PARAMETRIC EULER SUMS OF GENERALIZED HYPERHARMONIC NUMBERS

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Abstract

We analyze a parametric extension of Euler sums of generalized hyperharmonic numbers, and show that it exhibits a natural duality. This family of series includes many variants of harmonic number series and Euler-Apéry sums. We evaluate several families of such series involving odd harmonic numbers, binomial coefficients, and Catalan numbers explicitly in terms of Riemann zeta and Dirichlet beta values.

1. Introduction

The *generalized hyperharmonic number* $H_n^{[r]}(k)$ of depth k and order r is defined [32, 33] for positive integers n, k, r by the finite nested sum

$$H_n^{[r]}(k) := \sum_{\substack{0<n_{r+k-1}<\cdots<n_r \\ \leq n_{r-1}\leq\cdots\leq n_1\leq n}} \frac{1}{n_r n_{r+1}\cdots n_{r+k-1}}. \tag{1.1}$$

The *hyperharmonic Euler sums*

$$S(k,r;p) := \sum_{n=1}^{\infty} \frac{H_n^{[r]}(k)}{n^p}, \tag{1.2}$$

have been extensively studied [9, 10, 22, 32, 33] in recent years. These Dirichlet series are convergent for integers $p > r$ and their values may be expressed as rational polynomials in Riemann zeta values; this was demonstrated for $k = 1$ in [9, 22] and for general k in [32, 33]. In the case $k = r = 1$, this fact was known to Euler, for whom the sums are named.

Classically, an *Euler sum* [1, 11] refers to a convergent infinite series

$$\sum_{n=1}^{\infty} \frac{H_n^{(m_1)}\cdots H_n^{(m_k)}}{n^q} \tag{1.3}$$

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(for integers $q \geq 2$ and $m_1, \ldots, m_k \geq 1$) involving the generalized harmonic numbers

$$H_n^{(m)} := \sum_{j=1}^{n} \frac{1}{j^m}.$$

These sums may be expressed [11, 37] as $\mathbb{Q}$-linear combinations of *multiple zeta values*

$$\zeta(s_1, s_2, \ldots, s_j) = \sum_{n_1 > n_2 > \cdots > n_j > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_j^{s_j}} \tag{1.4}$$

where $s_1 > 1$ and $s_2, \ldots, s_j > 0$ are integers [39], which are multivariate generalizations of the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\Re(s) > 1).$$

The hyperharmonic Euler sums $S(k, r; p)$ are one of many variants of interest; others include alternating sums [27, 29], sums with odd harmonic numbers

$$O_n^{(m)} := 2^m \sum_{j=1}^{n} \frac{1}{(2j-1)^m} \tag{1.5}$$

[7, 19, 26, 28, 29, 36], sums with shifted denominators $(n+a)^q$ [14, 19, 25, 26, 27, 28, 31, 36], and the *Euler-Apéry sums* [3, 6, 7, 14, 18, 20, 30, 36] which include factors of $a_n := \binom{2n}{n}/4^n$ in the summands, after Apéry's proof [2] of the irrationality of $\zeta(3)$.

Two fundamental questions are to express such series, or families of such series, in terms of known constants, such as Riemann zeta values; and to determine arithmetic relations between such sums. Following [5], in this article we consider the parametric variant

$$S(k, r; p, a) := \sum_{n=k}^{\infty} \frac{H_n^{[r]}(k)}{(n+a)^p} \tag{1.6}$$

where r and a may be considered as complex variables. These sums $S(k, r; p, a)$ exhibit a natural duality (Corollary 1 below), and for $p \in \mathbb{N}$ and $r, a \in \mathbb{Q}$, include sums considered in [26, 30, 29, 31], odd harmonic variants of those in [3, 23, 24, 26, 25, 30], and Apéry-type variants of those in [19, 26, 27, 28, 31, 36].

An approach taken by several authors [3, 38] has been the expression of such sums in terms of *cyclotomic multiple zeta values* (also called *colored multiple zeta values*) [39], which refer to convergent series of the form

$$\sum_{n_1 > n_2 > \cdots > n_j > 0} \frac{\epsilon_1^{n_1} \cdots \epsilon_j^{n_j}}{n_1^{s} n_2^{s_2} \cdots n_j^{s_j}}$$

with each ϵ_i a root of unity. The *level* of the CMZV is the minimal $N \in \mathbb{N}$ such that all $\epsilon_i^N = 1$; thus in (1.4), MZVs are CMZVs of level one. For example, in [38, Corollary 9.8], a large class of Euler-Apéry sums, including sums of the form

