define

$$f^\kappa := \prod_{\mathcal{P}} P^{\nu_P(f)\kappa(P)}.$$

In addition, we say that $f^{\frac{1}{\kappa}} \in \mathcal{M}$ if $\kappa(P) \mid \nu_P(f)$ for all $P \in \mathcal{P}$ and in this case, we set

$$f^{\frac{1}{\kappa}} := \prod_{\mathcal{P}} P^{\frac{\nu_P(f)}{\kappa(P)}}.$$

The following statement corresponds to [9, Lemma 5].

Lemma 5. *Let $f \in \mathcal{M}$ and let $\{\kappa(P)\}_P$ be a sequence of non-zero integers. We have that*

$$(\mathbb{1}_{B_\kappa} * \mu)(f) = \begin{cases} \mu(f^{\frac{1}{\kappa}}) & \text{if } f^{\frac{1}{\kappa}} \in \mathcal{M}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $f = \prod_{\mathcal{P}} P^{\nu_P(f)}$ be the decomposition of f into prime factors. Write $f = gh$ with $g, h \in \mathcal{M}$. Note that, if $\kappa(P) < \nu_P(f)$ for some P, then either $P^{\kappa(P)} \mid g$ or $P^2 \mid h$. In this case, we get

$$(\mathbb{1}_{B_\kappa} * \mu)(f) = \sum_{\substack{g,h\in\mathcal{M}\\ gh=f}} \mathbb{1}_{B_\kappa}(g)\mu(h) = 0.$$

Thus, assume that $\kappa(P) \geq \nu_P(f)$ for all P. If $f = gh$ with $\mathbb{1}_{B_\kappa}(g) \neq 0$ and $\mu(h) \neq 0$, then the power of the primes in the decomposition of g should be all less than $\kappa(P)$, and h should be square-free.

We decompose the set $\{P \in \mathcal{P} : \nu_P(f) > 0\} = A \sqcup B$, where $A = \{P \in \mathcal{P} : \nu_P(f) = \kappa(P)\}$ and $B = \{P \in \mathcal{P} : 0 < \nu_P(f) < \kappa(P)\}$. Then we see that g and h should have the form

$$g = g' \prod_{P\in A} P^{\nu_P(f)-1}, \qquad h = h' \prod_{P\in A} P, \quad \text{where} \quad g'h' = \prod_{P\in B} P^{\nu_P(f)} =: f'.$$

Hence,

$$\begin{aligned}(\mathbb{1}_{B_\kappa} * \mu)(f) = \sum_{\substack{g,h\in\mathcal{M}\\ gh=f}} \mathbb{1}_{B_\kappa}(g)\mu(h) = \sum_{\left(\prod_{P\in A} P\right)\left|h\right|f} \mu(h) = \mu\left(\prod_{P\in A} P\right) \sum_{h'|f'} \mu(h') \\ =\mu\left(\prod_{P\in A} P\right)\delta_{f'=1}.\end{aligned}$$

This shows that $(\mathbb{1}_{B_\kappa} * \mu)(f) \neq 0$ precisely when $B = \varnothing$ and $f = \prod_{P\in A} P^{\kappa(P)}$ and in this case $(\mathbb{1}_{B_\kappa} * \mu)(f) = \mu\left(\prod_{P\in A} P\right) = \mu(f^{\frac{1}{\kappa}})$. To conclude with the proof, note that $f^{\frac{1}{\kappa}} \in \mathcal{M}$ precisely when $f = \prod_{P\in A} P^{\kappa(P)v(P)}$, where $v(P) \in \mathbb{Z}_{\geq 0}$. But if $v(P) > 1$ for some P, then $(\mathbb{1}_{B_\kappa} * \mu)(f) = 0$, and in this case, $\mu(f^{\frac{1}{\kappa}})$ is also vanishing and the result still holds. □

4. Some Auxiliary Results

In this section, we will prove some auxiliary statements that are needed for our main results. We fix a sequence $\kappa : \mathcal{P} \to \mathbb{Z}_{\geq 2}$ and we let $\kappa_{\min}$ be the minimum value of κ.

We start by proving the following elementary result.

Lemma 6. *For $f \in \mathcal{M}$, we have that*

$$\sum_{\substack{g \in \mathcal{M} \\ g \mid f}} \frac{\mu(g)}{|g|} = \frac{\Phi(f)}{|f|}.$$

Remark 1. We will often write $\sum_g$ to indicate a sum over $g \in \mathcal{M}$ and $\sum_P$ or $\prod_P$ to indicate a sum or product over $P \in \mathcal{P}$.

Proof of Lemma 6. Indeed, both sides of the equality are multiplicative functions, and therefore it suffices to prove the identity for powers of primes. Let $f = P^k$. We have

$$\sum_{g \mid P^k} \frac{\mu(g)}{|g|} = 1 - \frac{1}{|P|} = \frac{\Phi(P^k)}{|P^k|}.$$

Thus, the statement is true by multiplicativity. □

Lemma 7. *For $f \in \mathcal{M}$, we have that*

$$\sum_{\substack{g \in \mathcal{M} \\ g \mid f}} \frac{\mu(g)\deg(g)}{|g|} = \frac{\Phi(f)}{|f|} \sum_{\substack{P \in \mathcal{P} \\ P \mid f}} \frac{\deg(P)}{1 - |P|}.$$

Proof. We consider the generating function

$$\mathcal{B}(u) = \sum_{g \mid f} \frac{\mu(g) u^{\deg(g)}}{|g|} = \prod_{P \mid f} \left(1 - \frac{u^{\deg(P)}}{|P|}\right).$$

Taking the logarithmic derivative, we have

$$\frac{u\mathcal{B}'(u)}{\mathcal{B}(u)} = \sum_{P \mid f} \frac{\deg(P) u^{\deg(P)}}{u^{\deg(P)} - |P|}. \tag{4}$$

Notice that

$$\mathcal{B}'(1) = \sum_{g \mid f} \frac{\mu(g)\deg(g)}{|g|}.$$

Combining this with (4) and Lemma 6, we conclude that

$$\sum_{g \mid f} \frac{\mu(g)\deg(g)}{|g|} = \mathcal{B}(1) \sum_{P \mid f} \frac{\deg(P)}{1 - |P|} = \frac{\Phi(f)}{|f|} \sum_{P \mid f} \frac{\deg(P)}{1 - |P|}.$$

□

The following result will be applied to the general divisor function d_h.

Proposition 1. *Let $m \in \mathcal{M}$ be fixed, $h \in \mathbb{Z}_{\geq 1}$, and $\varepsilon > 0$. Then, as $N \to \infty$, we have*

$$\sum_{\substack{r\in\mathcal{M}\\ \deg(r^\kappa)\le N\\ (r,m)=1}} \frac{\mu(r)}{|r^\kappa|} \sum_{\substack{f\in\mathcal{M}\\ f|r^\kappa}} d_{h-1}(f) \sum_{\substack{g\in\mathcal{M}\\ g|f}} \frac{\mu(g)}{|g|} = \prod_{\substack{P\in\mathcal{P}\\ P\nmid m}} \left[1 - \frac{1}{|P^{\kappa(P)}|}\left(\binom{h+\kappa(P)-1}{\kappa(P)} + \frac{1}{|P|}\left[1-\binom{h+\kappa(P)-1}{\kappa(P)}\right]\right)\right] + O\left(q^{(\frac{1}{\kappa_{\min}}+\varepsilon-1)(N+1)}\right).$$