$$S(k, \tfrac{1}{2}; p, \tfrac{1}{2}) = \sum_{n=k}^{\infty} \frac{\binom{2n}{n} O_n(k)}{4^n (n+\frac{1}{2})^p}, \tag{1.7}$$

where

$$O_n(k) := \sum_{1 \le i_1 < \cdots < i_k \le n} \frac{2^k}{(2i_1 - 1) \cdots (2i_k - 1)} \tag{1.8}$$

is the depth k odd harmonic number, were recently shown to lie in the $\mathbb{Q}$-vector space spanned by the imaginary parts of level 4 CMZVs. In Corollary 7 below, we give a refinement showing that these sums (1.7) also belong to a large family which more specifically lie in the ring $\mathbb{Q}[\log 2, \pi, \zeta(2j+1)_{j\ge1}]$. Corollary 4 gives another family of sums $S(k, r; p, a)$ which lie in this ring, while Corollary 6 gives a family that lies in $\mathbb{Q}[\log 2, \pi, \zeta(2j+1)_{j\ge1}, \beta(2j)_{j\ge1}]$, where $\beta(s)$ denotes the Dirichlet beta function. Corollary 11 gives a surprising family of *products* of sums $S(k, r; p, a)$ which lie in this latter ring.

Although our results apply as well for integer parameters r, a in the sums (1.6), our paper is focused primarily on cases where at least one parameter is not an integer, and particularly often lies in $\frac{1}{2} + \mathbb{Z}$. By a combination of analytic and combinatorial techniques, we find several families of such series which have relatively concise expression, including Corollaries 3 and 9 below for certain Apéry-type odd harmonic sums, and Corollaries 5 and 8 for Catalan number harmonic sums. We illustrate each such family with several explicit examples.

2. Evaluation of Parametric Hyperharmonic Euler Sums

Throughout this paper we adopt the convention that in any variant of harmonic or hyperharmonic numbers, when the order r or depth k is 1, it may be omitted from the notation. From the definition (1.1) of generalized hyperharmonic numbers, we observe that for order $r > 1$, $H_n^{[r]}(k) - H_{n-1}^{[r]}(k)$ is equal to the sum (1.1) summed over $n_1 = n$ only, and is therefore equal to $H_n^{[r-1]}(k)$. With the initial conditions $H_0^{[r]}(k) = 0$, this recursion

$$H_n^{[r]}(k) - H_{n-1}^{[r]}(k) = H_n^{[r-1]}(k) \tag{2.1}$$

implies the sum formula

$$H_n^{[r]}(k) = \sum_{i=1}^{n} H_i^{[r-1]}(k). \tag{2.2}$$

Taking the order $r = 1$ in (1.1), we have the *multiple harmonic number of depth* k,

$$H_n^{[1]}(k) := H_n(k) := \sum_{0<n_k<\cdots<n_1\le n} \frac{1}{n_1 n_2 \cdots n_k}, \tag{2.3}$$

which becomes the classical harmonic number $H_n(1) = H_n := \sum_{j=1}^{n} 1/j$ when $k = 1$. The relation (2.2) in the case $k = 1$ formed the original definition [4, 8] of hyperharmonic numbers. Thus, from (1.6) and (2.3) we conclude that the family of sums $S(k, r; p, a)$ include as special cases the multiple zeta values

$$S(k, 1; p, 1) = \zeta(p, \underbrace{1, \ldots, 1}_{k})$$

for $p > 1$.

The hyperharmonic numbers are also given by the combinatorial formula

$$H_n^{[r]}(k) = \frac{1}{k!} D_r^k \binom{n+r-1}{n} \tag{2.4}$$

[33, eq. (2.8)], where $D_r = \frac{d}{dr}$ is the derivative and $\binom{n+r-1}{n}$ is the binomial coefficient. This permits us to consider $H_n^{[r]}(k)$ as a polynomial of degree $n-k$ in r, and consequently regard r as a complex variable. With this definition, the recurrence (2.1) extends to all $r \in \mathbb{C}$; therefore (for example) we obtain

$$H_n^{[0]}(k) = \frac{1}{n} H_{n-1}(k-1) \tag{2.5}$$

for positive integers n, k, from (2.3) and the $r = 1$ case of (2.1). Since $H_n^{[r]}(k) \sim n^{r-1}(\log n)^k/(r-1)!$ as $n \to \infty$ [12, Theorem VI.2], the series (1.6) is absolutely convergent when $\Re(r) < p$, defining $S(k, r; p, a)$ as an analytic function of r on that half-plane for fixed k, p, a, and an analytic function of a on $\mathbb{C} \setminus \{-k, -k-1, \ldots\}$ for fixed k, r, p. The recursion (2.1) implies the recursion

$$S(k, r; p; a) - S(k, r; p; a+1) = S(k, r-1; p; a) \tag{2.6}$$

for the hyperharmonic Euler sums. Our first main theorem expresses these sums in terms of the Euler beta function

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt, \qquad (\Re(a), \Re(b) > 0). \tag{2.7}$$

Theorem 1. *For nonnegative integers k and p, for $\Re(r) < p+1$ and $\Re(a) > -k$ we have the evaluation*

$$p!k!S(k, r; p+1; a) = (-1)^p D_a^p D_r^k B(a, 1-r)$$

in terms of derivatives of the Euler beta function. Thus for $\Re(r) < p$, $S(k, r; p; a)$ continues analytically to $a \in \mathbb{C} \setminus \{-k, -k-1, \ldots\}$.

Proof. The beta function $B(a, 1-r)$ is defined for $\Re(a) > 0$, $\Re(r) < 1$ by (2.7), and binomial expansion of $(1-t)^{-r}$ in the integrand produces the series

$$B(a, 1-r) = \sum_{m=0}^{\infty} \binom{m+r-1}{m} (m+a)^{-1} = S(0, r; 1; a) \tag{2.8}$$

which is convergent for $\Re(r) < 1$ and $\Re(a) > 0$. By (2.4), we see that this proves the first statement in the case $k = p = 0$.

Differentiating with respect to r under the integral sign in (2.7), the corresponding integral formula for $D_r^k B(a, 1-r)$ continues analytically to $\Re(a) > -k$, due to the introduction of factors of $\log(1-t)$ in the integrand. Therefore differentiating termwise in (2.8) produces

$$D_r^k B(a, 1-r) = \sum_{m=k}^{\infty} H_m^{[r]}(k)(m+a)^{-1} = k! S(k, r; 1; a) \tag{2.9}$$

with both members analytic for $\Re(r) < 1$ and $\Re(a) > -k$. Finally, differentiating with respect to a, termwise in (2.9) and under the integral sign in (2.7), produces

$$D_a^p D_r^k B(a, 1-r) = (-1)^p p! k! S(k, r; p+1; a)$$

with both members analytic for $\Re(r) < p+1$ and $\Re(a) > -k$. Since $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is an analytic function of a when a is not a nonpositive integer, $S(k, r; p, a)$ continues analytically as described. □

This evaluation by differentiation under the integral sign in (2.7) demonstrates that $S(k, r; p, a)$ is a real period (in the sense of [16]) when a, r are real algebraic numbers. Since $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is symmetric in a and b, this evaluation of $S(k, r; p, a)$ implies a duality relation for these sums, which applies as long as both series are convergent.

Corollary 1. *For $\Re(r) < p+1$ and $\Re(t) < k+1$ we have the duality*

$$S(k, r; p+1; 1-t) = S(p, t; k+1; 1-r),$$

that is,

$$\sum_{n=k}^{\infty} \frac{H_n^{[r]}(k)}{(n+1-t)^{p+1}} = \sum_{n=p}^{\infty} \frac{H_n^{[t]}(p)}{(n+1-r)^{k+1}}.$$

While this corollary establishes a useful symmetry for the hyperharmonic Euler sums $S(k, r; p, a)$ for integer parameters k, r, p, a, it also provides intriguing relations among the many variants of Euler sums, especially by taking either r or a in $\frac{1}{2} + \mathbb{Z}$. Using the well known expression of the coefficients of a polynomial as symmetric functions of its zeros, for $r \notin \{0, -1, \ldots, 1-n\}$, the combinatorial formula (2.4) implies an explicit expression

$$H_n^{[r]}(k) = \binom{n+r-1}{n} H_n(k;r) \tag{2.10}$$

in terms of the binomial coefficient $\binom{n+r-1}{n}$ and the depth k harmonic sum starting at r,

$$H_n(k;r) := \sum_{1 \le i_1 < \cdots < i_k \le n} \frac{1}{(r+i_1-1)\cdots(r+i_k-1)}, \tag{2.11}$$

with convention $H_n(0;r) := 1$. Specifically, write the binomial coefficient

$$F(x) = \binom{n+r-x-1}{n} = \sum_{k=0}^{n} a_k(r) x^k$$

as a polynomial in x. Setting $y = r - x$ and differentiating with respect to x at $x = 0$ using (2.4) gives

$$\frac{1}{k!} F^{(k)}(0) = \frac{1}{k!} D_x^k \binom{n+r-x-1}{n} \bigg|_{x=0} = (-1)^k H_n^{[r]}(k) = a_k(r).$$