Proof. We consider the generating function and apply Lemma 6 to obtain

$$\begin{aligned}
\mathcal{A}_{h,m}(u) &= \sum_{\substack{r\in\mathcal{M}\\ (r,m)=1}} \frac{\mu(r)}{|r^\kappa|} \sum_{\substack{f\in\mathcal{M}\\ f|r^\kappa}} d_{h-1}(f) \sum_{\substack{g\in\mathcal{M}\\ g|f}} \frac{\mu(g)}{|g|} u^{\deg(r^\kappa)} \\
&= \sum_{(r,m)=1} \frac{\mu(r)}{|r^\kappa|} \sum_{f|r^\kappa} d_{h-1}(f) \frac{\Phi(f)}{|f|} u^{\deg(r^\kappa)} \\
&= \prod_{\substack{P\in\mathcal{P}\\ P\nmid m}} \left(1 - \frac{u^{\kappa(P)\deg(P)}}{|P^{\kappa(P)}|}\left(1 + \left(1-\frac{1}{|P|}\right)\sum_{j=1}^{\kappa(P)} d_{h-1}(P^j)\right)\right) \\
&= \prod_{\substack{P\in\mathcal{P}\\ P\nmid m}} \left(1 - \frac{u^{\kappa(P)\deg(P)}}{|P^{\kappa(P)}|}\left(1 + \left(1-\frac{1}{|P|}\right)\left(\binom{h+\kappa(P)-1}{\kappa(P)} - 1\right)\right)\right),
\end{aligned}$$

where we have applied the fact that

$$\sum_{j=0}^{\kappa(P)} \binom{j+h-2}{j} = \binom{h+\kappa(P)-1}{\kappa(P)}.$$

We notice that $\mathcal{A}_{h,m}(u)$ converges absolutely for $|u| < q^{1-\frac{1}{\kappa_{\min}}}$. To see this, we compare with the sum

$$\sum_P \left(\frac{u}{q}\right)^{\kappa(P)\deg(P)} \binom{h+\kappa(P)-1}{\kappa(P)}.$$

Indeed, if we take $|u| = q^{1-\frac{1}{\kappa_{\min}}-\varepsilon}$ with $\varepsilon > 0$ fixed, then we have

$$\begin{aligned}
\left|\sum_P \left(\frac{u}{q}\right)^{\kappa(P)\deg(P)} \binom{h+\kappa(P)-1}{\kappa(P)}\right| &\le \sum_P \frac{q^{-\frac{\varepsilon}{2}\kappa(P)\deg(P)}}{q^{(\frac{1}{\kappa_{\min}}+\frac{\varepsilon}{2})\kappa(P)\deg(P)}} \binom{h+\kappa(P)-1}{\kappa(P)} \\
&\le \sum_P \frac{q^{-\frac{\varepsilon}{2}\kappa(P)}}{q^{(1+\frac{\varepsilon}{2})\deg(P)}} \binom{h+\kappa(P)-1}{\kappa(P)}.
\end{aligned}$$

Let

$$f(x) := q^{-\frac{\varepsilon}{2}x}\binom{x+h-1}{h-1}.$$

We have that f is a continuous function such that $f(0)=1$, $f(x)\geq 0$ for x real positive, and $\lim_{x\to\infty} f(x)=0$. Thus, $f(x)$ is bounded by some constant $C_{q,\varepsilon}$. Therefore, denoting by π_n the number of monic irreducible polynomials of degree n,

$$\begin{aligned}\left|\sum_P \left(\frac{u}{q}\right)^{\kappa(P)\deg(P)}\binom{h+\kappa(P)-1}{\kappa(P)}\right| &\leq C_{q,\varepsilon}\sum_P \frac{1}{q^{(1+\frac{\varepsilon}{2})\deg(P)}}\\ &\ll \sum_{n=1}^{\infty}\frac{\pi_n}{q^{(1+\frac{\varepsilon}{2})n}}\\ &\ll \sum_{n=1}^{\infty}\frac{q^n}{nq^{(1+\frac{\varepsilon}{2})n}}\\ &= O_\varepsilon(1),\end{aligned}$$

where we have used the estimate $\pi_n = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right)$ ([22, Theorem 2.2]). By applying Lemma 1, we have

$$\sum_{\substack{r\in\mathcal{M}\\ \deg(r^\kappa)\leq N\\ (r,m)=1}}\frac{\mu(r)}{|r^\kappa|}\sum_{\substack{f\in\mathcal{M}\\ f|r^\kappa}} d_{h-1}(f)\sum_{\substack{g\in\mathcal{M}\\ g|f}}\frac{\mu(g)}{|g|} = \frac{1}{2\pi i}\int_{|u|=\rho}\frac{\mathcal{A}_{h,m}(u)}{u^N(1-u)}\frac{du}{u},$$

where ρ is a small positive number. Shifting the integral to a circle of radius $q^{1-\frac{1}{\kappa_{\min}}-\varepsilon}$, we encounter a pole at $u=1$ and we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{|u|=\rho}\frac{\mathcal{A}_{h,m}(u)}{u^N(1-u)}\frac{du}{u} &= -\operatorname{Res}_{u=1}\frac{\mathcal{A}_{h,m}(u)}{u^N(1-u)}\\ &\quad + \frac{1}{2\pi i}\int_{|u|=q^{1-\frac{1}{\kappa_{\min}}-\varepsilon}}\frac{\mathcal{A}_{h,m}(u)}{u^N(1-u)}\frac{du}{u}.\end{aligned}$$

Notice that

$$\operatorname{Res}_{u=1}\frac{\mathcal{A}_{h,m}(u)}{u^N(1-u)} = -\mathcal{A}_{h,m}(1)$$

and

$$\frac{1}{2\pi i}\int_{|u|=q^{1-\frac{1}{\kappa_{\min}}-\varepsilon}}\frac{\mathcal{A}_{h,m}(u)}{u^N(1-u)}\frac{du}{u} \ll q^{(\frac{1}{\kappa_{\min}}+\varepsilon-1)(N+1)}.$$

Combining the above, we get the result. □

Proposition 2. *Let $m \in \mathcal{M}$ be fixed and let $\varepsilon > 0$. Then, as $N \to \infty$, we have*

$$\sum_{\substack{r\in\mathcal{M}\\ \deg(r^\kappa)\le N\\ (r,m)=1}} \frac{\mu(r)}{|r^\kappa|} \sum_{\substack{f\in\mathcal{M}\\ f|r^\kappa}} \sum_{\substack{g\in\mathcal{M}\\ g|f}} \frac{\mu(g)\deg(g)}{|g|} = \prod_{\substack{P\in\mathcal{P}\\ P\nmid m}} \left(1 - \frac{\kappa(P)+1}{|P^{\kappa(P)}|} + \frac{\kappa(P)}{|P^{\kappa(P)+1}|}\right)$$
$$\times \sum_{\substack{P\in\mathcal{P}\\ P\nmid m}} \frac{\kappa(P)\deg(P)}{|P^{\kappa(P)+1}| - |P|(\kappa(P)+1) + \kappa(P)}$$
$$+ O\left(q^{(\frac{1}{\kappa_{\min}}+\varepsilon-1)(N+1)}\right).$$

Proof. Given $f \in \mathbb{F}_q[T]$, we define the function $\gamma : \mathbb{F}_q[T] \to \mathbb{Q}$ by

$$\gamma(f) := \sum_{P|f} \frac{\deg(P)}{1-|P|}.$$

Introducing a variable v, we remark that the function $\Gamma_v(f) := v^{\gamma(f)}$ when $\gamma(f) \neq 0$ and $\Gamma(f) = 1$ when $\gamma(f) = 0$, is a multiplicative function when we evaluate v conveniently. We will take v such that $|v-1| < \frac{1}{2}$ so that the principal branch of the logarithm allows us to define $v^{\gamma(f)}$ without ambiguity.

Recall from Lemma 7 that

$$\sum_{\substack{\deg(r^\kappa)\le N\\ (r,m)=1}} \frac{\mu(r)}{|r^\kappa|} \sum_{f|r^\kappa} \sum_{g|f} \frac{\mu(g)\deg(g)}{|g|} = \sum_{\substack{\deg(r^\kappa)\le N\\ (r,m)=1}} \frac{\mu(r)}{|r^\kappa|} \sum_{f|r^\kappa} \frac{\Phi(f)}{|f|}\gamma(f). \tag{5}$$

We now consider the generating function and apply Lemma 6 to obtain

$$\begin{aligned}\mathcal{A}_m(u,v) &= \sum_{\substack{r\in\mathcal{M}\\ (r,m)=1}} \frac{\mu(r)}{|r^\kappa|} \sum_{f|r^\kappa} \sum_{g|f} \frac{\mu(g)}{|g|} v^{\gamma(f)} u^{\deg(r^\kappa)}\\ &= \sum_{\substack{r\in\mathcal{M}\\ (r,m)=1}} \frac{\mu(r)}{|r^\kappa|} \sum_{f|r^\kappa} \frac{\Phi(f)}{|f|} v^{\gamma(f)} u^{\deg(r^\kappa)}\\ &= \prod_{P\nmid m} \left(1 - \frac{u^{\kappa(P)\deg(P)}}{|P^{\kappa(P)}|}\left(1+\kappa(P)\left(1-\frac{1}{|P|}\right)v^{\gamma(P)}\right)\right).\end{aligned}$$

As in the proof of Proposition 1, we can see that $\mathcal{A}_m(u,v)$ converges absolutely in the region $|u| < q^{1-\frac{1}{\kappa_{\min}}}$ and $|v-1| < \frac{1}{2}$. Indeed, letting $d = \deg(P)$, we have that

$$\left|\left(1-\frac{1}{|P|}\right)v^{\gamma(P)}\right| = \left(1-\frac{1}{q^d}\right)\left|\frac{1}{v}\right|^{\frac{d}{q^d-1}} \le 2^{\frac{d}{q^d-1}} \le 2.$$

Thus, it suffices to compare with the sum

$$\sum_{P}\left(\frac{u}{q}\right)^{\kappa(P)\deg(P)}(2\kappa(P)+1)\ll\sum_{P}\left(\frac{u}{q}\right)^{\kappa(P)\deg(P)}(\kappa(P)+1),$$

and we can proceed as in the proof of Proposition 1.

The logarithmic derivative of $\mathcal{A}_m(u,v)$ with respect to v at $v=1$ is given by

$$\frac{\frac{\partial\mathcal{A}_m(u,v)}{\partial v}\Big|_{v=1}}{\mathcal{A}_m(u,1)}=\sum_{P\nmid m}\frac{\frac{\kappa(P)\deg(P)u^{\kappa(P)\deg(P)}}{\left|P^{\kappa(P)+1}\right|}}{1-\frac{u^{\kappa(P)\deg(P)}}{\left|P^{\kappa(P)}\right|}\left(1+\kappa(P)\left(1-\frac{1}{|P|}\right)\right)},$$

so that

$$\mathcal{B}_m(u):=\frac{\partial\mathcal{A}_m(u,v)}{\partial v}\Big|_{v=1}=\prod_{P\nmid m}\left(1-\frac{u^{\kappa(P)\deg(P)}}{\left|P^{\kappa(P)}\right|}\left(1+\kappa(P)\left(1-\frac{1}{|P|}\right)\right)\right)$$
$$\times\sum_{P\nmid m}\frac{\frac{\kappa(P)\deg(P)u^{\kappa(P)\deg(P)}}{\left|P^{\kappa(P)+1}\right|}}{1-\frac{u^{\kappa(P)\deg(P)}}{\left|P^{\kappa(P)}\right|}\left(1+\kappa(P)\left(1-\frac{1}{|P|}\right)\right)}$$

is the generating function

$$\sum_{(r,m)=1}\frac{\mu(r)}{|r^\kappa|}\sum_{f|r^\kappa}\sum_{g|f}\frac{\mu(g)}{|g|}\gamma(f)u^{\deg(r^\kappa)}=\sum_{(r,m)=1}\frac{\mu(r)}{|r^\kappa|}\sum_{f|r^\kappa}\frac{\Phi(f)}{|f|}\gamma(f)u^{\deg(r^\kappa)}$$

corresponding to the sum (5) we wish to evaluate. By applying Lemma 1, we have

$$\sum_{\substack{\deg(r^\kappa)\leq N\\(r,m)=1}}\frac{\mu(r)}{|r^\kappa|}\sum_{f|r^\kappa}\sum_{g|f}\frac{\mu(g)\deg(g)}{|g|}=\frac{1}{2\pi i}\int_{|u|=\rho}\frac{\mathcal{B}_m(u)}{u^N(1-u)}\frac{du}{u},$$

where ρ is a small positive number. Shifting the integral to a circle of radius $q^{1-\frac{1}{\kappa_{\min}}-\varepsilon}$, we encounter a pole at $u=1$ and we get

$$\frac{1}{2\pi i}\int_{|u|=\rho}\frac{\mathcal{B}_m(u)}{u^N(1-u)}\frac{du}{u}=\mathcal{B}_m(1)+\frac{1}{2\pi i}\int_{|u|=q^{1-\frac{1}{\kappa_{\min}}-\varepsilon}}\frac{\mathcal{B}_m(u)}{u^N(1-u)}\frac{du}{u}.$$

We remark that

$$\frac{1}{2\pi i}\int_{|u|=q^{1-\frac{1}{\kappa_{\min}}-\varepsilon}}\frac{\mathcal{B}_m(u)}{u^N(1-u)}\frac{du}{u}\ll q^{(\frac{1}{\kappa_{\min}}+\varepsilon-1)(N+1)}.$$

By combining the above statements, we obtain the desired result. □

Proposition 3. *Let $m \in \mathcal{M}$ be fixed and let $\varepsilon > 0$. Then, as $N \to \infty$, we have*

$$\sum_{\substack{r\in\mathcal{M}\\ \deg(r^\kappa)\le N\\ (r,m)=1}} \frac{\mu(r)\deg(r^\kappa)}{|r^\kappa|}\sum_{\substack{f\in\mathcal{M}\\ f|r^\kappa}}\sum_{\substack{g\in\mathcal{M}\\ g|f}}\frac{\mu(g)}{|g|} = -\prod_{\substack{P\in\mathcal{P}\\ P\nmid m}}\left(1-\frac{\kappa(P)+1}{|P^{\kappa(P)}|}+\frac{\kappa(P)}{|P^{\kappa(P)+1}|}\right)$$
$$\times\sum_{\substack{P\in\mathcal{P}\\ P\nmid m}}\frac{\kappa(P)\deg(P)\left(|P|(\kappa(P)+1)-\kappa(P)\right)}{|P^{\kappa(P)+1}|-|P|(\kappa(P)+1)+\kappa(P)}$$
$$+O\left(q^{(\frac{1}{\kappa_{\min}}+\varepsilon-1)(N+1)}\right).$$

Proof. We consider the generating function and apply Lemma 6 to obtain

$$\begin{aligned}\mathcal{A}_m(u) &= \sum_{(r,m)=1}\frac{\mu(r)}{|r^\kappa|}\sum_{f|r^\kappa}\sum_{g|f}\frac{\mu(g)}{|g|}u^{\deg(r^\kappa)}\\ &= \prod_{\substack{P\in\mathcal{P}\\ P\nmid m}}\left(1-\frac{u^{\kappa(P)\deg(P)}}{|P^{\kappa(P)}|}\left(1+\kappa(P)\left(1-\frac{1}{|P|}\right)\right)\right),\end{aligned}$$

as in the proof of Proposition 1. We compute

$$u\mathcal{A}'_m(u) = \sum_{(r,m)=1}\frac{\mu(r)\deg(r^\kappa)}{|r^\kappa|}\sum_{f|r^\kappa}\sum_{g|f}\frac{\mu(g)}{|g|}u^{\deg(r^\kappa)}.$$

Thus, $u\mathcal{A}'_m(u)$ yields the generating function for our problem. Applying Lemma 1, we have that

$$\sum_{\substack{\deg(r^\kappa)\le N\\ (r,m)=1}}\frac{\mu(r)\deg(r^\kappa)}{|r^\kappa|}\sum_{f|r^\kappa}\sum_{g|f}\frac{\mu(g)}{|g|} = \frac{1}{2\pi i}\int_{|u|=\rho}\frac{u\mathcal{A}'_m(u)}{u^N(1-u)}\frac{du}{u},$$

where ρ is a small positive number. Shifting the integral to a circle of radius $q^{1-\frac{1}{\kappa_{\min}}-\varepsilon}$, we encounter a pole at $u=1$ and we get

$$\frac{1}{2\pi i}\int_{|u|=\rho}\frac{u\mathcal{A}'_m(u)}{u^N(1-u)}\frac{du}{u} = \mathcal{A}'_m(1)+\frac{1}{2\pi i}\int_{|u|=q^{1-\frac{1}{\kappa_{\min}}-\varepsilon}}\frac{u\mathcal{A}'_m(u)}{u^N(1-u)}\frac{du}{u}$$