But since $F(x)$ is a polynomial in x with leading coefficient $\frac{(-1)^n}{n!}$ and zeros $x \in \{r, r+1, \ldots, r+n-1\}$, we have

$$F(x) = \frac{(-1)^n}{n!} \prod_{i=0}^{n-1} (x - (r+i)),$$

so that $a_k(r)$ is $(-1)^k/n!$ times the $(n-k)$-th elementary symmetric function of $\{r, r+1, \ldots, r+n-1\}$. Therefore $H_n^{[r]}(k)$ is equal to

$$\frac{1}{n!} \sum_{0 \le j_1 < \cdots < j_{n-k} \le n-1} (r+j_1)\cdots(r+j_{n-k}) = \frac{r(r+1)\cdots(r+n-1)}{n!} H_n(k;r)$$

as claimed in (2.10). Taking $r = 1$ in (2.11) yields the depth k multiple harmonic number $H_n(k;1) = H_n(k)$ in (2.3), while taking $r = \frac{1}{2}$ yields

$$H_n^{[1/2]}(k) = \frac{\binom{2n}{n} O_n(k)}{4^n}, \tag{2.12}$$

where $O_n(k) := H_n(k; \frac{1}{2})$ is the depth k odd harmonic number defined in (1.8). Therefore, among the $r = \frac{1}{2}$ cases of Corollary 1 we find the relations

$$\sum_{n=k}^{\infty} \frac{\binom{2n}{n} O_n(k)}{4^n (n+1)^{p+1}} = \sum_{n=p}^{\infty} \frac{H_{n-1}(p-1)}{n(n+\frac{1}{2})^{k+1}},$$

$$\sum_{n=k}^{\infty} \frac{\binom{2n}{n} O_n(k)}{4^n (n+\frac{1}{2})^{p+1}} = \sum_{n=p}^{\infty} \frac{\binom{2n}{n} O_n(p)}{4^n (n+\frac{1}{2})^{k+1}}$$

among these variant sums, for any positive integers k and p.

3. Expression as Hurwitz Zeta Values

3.1. One Non-integer Parameter

It is known [9, 10, 22, 32, 33] that for positive integers k, r, p with $p > r$, the series $S(k, r; p, 0)$ is a rational polynomial in Riemann zeta values. First we extend this principle to the case where one of (r, a) is a positive integer and one is not; this entails an evaluation of the analytic continuation of $D_a^p D_b^k B(a, b)$ at values where $B(a, b)$ itself has a pole. Our expression of the sums $S(k, r; p, a)$ involves the digamma function $\psi(a) = \Gamma'(a)/\Gamma(a)$ and its derivatives $\psi^{(m)}(a)$, which are called polygamma functions. These functions satisfy $\psi^{(m)}(a) = (-1)^{m+1} m! \zeta(m+1, a)$ for positive integers m, where

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad (\Re(a) > 0,\ \Re(s) > 1)$$

denotes the Hurwitz zeta function.

Theorem 2. *For $r = 1, \ldots, p$ and $a \in \mathbb{C} \setminus \mathbb{Z}$, the generalized hyperharmonic Euler sum $S(k, r; p + 1; a)$ is a polynomial in $\psi(a) + \gamma$, polygamma values, and Riemann zeta values, that is,*

$$S(k, r; p + 1; a) \in \mathbb{Q}(a)[\psi(a) + \gamma, \psi^{(j)}(a)_{j \geq 1}, \zeta(j)_{j \geq 2}].$$

This extends as well to integers a with $a > -k$ by continuity.

Proof. By Theorem 1, we need to evaluate $D_a^p D_r^k B(a, 1 - r)$ for $r = 1, \ldots, p$, which, up to sign, amounts to $D_a^p D_b^k B(a, b)$ for $b = 0, \ldots, 1 - p$. Using the derivative formula