Notice that,

$$\frac{1}{2\pi i}\int_{|u|=q^{\kappa_{\min}-1-\varepsilon}}\frac{u\mathcal{A}'_m(u)}{u^N(1-u)}\frac{du}{u} \ll q^{(\frac{1}{\kappa_{\min}}+\varepsilon-1)(N+1)}.$$

To compute $\mathcal{A}'_m(1)$ we find the logarithmic derivative of $\mathcal{A}_m(u)$:

$$\frac{u\mathcal{A}'(u)}{\mathcal{A}(u)} = -\sum_{P\nmid m}\frac{\kappa(P)\deg(P)\left(1+\kappa(P)\left(1-\frac{1}{|P|}\right)\right)\frac{u^{\kappa(P)\deg(P)}}{|P^{\kappa(P)}|}}{1-\frac{u^{\kappa(P)\deg(P)}}{|P^{\kappa(P)}|}\left(1+\kappa(P)\left(1-\frac{1}{|P|}\right)\right)}.$$

Therefore, we obtain

$$\mathcal{A}'(1) = -\prod_{\substack{P\in\mathcal{P}\\ P\nmid m}}\left(1-\frac{1}{|P^{\kappa(P)}|}\left(1+\kappa(P)\left(1-\frac{1}{|P|}\right)\right)\right)$$
$$\times\sum_{P\nmid m}\frac{\kappa(P)\deg(P)\left(1+\kappa(P)\left(1-\frac{1}{|P|}\right)\right)\frac{1}{|P^{\kappa(P)}|}}{1-\frac{1}{|P^{\kappa(P)}|}\left(1+\kappa(P)\left(1-\frac{1}{|P|}\right)\right)}.$$

Combining the above identities, we get the result. □

We will also need the following elementary bounds.

Lemma 8. *Let $r\in\mathcal{M}$ and N be a positive integer. Then*

$$\left|\sum_{\substack{r\in\mathcal{M}\\ \deg(r^{\kappa})\le N}}\mu(r)\sum_{f\mid r^{\kappa}}\sum_{\ell\mid f}\mu(\ell)\right|\le q^{\frac{N}{\kappa_{\min}}} \tag{6}$$

and

$$\sum_{\substack{r\in\mathcal{M}\\ \deg(r^{\kappa})\le N}}\sum_{f\mid r^{\kappa}}\sum_{\ell\mid f}1\ll q^{(\frac{1}{\kappa_{\min}}+\varepsilon)N}. \tag{7}$$

Proof. The innermost sum in (6) is nonzero only when $f=1$. Therefore, we have that

$$\sum_{\deg(r^{\kappa})\le N}\mu(r)\sum_{f\mid r^{\kappa}}\sum_{\ell\mid f}\mu(\ell)=\sum_{\deg(r^{\kappa})\le N}\mu(r).$$

Now we bound this trivially to get

$$\left|\sum_{\deg(r^{\kappa})\le N}\mu(r)\right|\le\sum_{\deg(r^{\kappa})\le N}1\le\sum_{\deg(r^{\kappa_{\min}})\le N}1=\sum_{\deg(r)\le\frac{N}{\kappa_{\min}}}1\le q^{\frac{N}{\kappa_{\min}}}.$$

For (7), we apply the bound $d(f)\ll q^{\varepsilon\deg(f)}$ to get

$$\sum_{\deg(r^{\kappa})\le N}\sum_{f\mid r^{\kappa}}\sum_{\ell\mid f}1=\sum_{\deg(r^{\kappa})\le N}\sum_{f\mid r^{\kappa}}d(f)\ll\sum_{\deg(r^{\kappa})\le N}\sum_{f\mid r^{\kappa}}q^{\varepsilon\deg(f)}$$
$$\ll q^{\varepsilon N}\sum_{\deg(r^{\kappa})\le N}d(r^{\kappa})\ll q^{2\varepsilon N}\sum_{\deg(r^{\kappa})\le N}1\ll q^{(\frac{1}{\kappa_{\min}}+\varepsilon)N},$$

which proves the desired statement. □

Let χ_1 and χ_2 be two characters and $f \in \mathcal{M}$. Recall that the Dirichlet convolution of χ_1 and χ_2 is given by

$$(\chi_1 * \chi_2)(f) = \sum_{g|f} \chi_1(g)\chi_2\left(\frac{f}{g}\right) = \sum_{gh=f} \chi_1(g)\chi_2(h).$$

We will denote by $\chi^{(h)}$ the h-fold Dirichlet convolution of χ with itself, namely

$$\chi^{(h)}(f) = \sum_{\substack{g_1,\ldots,g_h \in \mathcal{M} \\ g_1 \cdots g_h = f}} \chi(g_1) \cdots \chi(g_h) = \sum_{\substack{g_1,\ldots,g_h \in \mathcal{M} \\ g_1 \cdots g_h = f}} \chi(f) = \chi(f) d_h(f). \tag{8}$$

We have the following estimate for the h-fold convolution of the trivial character.

Proposition 4. *Let $m \in \mathcal{M}$ be fixed such that $m \neq 1$ and let χ_0 denote the principal character modulo m. Let $h \in \mathbb{Z}_{\geq 1}$. Then we have*

$$\sum_{f \in \mathcal{M}_{\leq N}} \chi_0^{(h)}(f) = \frac{N^{h-1} q^N}{(h-1)!} \left(\frac{\Phi(m)}{|m|}\right)^h \left(1 - \frac{1}{q}\right)^{-1} \left(1 + O\left(\frac{1}{N}\right)\right). \tag{9}$$

In the particular case that $m = 1$, then χ_0 is the identity function, and we have

$$\sum_{f \in \mathcal{M}_{\leq N}} 1^{(h)}(f) = N^{h-1} q^N \left(1 - \frac{1}{q}\right)^{-1} \left(1 + O\left(\frac{1}{N}\right)\right). \tag{10}$$

For $h = 1$, we have

$$\sum_{f \in \mathcal{M}_{\leq N}} \chi_0(f) = \frac{q^{N+1}}{q-1} \frac{\Phi(m)}{|m|}$$

and, for $m = 1$,

$$\sum_{f \in \mathcal{M}_{\leq N}} 1 = \frac{q^{N+1} - 1}{q - 1}.$$

For $h = 2$, we have

$$\begin{aligned}\sum_{f \in \mathcal{M}_{\leq N}} (\chi_0 * \chi_0)(f) = q^N \left(\frac{\Phi(m)}{|m|}\right)^2 \Bigg[& \left(1 - \frac{1}{q}\right)^{-1} \left(N + 1 + 2 \sum_{P|m} \frac{\deg P}{|P| - 1}\right) \\ & - \frac{1}{q}\left(1 - \frac{1}{q}\right)^{-2} \Bigg],\end{aligned}$$

and, for $m = 1$,

$$\sum_{f \in \mathcal{M}_{\leq N}} (1 * 1)(f) = \frac{(N+1) q^{N+2} - (N+2) q^{N+1} + 1}{(q-1)^2}.$$

Proof. We treat the general case of Equations (9) and (10) only, as the cases $h = 1, 2$ are similar. We have that

$$\sum_{f \in \mathcal{M}_{\leq N}} \chi_0^{(h)}(f) = \sum_{\substack{f \in \mathcal{M}_{\leq N} \\ (f,m)=1}} d_h(f).$$

The generating function is given by

$$\mathcal{A}_h(u) = \prod_{P \nmid m} (1 - u^{\deg(P)})^{-h} = \prod_{P \mid m} (1 - u^{\deg(P)})^h \mathcal{Z}_q(u)^h.$$

For the case $m = 1$ we have

$$\begin{aligned} \sum_{f \in \mathcal{M}_{\leq N}} d_h(f) =& \sum_{n=0}^{N} \binom{h+n-1}{h-1} q^n = \frac{1}{(h-1)!} \frac{d^{h-1}}{dq^{h-1}} \left(\frac{q^{N+h}-1}{q-1} \right) \\ =& \frac{N^{h-1}}{q-1} q^{N+1} \left(1 + O\left(\frac{1}{N} \right) \right), \end{aligned} \tag{11}$$

where we have used the identity

$$\frac{d^n}{dx^n} \left(\frac{x^{N+1}-1}{x-1} \right) = n! \sum_{j=0}^{n} \frac{(-1)^j \binom{N+1}{n-j} x^{N-n+j+1}}{(x-1)^{j+1}} + (-1)^{n+1} \frac{n!}{(x-1)^{n+1}}.$$

This concludes the proof of (10).

Now let $m \neq 1$. We have

$$\begin{aligned} \sum_{\substack{f \in \mathcal{M}_{\leq N} \\ (f,m)=1}} d_h(f) =& \frac{1}{2\pi i} \int_{|u|=\rho} \frac{\prod_{P \mid m} (1 - u^{\deg P})^h \mathcal{Z}_q(u)^h}{u^N (1-u)} \frac{du}{u} \\ =& \frac{1}{2\pi i} \int_{|u|=\rho} \frac{\prod_{P \mid m} (1 - u^{\deg P})^h}{(1-u)(1-qu)^h} \frac{du}{u^{N+1}}, \end{aligned}$$

where ρ is small and positive. We shift the integral to the radius $|u| = R > 1$ and encounter the poles at $u = 1$ and $u = \frac{1}{q}$. This gives

$$\begin{aligned} \sum_{\substack{f \in \mathcal{M}_{\leq N} \\ (f,m)=1}} d_h(f) =& -\operatorname{Res}_{u=1} \frac{\prod_{P \mid m} (1 - u^{\deg P})^h}{(1-u)(1-qu)^h u^{N+1}} - \operatorname{Res}_{u=\frac{1}{q}} \frac{\prod_{P \mid m} (1 - u^{\deg P})^h}{(1-u)(1-qu)^h u^{N+1}} \\ & + \frac{1}{2\pi i} \int_{|u|=R} \frac{\prod_{P \mid m} (1 - u^{\deg P})^h}{(1-u)(1-qu)^h} \frac{du}{u^{N+1}}. \end{aligned} \tag{12}$$

We remark that the first term above is zero when $m \neq 1$, since the numerator vanishes at $u = 1$.

For the second term in (12) we have

$$\begin{aligned}
\operatorname{Res}_{u=\frac{1}{q}} \frac{\prod_{P|m}(1-u^{\deg P})^h}{(1-u)(1-qu)^h u^{N+1}} &= \frac{(-1)^h}{(h-1)!q^h} \frac{\partial^{h-1}}{\partial u^{h-1}} \left(\frac{\prod_{P|m}(1-u^{\deg P})^h}{(1-u)u^{N+1}} \right)\Bigg|_{u=\frac{1}{q}} \\
&= \frac{(-1)^h}{(h-1)!q^h} (-1)^{h-1}(N+h-1)\cdots(N+1) \\
&\quad \times \left(\frac{\prod_{P|m}(1-u^{\deg P})^h}{(1-u)u^{N+h}} \right)\Bigg|_{u=\frac{1}{q}} \left(1 + O\left(\frac{1}{N}\right)\right) \\
&= -\frac{N^{h-1}q^N}{(h-1)!} \frac{\prod_{P|m}\left(1-\frac{1}{|P|}\right)^h}{1-\frac{1}{q}} \left(1 + O\left(\frac{1}{N}\right)\right) \\
&= -\frac{N^{h-1}q^N}{(h-1)!} \left(\frac{\Phi(m)}{|m|}\right)^h \frac{1}{1-\frac{1}{q}} \left(1 + O\left(\frac{1}{N}\right)\right).
\end{aligned} \tag{13}$$

Finally, letting $R \to \infty$, we can bound the third term in (12) by

$$\frac{1}{2\pi i} \int_{|u|=R} \frac{\prod_{P|m}(1-u^{\deg P})^h}{(1-u)(1-qu)^h} \frac{du}{u^{N+1}} \ll R^{-N-h-1} \to 0.$$

The final result then follows from (13). □

We have the following bound for the h-fold convolution of non-trivial characters.

Proposition 5. *Let χ be a non-principal Dirichlet character of conductor $m \in \mathcal{M}$ and $h \in \mathbb{Z}_{\geq 1}$. Then*

$$\left| \sum_{f \in \mathcal{M}_{\leq N}} \chi^{(h)}(f) \right| \leq (\sqrt{q}+1)^{h(\deg(m)-1)} = O\left(q^{\frac{h\deg(m)}{2}}\right).$$

Proof. Considering the generating series, we have

$$\begin{aligned}
\sum_{f \in \mathcal{M}} \chi^{(h)}(f) u^{\deg(f)} &= \sum_{f \in \mathcal{M}} \sum_{g_1 \cdots g_h = f} \chi(g_1) \cdots \chi(g_h) u^{\deg(f)} \\
&= \left(\sum_{g_1 \in \mathcal{M}} \chi(g_1) u^{\deg(g_1)} \right) \cdots \left(\sum_{g_h \in \mathcal{M}} \chi(g_h) u^{\deg(g_h)} \right) = \mathcal{L}(u,\chi)^h,
\end{aligned}$$

where $\mathcal{L}(u,\chi)$ denotes the Dirichlet L-function associated to χ, which is a polynomial of degree at most $\deg(m)-1$ for χ non-principal ([22, Proposition 4.3]). Let $M=\deg(m)-1$ and

$$\mathcal{L}(u,\chi)=\sum_{k=0}^{M} a_k u^k.$$

By the Riemann Hypothesis, we have

$$|a_k| \le \binom{M}{k} q^{\frac{k}{2}}.$$

Using this, we have,

$$\begin{aligned}\left|\sum_{\deg(f)\le N} \chi^{(h)}(f)\right| \le& \sum_{k=0}^{N} \sum_{\substack{j_1,\dots,j_h\\ j_1+\cdots+j_h=k}} |a_{j_1}\cdots a_{j_h}| \\ \le& \sum_{k=0}^{N} q^{\frac{k}{2}} \sum_{\substack{j_1,\cdots,j_h\\ j_1+\cdots+j_h=k}} \binom{M}{j_1}\cdots\binom{M}{j_h} = \sum_{k=0}^{N} q^{\frac{k}{2}}\binom{hM}{k} \\ \le& (\sqrt{q}+1)^{hM}.\end{aligned}$$

□

5. Proofs of the Main Results

5.1. Proof of Theorems 1 and 2

We notice that $\tilde{D}_{B_\kappa,h}(N)=D[\mathbb{1}_{B_\kappa},d_{h-1},1](N)$. Thus, by Theorem 6,

$$\tilde{D}_{B_\kappa,h}(N)=\sum_{f\in\mathcal{M}_{\le N}} (\mathbb{1}_{B_\kappa}*\mu)(f)\sum_{g|f} d_{h-1}(g)\sum_{\ell|g}\mu(\ell)D[d_{h-1},1](N-\deg(f\ell)).$$

By Lemma 5, $(\mathbb{1}_{B_\kappa}*\mu)(f)$ is non-vanishing only when f takes the form $f=r^\kappa$ and in this case $(\mathbb{1}_{B_\kappa}*\mu)(f)=\mu(r)$. Hence,

$$\tilde{D}_{B_\kappa,h}(N)=\sum_{\deg(r^\kappa)\le N}\mu(r)\sum_{g|r^\kappa} d_{h-1}(g)\sum_{\ell|g}\mu(\ell)\sum_{\deg(s)\le N-\deg(r^\kappa\ell)} d_h(s), \tag{14}$$

where we have used that $d_h=d_{h-1}*1$.

Proof of Theorem 1. We first consider the case $h=1$. Applying the case $m=1$ in

Proposition 4 for the last sum, we have

$$\begin{aligned}\tilde{D}_{B_\kappa,1}(N) &= \sum_{\deg(r^\kappa)\le N} \mu(r) \sum_{g|r^\kappa} \sum_{\ell|g} \mu(\ell) \left(\frac{q^{N-\deg(r^\kappa \ell)+1}}{q-1} - \frac{1}{q-1} \right) \\ &= \frac{q^{N+1}}{q-1} \sum_{\deg(r^\kappa)\le N} \frac{\mu(r)}{|r^\kappa|} \sum_{g|r^\kappa} \sum_{\ell|g} \frac{\mu(\ell)}{|\ell|} - \frac{1}{q-1} \sum_{\deg(r^\kappa)\le N} \mu(r) \sum_{g|r^\kappa} \sum_{\ell|g} \mu(\ell).\end{aligned}$$

Applying Proposition 1 for $h = 1$ and Lemma 8, we obtain the result.