$$D_a B(a, b) = B(a, b)(\psi(a) - \psi(a + b)) \tag{3.1}$$

and its symmetric counterpart formula for $D_b B(a, b)$, we may write

$$D_a^p D_b^k B(a, b) = B(a, b) \cdot P_{p,k}(a, b) \tag{3.2}$$

where the meromorphic function $P_{p,k}(a, b)$ is an integral polynomial in the derivatives $\psi^{(i)}(a), \psi^{(i)}(b), \psi^{(i)}(a+b)$ for $i \geq 0$. Fixing $a \in \mathbb{C} \backslash \mathbb{Z}$, the meromorphic function $D_a^p D_b^k B(a, b)$ is analytic at $b = 0, \ldots, 1 - p$ where $B(a, b)$ itself has simple poles; therefore $P_{p,k}(a, b)$ has zeros at $b = 0, \ldots, 1 - p$. Specifically, using the translation functional equation

$$B(a, b) = \frac{(a + b)_{j+1}}{(b)_{j+1}} B(a, b + j + 1), \tag{3.3}$$

where $(b)_j = \Gamma(b + j)/\Gamma(b)$ denotes the rising factorial, we find that

$$\lim_{b \to -j} (b + j) B(a, b) = (-1)^j \binom{a - 1}{j} \in \mathbb{Q}[a], \tag{3.4}$$

which implies that if $a \in \mathbb{C} \setminus \mathbb{Z}$ then

$$\lim_{b \to -j} \frac{P_{p,k}(a,b)}{b+j} \quad \text{exists for } j = 0, \ldots, p-1. \tag{3.5}$$

Recalling the translation functional equation

$$\psi(a+1) = \psi(a) + \frac{1}{a},$$

we let j be a nonnegative integer and consider the Taylor series of $P_{1,0}(a,b)$ at $b = -j$,

$$\begin{aligned} P_{1,0}(a,b) &= \psi(a) - \psi(a+b) \\ &= \sum_{i=0}^{j-1} (a+b+i)^{-1} + \psi(a) - \psi(a+b+j) \\ &= \sum_{i=0}^{j-1} (a+b+i)^{-1} - \psi'(a)(b+j) - \psi''(a)(b+j)^2/2 - \cdots \\ &\in \mathbb{Q}(a)[\psi^{(m)}(a)_{m \geq 1}][[b+j]], \end{aligned} \tag{3.6}$$

since the Taylor series of each term $(a+b+i)^{-1}$ lies in $\mathbb{Q}(a)[[b+j]]$, the ring of formal power series in indeterminate $b+j$ with coefficients in $\mathbb{Q}(a)$. The definition (3.2) of $P_{p,k}$ implies the recursion

$$P_{p+1,0}(a,b) = D_a P_{p,0}(a,b) + (\psi(a) - \psi(a+b)) P_{p,0}(a,b), \tag{3.7}$$

so induction using (3.6) and (3.7) shows that

$$P_{p,0}(a,b) \in \mathbb{Q}(a)[\psi^{(m)}(a)_{m \geq 1}][[b+j]] \quad \text{for } p > j.$$

Since the limit (3.5) is known to exist, we in fact have

$$\frac{P_{p,0}(a,b)}{b+j} \in \mathbb{Q}(a)[\psi^{(m)}(a)_{m \geq 1}][[b+j]] \quad \text{for } p > j. \tag{3.8}$$

Next, the definition (3.2) of $P_{p,k}$ implies the recursion

$$P_{p,k+1}(a,b) = D_b P_{p,k}(a,b) + (\psi(b) - \psi(a+b)) P_{p,k}(a,b). \tag{3.9}$$

Writing $\psi(b) = \psi(b+j+1) - \sum_{i=0}^{j} (b+i)^{-1}$, we have

$$\begin{aligned} P_{p,k+1}(a,b) &= D_b P_{p,k}(a,b) - \frac{P_{p,k}(a,b)}{b+j} \\ &\quad + \left(\psi(b+j+1) - \psi(a+b) + \sum_{i=0}^{j-1} (b+i)^{-1} \right) P_{p,k}(a,b) \end{aligned}$$

so that

$$\frac{P_{p,k+1}(a,b)}{b+j} = D_b\left(\frac{P_{p,k}(a,b)}{b+j}\right) + \left(\psi(b+j+1)-\psi(a+b)+\sum_{i=0}^{j-1}(b+i)^{-1}\right)\frac{P_{p,k}(a,b)}{b+j}$$

is analytic at $b=-j$ for $p>j$ and all $k\in\mathbb{N}$. The i-th coefficient of the Taylor expansion of $\psi(b+j+1)-\psi(a+j)$ at $b=-j$ is $(\psi^{(i)}(1)-\psi^{(i)}(a-j))/j!$. Since $\psi^{(i)}(1)=(-1)^{i+1}i!\zeta(i+1)$, we conclude that

$$\frac{P_{p,k}(a,b)}{b+j}\in\mathbb{Q}(a)[\psi(a)+\gamma,\psi^{(m)}(a)_{m\geq 1},\zeta(i)_{i\geq 2}][[b+j]] \tag{3.10}$$

for $p>j$ and for all $k\in\mathbb{N}$. The stated result then follows from (3.2), (3.4), and (3.10). □