Now we consider the case $h = 2$. Using the particular case $m = 1$ of Proposition 4 for the last sum, we get

$$\begin{aligned}\tilde{D}_{B_\kappa,2}(N) = &\sum_{\deg(r^\kappa)\le N} \mu(r) \sum_{g|r^\kappa} \sum_{\ell|g} \mu(\ell) \Bigg(\frac{(N-\deg(r^\kappa \ell)+1) q^{N-\deg(r^\kappa \ell)+1}}{q-1} \\ &- \frac{q^{N-\deg(r^\kappa \ell)+1}}{(q-1)^2} + \frac{1}{(q-1)^2} \Bigg) \\ = &\frac{q^{N+1}}{q-1} \sum_{\deg(r^\kappa)\le N} \frac{\mu(r)}{|r^\kappa|} \sum_{g|r^\kappa} \sum_{\ell|g} \frac{\mu(\ell)}{|\ell|} (N - \deg(r^\kappa \ell) + 1) \\ &- \frac{q^{N+1}}{(q-1)^2} \sum_{\deg(r^\kappa)\le N} \frac{\mu(r)}{|r^\kappa|} \sum_{g|r^\kappa} \sum_{\ell|g} \frac{\mu(\ell)}{|\ell|} \\ &+ \frac{1}{(q-1)^2} \sum_{\deg(r^\kappa)\le N} \mu(r) \sum_{g|r^\kappa} \sum_{\ell|g} \mu(\ell) \\ = &\left(\frac{(N+1) q^{N+1}}{q-1} - \frac{q^{N+1}}{(q-1)^2} \right) \sum_{\deg(r^\kappa)\le N} \frac{\mu(r)}{|r^\kappa|} \sum_{g|r^\kappa} \sum_{\ell|g} \frac{\mu(\ell)}{|\ell|} \\ &- \frac{q^{N+1}}{q-1} \sum_{\deg(r^\kappa)\le N} \frac{\mu(r)}{|r^\kappa|} \deg(r^\kappa) \sum_{g|r^\kappa} \sum_{\ell|g} \frac{\mu(\ell)}{|\ell|} \\ &- \frac{q^{N+1}}{q-1} \sum_{\deg(r^\kappa)\le N} \frac{\mu(r)}{|r^\kappa|} \sum_{g|r^\kappa} \sum_{\ell|g} \frac{\mu(\ell)}{|\ell|} \deg(\ell) \\ &+ \frac{1}{(q-1)^2} \sum_{\deg(r^\kappa)\le N} \mu(r) \sum_{g|r^\kappa} \sum_{\ell|g} \mu(\ell).\end{aligned}$$

By replacing each sum by its value in Propositions 1, 2, and 3 respectively, and bounding the last term using Lemma 8, we obtain the result. □

Proof of Theorem 2. Applying (11) to (14), we obtain

$$\begin{aligned}\tilde{D}_{B_\kappa,h}(N) =&\frac{q^{N+1}}{q-1}\sum_{\deg(r^\kappa)\le N}\frac{\mu(r)}{|r^\kappa|}\sum_{g|r^\kappa}d_{h-1}(g)\sum_{\ell|g}\frac{\mu(\ell)}{|\ell|}(N-\deg(r^\kappa\ell))^{h-1}\\ &\times\left(1+O\left(\frac{1}{N}\right)\right)\\ =&\frac{q^{N+1}N^{h-1}}{q-1}\sum_{\deg(r^\kappa)\le N}\frac{\mu(r)}{|r^\kappa|}\sum_{g|r^\kappa}d_{h-1}(g)\sum_{\ell|g}\frac{\mu(\ell)}{|\ell|}\left(1+O\left(\frac{1}{N}\right)\right),\end{aligned}$$

and the result follows from Proposition 1. □

5.2. Proof of Theorems 3 and 4

By orthogonality (2), since $(a,m)=1$,

$$\begin{aligned}\tilde{D}_{B_\kappa,h}(N,a,m) =& \sum_{\substack{f\in\mathcal{M}_{\le N}\\ f\equiv a \pmod m}}\mathbb{1}_{B_\kappa}(f)d_h(f)\\ =&\frac{1}{\Phi(m)}\sum_{\chi}\overline{\chi}(a)\sum_{f\in\mathcal{M}_{\le N}}\mathbb{1}_{B_\kappa}(f)\chi(f)d_h(f)\\ =&\frac{1}{\Phi(m)}\sum_{\chi}\overline{\chi}(a)D[\mathbb{1}_{B_\kappa},\chi^{(h-1)},\chi](N),\end{aligned}$$

where we have also applied Equation (8). By Theorem 6,

$$\begin{aligned}D[\mathbb{1}_{B_\kappa},\chi^{(h-1)},\chi](N) =& \sum_{\deg(f)\le N}(\mathbb{1}_{B_\kappa}*\mu)(f)\sum_{g|f}\chi^{(h-1)}(g)\chi\left(\frac{f}{g}\right)\\ &\times\sum_{\ell|g}\mu(\ell)\chi(\ell)D[\chi^{(h-1)},\chi](N-\deg(f\ell))\\ =&\sum_{\deg(r^\kappa)\le N}\mu(r)\sum_{g|r^\kappa}\chi^{(h-1)}(g)\chi\left(\frac{r^\kappa}{g}\right)\sum_{\ell|g}\mu(\ell)\chi(\ell)\\ &\times\sum_{\deg(s)\le N-\deg(r^\kappa\ell)}\chi^{(h)}(s),\end{aligned}$$

where we have applied Lemma 5 to $(\mathbb{1}_{B_\kappa}*\mu)(f)$.

If χ is non-principal, we apply Equation (8) and write $\chi^{(h-1)}(g)=\chi(g)d_{h-1}(g)$. Trivially bounding all the characters and the Möbius functions other than the innermost sum, we have,

$$\begin{aligned}\left|D[\mathbb{1}_{B_\kappa},\chi^{(h-1)},\chi](N)\right| &\le \sum_{\deg(r^\kappa)\le N}\sum_{g|r^\kappa} d_{h-1}(g)\sum_{\ell|g}\left|\sum_{\deg(s)\le N-\deg(r^\kappa\ell)}\chi^{(h)}(s)\right| \\ &\le C_1 q^{\frac{h\deg(m)}{2}}\sum_{\deg(r^\kappa)\le N}\sum_{g|r^\kappa} d_{h-1}(g)\sum_{\ell|g}1 \\ &\ll C_1 q^{\frac{h\deg(m)}{2}} q^{\epsilon N}\sum_{\deg(r^\kappa)\le N}\sum_{g|r^\kappa}\sum_{\ell|g}1,\end{aligned}$$

where we have applied Proposition 5 and the fact that $d_{h-1}(g)\ll q^{\varepsilon\deg(g)}$. Applying Lemma 8, we finally obtain

$$D[\mathbb{1}_{B_\kappa},\chi^{(h-1)},\chi](N) = O(q^{\frac{h\deg(m)}{2}+(\frac{1}{\kappa_{\min}}+\varepsilon)N}). \tag{15}$$

When $\chi=\chi_0$ is principal, we have, by Theorem 6 and Lemma 5,

$$\begin{aligned}D[\mathbb{1}_{B_\kappa},\chi_0^{(h-1)},\chi_0](N) = &\sum_{\deg(r^\kappa)\le N}\mu(r)\sum_{g|r^\kappa}\chi_0^{(h-1)}(g)\chi_0\left(\frac{r^\kappa}{g}\right)\sum_{\ell|g}\mu(\ell)\chi_0(\ell) \\ &\times\sum_{\deg(s)\le N-\deg(r^\kappa\ell)}\chi_0^{(h)}(s).\end{aligned}$$

Notice that $\chi_0(r)=1$ if and only if $(r,m)=1$ and zero otherwise, and since $\ell\mid g$ and $g\mid r^\kappa$, we have that $(\ell,m)=1$ when $\chi_0(r)=1$, which means that $\chi_0(\ell)=1$. Thus, we have

$$\begin{aligned}D[\mathbb{1}_{B_\kappa},\chi_0^{(h-1)},\chi_0](N) = &\sum_{\substack{\deg(r^\kappa)\le N\\(r,m)=1}}\mu(r)\sum_{g|r^\kappa} d_{h-1}(g)\sum_{\ell|g}\mu(\ell) \\ &\times\sum_{\deg(s)\le N-\deg(r^\kappa\ell)}\chi_0^{(h)}(s).\end{aligned}$$

Proof of Theorem 3. For the case $h=1$, replacing $\sum_{\deg(h)\le N-\deg(r^\kappa\ell)}\chi_0(h)$ by its value in Proposition 4, we get

$$\begin{aligned}D[\mathbb{1}_{B_\kappa},1,\chi_0](N) &= \sum_{\substack{\deg(r^\kappa)\le N\\(r,m)=1}}\mu(r)\sum_{g|r^\kappa}\sum_{\ell|g}\mu(\ell)\frac{q^{N-\deg(r^\kappa\ell)+1}}{q-1}\left(\frac{\Phi(m)}{|m|}\right) \\ &= \frac{q^{N+1}}{q-1}\frac{\Phi(m)}{|m|}\sum_{\substack{\deg(r^\kappa)\le N\\(r,m)=1}}\frac{\mu(r)}{|r^\kappa|}\sum_{g|r^\kappa}\sum_{\ell|g}\frac{\mu(\ell)}{|\ell|}.\end{aligned}$$

Replacing the last sum by its value in Proposition 1 for $h = 1$, we obtain

$$D[\mathbb{1}_{B_\kappa}, 1, \chi_0](N) = \frac{q^{N+1}}{q-1}\frac{\Phi(m)}{|m|}\prod_{P\nmid m}\left(1 - \frac{1}{\left|P^{\kappa(P)}\right|}\right) + O\left(q^{(\frac{1}{\kappa_{\min}}+\varepsilon-1)(N+1)}\right).$$

The result then follows by combining with (15).

Now we assume that $h = 2$. Replacing $\sum\limits_{\deg(h)\le N-\deg(r^\kappa\ell)} (\chi_0 * \chi_0)(h)$ by its value in Proposition 4, we get

$$\begin{aligned}
D[\mathbb{1}_{B_\kappa}, \chi_0, \chi_0](N) = &\sum_{\substack{\deg(r^\kappa)\le N\\(r,m)=1}} \mu(r)\sum_{g|r^\kappa}\sum_{\ell|g}\mu(\ell)q^{N-\deg(r^\kappa\ell)}\left(\frac{\Phi(m)}{|m|}\right)^2\\
&\times\left[\left(1-\frac{1}{q}\right)^{-1}\left(N - \deg(r^\kappa\ell) + 1 + 2\sum_{P|m}\frac{\deg P}{|P|-1}\right)\right.\\
&\left.-\frac{1}{q}\left(1-\frac{1}{q}\right)^{-2}\right]\\
=&\frac{q^N\left(\frac{\Phi(m)}{|m|}\right)^2\left[(1-\frac{1}{q})\left(N+1+2\sum\limits_{P|m}\frac{\deg P}{|P|-1}\right) - \frac{1}{q}\right]}{(1-\frac{1}{q})^2}\\
&\times\sum_{\substack{\deg(r^\kappa)\le N\\(r,m)=1}}\frac{\mu(r)}{|r^\kappa|}\sum_{g|r^\kappa}\sum_{\ell|g}\frac{\mu(\ell)}{|\ell|}\\
&-\frac{q^N\left(\frac{\Phi(m)}{|m|}\right)^2}{(1-\frac{1}{q})}\sum_{\substack{\deg(r^\kappa)\le N\\(r,m)=1}}\frac{\mu(r)}{|r^\kappa|}\sum_{g|r^\kappa}\sum_{\ell|g}\frac{\mu(\ell)\deg(\ell)}{|\ell|}\\
&-\frac{q^N\left(\frac{\Phi(m)}{|m|}\right)^2}{(1-\frac{1}{q})}\sum_{\substack{\deg(r^\kappa)\le N\\(r,m)=1}}\frac{\mu(r)\deg(r^\kappa)}{|r^\kappa|}\sum_{g|r^\kappa}\sum_{\ell|g}\frac{\mu(\ell)}{|\ell|}.
\end{aligned}$$

Replacing each sum by its value in Propositions 1, 2, and 3 respectively, we obtain

$$D[\mathbb{1}_{B_\kappa},\chi_0,\chi_0](N)=\frac{q^N\left(\frac{\Phi(m)}{|m|}\right)^2}{(1-\frac{1}{q})}\prod_{\substack{P\in\mathcal{P}\\P\nmid m}}\left(1-\frac{\kappa(P)+1}{|P^{\kappa(P)}|}+\frac{\kappa(P)}{|P^{\kappa(P)+1}|}\right)$$
$$\times\left[N+1-\frac{1}{q-1}+2\sum_{\substack{P\in\mathcal{P}\\P|m}}\frac{\deg P}{|P|-1}\right.$$
$$\left.+\sum_{\substack{P\in\mathcal{P}\\P\nmid m}}\frac{\kappa(P)(\kappa(P)+1)\deg(P)\,(|P|-1)}{|P^{\kappa(P)+1}|-|P|(\kappa(P)+1)+\kappa(P)}\right]+O\left(q^{(\frac{1}{\kappa_{\min}}+\varepsilon)N}\right).$$

Combining with the estimate (15), we obtain the result. □

Proof of Theorem 4. Replacing $\sum\limits_{\deg(h)\le N-\deg(r^\kappa\ell)}\chi_0^{(h)}(s)$ by its value in Proposition 4, we get

$$D[\mathbb{1}_{B_\kappa},\chi_0^{(h-1)},\chi_0](N)=\frac{q^N}{(h-1)!}\left(\frac{\Phi(m)}{|m|}\right)^h\left(1-\frac{1}{q}\right)^{-1}\sum_{\substack{\deg(r^\kappa)\le N\\(r,m)=1}}\frac{\mu(r)}{|r^\kappa|}\sum_{g|r^\kappa}d_{h-1}(g)$$
$$\times\sum_{\ell|g}\frac{\mu(\ell)}{|\ell|}(N-\deg(r^\kappa\ell))^{h-1}\left(1+O\left(\frac{1}{N}\right)\right)$$
$$=\frac{q^NN^{h-1}}{(h-1)!}\left(\frac{\Phi(m)}{|m|}\right)^h\left(1-\frac{1}{q}\right)^{-1}\sum_{\substack{\deg(r^\kappa)\le N\\(r,m)=1}}\frac{\mu(r)}{|r^\kappa|}$$
$$\times\sum_{g|r^\kappa}d_{h-1}(g)\sum_{\ell|g}\frac{\mu(\ell)}{|\ell|}\left(1+O\left(\frac{1}{N}\right)\right).$$

The result follows from Proposition 1. □

5.3. Proof of Theorem 5

We start by writing

$$A(T)=T^N+a_{N-1}T^{N-1}+\cdots+a_0$$

and by considering

$$B(T):=T^{N-\Delta-1}+a_{N-1}T^{N-\Delta-2}+\cdots+a_{\Delta+1}.$$

Then $I(A;\Delta)=I(BT^{\Delta+1};\Delta)$ and therefore

$$\tilde{N}_{B_\kappa,h}(A,\Delta)=\tilde{N}_{B_\kappa,h}(BT^{\Delta+1},\Delta).$$

Let $\mathbb{F}_q[T]_{\leq n}$ be the set of polynomials of degree less than or equal to n and let $\theta_n : \mathbb{F}_q[T]_{\leq n} \to \mathbb{F}_q[T]_{\leq n}$ be given by

$$\theta_n(f)(T) = T^n f\left(\frac{1}{T}\right).$$

Thus, $\theta_n(f) = f^*$ when $f(0) \neq 0$.

Proof of Theorem 5. Applying [16, Lemma 5.3], we have

$$\begin{aligned}\tilde{N}_{B_\kappa,h}(A,\Delta) = \left\langle \tilde{N}_{B_\kappa,h}(\cdot,\Delta)\right\rangle + \frac{1}{\Phi_{\text{even}}(T^{N-\Delta})}\sum_{\ell=0}^{N} \mathbb{1}_{B_\kappa}(T^{N-\ell})d_h(T^{N-\ell})\\ \times \sum_{\substack{\chi \pmod{T^{N-\Delta}} \\ \chi\neq\chi_0 \text{ even}}} \overline{\chi}(\theta_{N-\Delta-1}(B))\mathcal{M}(\ell;\mathbb{1}_{B_\kappa}\chi d_h),\end{aligned}$$

where

$$\Phi_{\text{even}}(T^n) = T^{n-1},$$

is the number of even characters modulo T^n, and

$$\mathcal{M}(\ell;\mathbb{1}_{B_\kappa}\chi d_h) = \sum_{f\in\mathcal{M}_\ell} \mathbb{1}_{B_\kappa}(f)\chi(f)d_h(f) = \sum_{f\in\mathcal{M}_\ell} \mathbb{1}_{B_\kappa}(f)\chi^{(h)}(f),$$

where the last equality follows from Equation (8).