We remark that if $a\in\mathbb{Q}$ and $b\in\mathbb{N}$, then $B(a,b)\in\mathbb{Q}$. For rational a and integer r, the sum $S(k,r;p,a)$ may be expressed in terms of Riemann and Hurwitz zeta values using (3.1), (3.2), and (3.3). In the case where a is a half-integer, we obtain the following corollary.

Corollary 2. *If $a\in\frac{1}{2}+\mathbb{Z}$, then for integers r,k,p with $k\geq 0$ and $1\leq r\leq p$ we have*

$$\sum_{n=k}^{\infty}\frac{H_n^{[r]}(k)}{(n+a)^p}\in\mathbb{Q}[\log 2,\pi^2,\zeta(2j+1)_{j\geq 1}].$$

Furthermore, for integers a,k,p with $k\geq 0$, $p>0$ and $a>-k$, we have

$$\sum_{n=k}^{\infty}\frac{\binom{2n}{n}O_n(k)}{4^n(n+a)^p}\in\mathbb{Q}[\log 2,\pi^2,\zeta(2j+1)_{j\geq 1}],$$

where $O_n(k)$ is the multiple odd harmonic number defined in (1.8).

Proof. The first statement follows from the identities $\psi(\frac{1}{2})+\gamma=-2\log 2$ and $\psi^{(m)}(a)=(-1)^{m+1}m!\zeta(m+1,a)$. We use the well known facts that $\zeta(n,\frac{1}{2})=(2^n-1)\zeta(n)$, and that $\zeta(2j)$ is a rational multiple of π^{2j}. The second statement follows using the duality (Corollary 1) to obtain the result for $r=\frac{1}{2}$ and $a\in\mathbb{Z}$, and the identity (2.12). □

Theorem 2 gives an algorithm for evaluating the indicated series by using (3.3) to translate (if necessary) away from the poles of $B(a,1-r)$, and then (3.2) to express the sum in terms of Hurwitz zeta values. We remark that the $a=0$ case of the second statement of this corollary was given in [38, Theorem 6.5]. In the following corollary we observe that for the choice $(p,a)=(2,1)$, the family of series $S(k,\frac{1}{2};2,1)$ have explicit expression as $\mathbb{Z}$-linear combinations of $\{\log 2,\zeta(j)_{j\geq 2}\}$.

Corollary 3. *For every integer $k \geq 0$ we have*

$$\sum_{n=k}^{\infty} \frac{\binom{2n}{n} O_n(k)}{4^n (n+1)^2} = 2^{k+2}(k+1-\log 2) - \sum_{j=1}^{k} (2^{k+2} - 2^j)\zeta(k+2-j).$$

Proof. We may evaluate $S(k, \frac{1}{2}, 2, 1)$ directly from (3.2); or by evaluating its dual sum

$$S(1, 0; k+1, \tfrac{1}{2}) = \sum_{n=1}^{\infty} \frac{1}{n(n+\frac{1}{2})^{k+1}}.$$

This sum has the partial fraction decomposition

$$2^{k+1} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+\frac{1}{2}} \right) - \sum_{j=1}^{k} \sum_{n=1}^{\infty} \frac{2^j}{(n+\frac{1}{2})^{k+2-j}},$$

where the sum on the left is the usual alternating series for $2 - 2\log 2$, and the j-th term of the sum on the right is the Hurwitz zeta value $2^j \zeta(k+2-j, \frac{3}{2})$. Since $\zeta(k, \frac{3}{2}) = (2^k - 1)\zeta(k) - 2^k$, the result follows. □