By [16, Lemma 5.2],

$$\begin{aligned}\left\langle \tilde{N}_{B_\kappa,h}(\cdot,\Delta)\right\rangle =&\frac{1}{q^{N-\Delta-1}}\sum_{f\in\mathcal{M}_N} \mathbb{1}_{B_\kappa}(f)d_h(f)\\ =&\frac{1}{q^{N-\Delta-1}}\left(\tilde{D}_{B_\kappa,h}(N) - \tilde{D}_{B_\kappa,h}(N-1)\right),\end{aligned}$$

and this gives the main term.

Bounding the non-trivial character sums by Equation (15), we obtain

$$\begin{aligned}\sum_{f\in\mathcal{M}_\ell} \mathbb{1}_{B_\kappa}(f)\chi^{(h)}(f) = &\sum_{f\in\mathcal{M}_{\leq\ell}} \mathbb{1}_{B_\kappa}(f)\chi^{(h)}(f) - \sum_{f\in\mathcal{M}_{\leq\ell-1}} \mathbb{1}_{B_\kappa}(f)\chi^{(h)}(f)\\ =&D[\mathbb{1}_{B_\kappa},\chi^{(h-1)},\chi](\ell) - D[\mathbb{1}_{B_\kappa},\chi^{(h-1)},\chi](\ell-1)\\ \ll&q^{\frac{h(N-\Delta)}{2}+(\frac{1}{\kappa_{\min}}+\varepsilon)\ell}.\end{aligned}$$

Thus, the contribution of non-trivial characters is bounded by

$$\left|\frac{1}{\Phi_{\text{even}}(T^{N-\Delta})}\sum_{\ell=0}^{N}\mathbb{1}_{B_\kappa}(T^{N-\ell})d_h(T^{N-\ell})\sum_{\substack{\chi \pmod{T^{N-\Delta}} \\ \chi\neq\chi_0 \text{ even}}}\overline{\chi}(\theta_{N-\Delta-1}(B))\mathcal{M}(\ell;\mathbb{1}_{B_\kappa}\chi d_h)\right|$$
$$\ll q^{\frac{h(N-\Delta)}{2}}\sum_{\ell=\max\{0,N-\kappa(T)\}}^{N}\binom{N-\ell+h-1}{h-1}q^{(\frac{1}{\kappa_{\min}}+\varepsilon)\ell}$$
$$\ll q^{\frac{h(N-\Delta)}{2}+(\frac{1}{\kappa_{\min}}+\varepsilon)N},$$

and this gives the error term. □

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References

[1] E. Alkan, Nonvanishing of Fourier coefficients of modular forms, *Proc. Amer. Math. Soc.* **131** (2003), no. 6, 1673–1680.

[2] E. Alkan and A. Zaharescu, B-free numbers in short arithmetic progressions, *J. Number Theory* **113** (2005), no. 2, 226–243.

[3] E. Alkan and A. Zaharescu, Nonvanishing of Fourier coefficients of newforms in progressions, *Acta Arith.* **116** (2005), no. 1, 81–98.

[4] E. Alkan and A. Zaharescu, A survey on the distribution of B-free numbers, *Turkish J. Math.* **30** (2006), no. 3, 293–308.

[5] J. C. Andrade, L. Bary-Soroker, and Z. Rudnick, Shifted convolution and the Titchmarsh divisor problem over $\mathbb{F}_q[t]$, *Philos. Trans. Roy. Soc. A* **373** (2015), no. 2040, 20140308, 18 pp.

[6] A. Balog and K. Ono, The Chebotarev density theorem in short intervals and some questions of Serre, *J. Number Theory* **91** (2001), no. 2, 356–371.

[7] G. Bantle and F. Grupp, On a problem of Erdős and Szemerédi, *J. Number Theory* **22** (1986), no. 3, 280–288.

[8] A. P. de Camargo, The Dirichlet divisor problem over square–free integers and unitary convolutions, *Notes Number Theory Discrete Math.* **29** (2023), no. 3, 549–556.

[9] A. P. de Camargo, Weighted summation of Dirichlet convolutions and applications to some divisor problems over k-free and B-free integers, *Integers* **24** (2024), #A75, 16 pp.

[10] C. David, A. Florea, and M. Lalín, Nonvanishing for cubic L-functions, *Forum Math. Sigma* **9** (2021), Paper No. e69, 58 pp.

[11] P. Erdős, On the difference of consecutive terms of sequences defined by divisibility properties, *Acta Arith.* **12** (1966/67), 175–182.

[12] M. Filaseta and O. Trifonov, On gaps between squarefree numbers. II, *J. London Math. Soc. (2)* **45** (1992), no. 2, 215–221.

[13] A. Granville, *ABC* allows us to count squarefrees, *Internat. Math. Res. Notices* (1998), no. 19, 991–1009.

[14] R. Jakimczuk and M. Lalín, Asymptotics of sums of divisor functions over sequences with restricted factorization structure, *Notes Number Theory Discrete Math.* **28** (2022), no. 4, 617–634.

[15] J. P. Keating, B. Rodgers, E. Roditty-Gershon, and Z. Rudnick, Sums of divisor functions in $\mathbb{F}_q[t]$ and matrix integrals, *Math. Z.* **288** (2018), no. 1-2, 167–198.

[16] J. Keating and Z. Rudnick, Squarefree polynomials and Möbius values in short intervals and arithmetic progressions, *Algebra Number Theory* **10** (2016), no. 2, 375–420.

[17] E. Kowalski, O. Robert, and J. Wu, Small gaps in coefficients of L-functions and $\mathcal{B}$-free numbers in short intervals, *Rev. Mat. Iberoam.* **23** (2007), no. 1, 281–326.

[18] V. Kuperberg and M. Lalín, Sums of divisor functions and von Mangoldt convolutions in $\mathbb{F}_q[T]$ leading to symplectic distributions, *Forum Math.* **34** (2022), no. 3, 711–747.

[19] K. Matomäki, On the distribution of $\mathcal{B}$-free numbers and non-vanishing Fourier coefficients of cusp forms, *Glasg. Math. J.* **54** (2012), no. 2, 381–397.

[20] M. R. Murty, *Problems in Analytic Number Theory*, 2nd ed., Graduate Texts in Mathematics, Vol. 206, Springer, New York, 2008.

[21] V. A. Plaksin, Distribution of $\mathcal{B}$-free numbers, *Mat. Zametki* **47** (1990), no. 2, 69–77, 159.

[22] M. Rosen, *Number Theory in Function Fields*, Graduate Texts in Mathematics, Vol. 210, Springer-Verlag, New York, 2002.

[23] P. Sargos and J. Wu, Multiple exponential sums with monomials and their applications in number theory, *Acta Math. Hungar.* **87** (2000), no. 4, 333–354.

[24] W. Sawin, Square-root cancellation for sums of factorization functions over short intervals in function fields, *Duke Math. J.* **170** (2021), no. 5, 997–1026.

[25] W. Sawin, Square-root cancellation for sums of factorization functions over squarefree progressions in $\mathbb{F}_q[t]$, *Acta Math.* **233** (2024), no. 2, 285–418.

[26] E. Szemerédi, On the difference of consecutive terms of sequences defined by divisibility properties. II, *Acta Arith.* **23** (1973), 359–361.

[27] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., The Clarendon Press, Oxford University Press, New York, 1986.

[28] J. Wu, Nombres B-libres dans les petits intervalles, *Acta Arith.* **65** (1993), no. 2, 97–116.

[29] W. Zhai, Number of B-free numbers in short intervals, *Chinese Sci. Bull.* **45** (2000), no. 3, 208–212.