Example 1. Here we give a few examples with $(p, a) = (2, 1)$ as in Corollary 3, and a few with $(p, a) \neq (2, 1)$ illustrating the general case of Corollary 2:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\binom{2n}{n} O_n}{4^n (n+1)^2} &= 16 - 8\log 2 - 6\zeta(2); \\
\sum_{n=2}^{\infty} \frac{\binom{2n}{n} O_n(2)}{4^n (n+1)^2} &= 48 - 16\log 2 - 14\zeta(3) - 12\zeta(2); \\
\sum_{n=3}^{\infty} \frac{\binom{2n}{n} O_n(3)}{4^n (n+1)^2} &= 128 - 32\log 2 - 30\zeta(4) - 28\zeta(3) - 24\zeta(2); \\
\sum_{n=1}^{\infty} \frac{\binom{2n}{n} O_n}{4^n (n+1)^3} &= 48 - 14\zeta(3) - 16\zeta(2) - 32\log 2 \\
&\quad +12\zeta(2)\log 2 + 8\log^2 2; \\
\sum_{n=2}^{\infty} \frac{\binom{2n}{n} O_n(2)}{4^n (n+1)^3} &= 192 - 45\zeta(4) - 56\zeta(3) - 56\zeta(2) + 9\zeta(2)^2 \\
&\quad -96\log 2 + 28\zeta(3)\log 2 + 24\zeta(2)\log 2 + 16\log^2 2; \\
\sum_{n=1}^{\infty} \frac{\binom{2n}{n} O_n}{4^n (n+2)^2} &= \tfrac{320}{27} - \tfrac{64}{9}\log 2 - 4\zeta(2).
\end{aligned}$$

The above examples include sums, or odd harmonic or Apéry-type variants of sums, considered in [3, 7, 24, 26, 29, 31]. When more generally the order $r \in \frac{1}{2} + \mathbb{Z}$, we obtain similar expressions for sums including other binomial coefficients as in [17, 19].

Corollary 4. *For any integers $k \geq 0$, $p > m \geq 0$, and $a > -k$, we have*

$$\sum_{n=k}^{\infty} \frac{\binom{n}{m}\binom{2n}{n} O_n(k)}{4^n (n+a)^p} \in \mathbb{Q}[\log 2, \pi^2, \zeta(2j+1)_{j\geq 1}],$$

and for integers $k \geq 0$, $p, m > 0$, and $a > -k$, we have

$$\sum_{n=k}^{\infty} \frac{\binom{2n}{n} O_n(k)}{4^n \binom{n+m}{m} (n+a)^p} \in \mathbb{Q}[\log 2, \pi^2, \zeta(2j+1)_{j\geq 1}].$$

Proof. This corollary follows from the dual of Theorem 2 by taking the order $r = \frac{1}{2} \pm m$. From the identity (2.10) we conclude

$$H_{n+m}^{[r-m]}(0) = \frac{\binom{r-1}{m}}{\binom{n+m}{m}} H_n^{[r]}(0), \qquad H_{n-m}^{[r+m]}(0) = \frac{\binom{n}{m}}{\binom{r+m-1}{m}} H_n^{[r]}(0). \tag{3.11}$$

Taking $r = \frac{1}{2}$, the $k = 0$ case of this corollary follows from the $r = \frac{1}{2} \pm m$ cases of Corollary 2. Then in (2.10) we observe that

$$H_{n+1}(k; r-1) = H_n(k; r) + \frac{1}{r-1} H_n(k-1; r),$$

which by (2.10) implies the recurrence

$$(n+1) H_{n+1}^{[r-1]}(k) = (r-1) H_n^{[r]}(k) + H_n^{[r]}(k-1). \tag{3.12}$$

Iterating this recurrence produces the two reduction formulas

$$H_{n+m}^{[r-m]}(k) = \frac{\binom{r-1}{m}}{\binom{n+m}{m}} \sum_{j=0}^{\min\{m,k\}} H_n^{[r]}(k-j) H_m(j; r-m), \tag{3.13}$$

$$H_{n-m}^{[r+m]}(k) = \frac{\binom{n}{m}}{\binom{r+m-1}{m}} \sum_{j=0}^{k} (-1)^j H_n^{[r]}(k-j) G_m(j; r), \tag{3.14}$$

where $G_m(j; r)$ is defined recursively by

$$G_1(j; r) = \frac{1}{r^j}, \qquad G_{m+1}(j; r) = \sum_{i=0}^{j} G_m(i; r+1) r^{i-j}.$$

By (2.12), these two recurrences express $H_{n+m}^{[1/2-m]}(k)$ (*resp.* $H_{n-m}^{[1/2+m]}(k)$) as $\mathbb{Q}$-linear combinations of $\binom{n+m}{m}^{-1} 4^{-n} \binom{2n}{n} O_n(j)$ (*resp.* $\binom{n}{m} 4^{-n} \binom{2n}{n} O_n(j)$) for $0 \leq j \leq k$. Using induction on k, dividing these combinations by $(n+a)^p$ expresses $S(k, \frac{1}{2} \pm m; p, a \pm m)$ (which lies in the indicated ring by Corollary 2) as a sum in which all except the $j = k$ term lie in this ring by the induction hypothesis. Thus the $j = k$ term also lies in the indicated ring, completing the proof. ∎

In the case $r = \frac{1}{2}$, $m = 1$, the reduction formula (3.12) expresses $H_{n+1}^{[-1/2]}(k)$ in terms of Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ and odd harmonic numbers $O_n(k)$. Similar to Corollary 3, we have a family of series with $(r,p,a) = (-\frac{1}{2},1,1)$ which may be expressed as explicit $\mathbb{Z}$-linear combinations of $\{\zeta(j)_{j\geq 2}\}$.

Corollary 5. *For every integer $k > 0$ we have*

$$\sum_{n=k}^{\infty} \frac{C_n O_n(k)}{4^n n} = (2^{k+1}-1)\zeta(k+1) - 2^{k+1},$$

where $C_n := \frac{1}{n+1}\binom{2n}{n}$ denotes the Catalan number.

Proof. Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, we first have

$$\sum_{n=1}^{\infty} \frac{C_n O_n}{4^n n} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n} O_n}{4^n n(n+1)} = S(1,\tfrac{1}{2},1,0) - S(1,\tfrac{1}{2},1,1).$$

On the right side, the second of these sums may be evaluated directly by (3.1) as $S(1,\frac{1}{2},1,1) = 4$. Also by (3.1), the first sum above is

$$S(1,\tfrac{1}{2},1,0) = -\lim_{a\to 0} \frac{\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})} \cdot a\Gamma(a) \cdot \frac{\psi(\frac{1}{2}) - \psi(a+\frac{1}{2})}{a} = \psi'(\tfrac{1}{2}) = 3\zeta(2).$$

This proves the corollary in the case $k = 1$. For $k > 1$, the reduction formula (3.13) in the case $r = \frac{1}{2}$, $m = 1$ gives

$$-2H_{n+1}^{[-1/2]}(k) = 4^{-n}\left(C_n O_n(k) - 2C_n O_n(k-1)\right), \tag{3.15}$$

so by duality we have

$$\sum_{n=k}^{\infty} \frac{C_n O_n(k)}{4^n n} = 2\sum_{n=k-1}^{\infty} \frac{C_n O_n(k-1)}{4^n n} - 2\sum_{n=p}^{\infty} \frac{n+1}{(n+\frac{3}{2})^{k+1}},$$

Writing $n+1 = (n+\frac{3}{2}) - \frac{1}{2}$, this rightmost sum may be expressed as

$$\zeta\left(k,\frac{3}{2}\right) - \frac{1}{2}\zeta\left(k+1,\frac{3}{2}\right)$$

in terms of the Hurwitz zeta function. Since $\zeta(k,\frac{3}{2}) = (2^k-1)\zeta(k) - 2^k$, this sum equals $(2^{k+1}-1)\zeta(k+1) - (2^{k+1}-2)\zeta(k)$, and the stated evaluation follows via induction on k. $\square$

Example 2. With $r = -\frac{1}{2}$, we give a few examples with $(p,a) = (1,1)$ as in Corollary 5, and a few with $(p,a) \neq (1,1)$ illustrating Corollary 4:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{C_nO_n}{4^n n} &= 3\zeta(2)-4;\\
\sum_{n=2}^{\infty}\frac{C_nO_n(2)}{4^n n} &= 7\zeta(3)-8;\\
\sum_{n=3}^{\infty}\frac{C_nO_n(3)}{4^n n} &= 15\zeta(4)-16;\\
\sum_{n=1}^{\infty}\frac{C_nO_n}{4^n n^2} &= 7\zeta(3)-6\zeta(2)\log 2+4-3\zeta(2);\\
\sum_{n=2}^{\infty}\frac{C_nO_n(2)}{4^n n^2} &= 8-7\zeta(3)+\tfrac{45}{2}\zeta(4)-\tfrac{9}{2}\zeta(2)^2-14\zeta(3)\log 2;\\
\sum_{n=0}^{\infty}\frac{C_n}{4^n (n+2)^2} &= \tfrac{8}{3}\log 2-\tfrac{14}{9};\\
\sum_{n=1}^{\infty}\frac{C_nO_n}{4^n (n+2)^2} &= 4\zeta(2)+\tfrac{64}{9}\log 2-\tfrac{308}{27}.
\end{aligned}$$

We remark that $\sum_{n=1}^{\infty} C_nO_n/4^n = 4$ was proved in [6].

Taking different rational values of a in Theorem 1 produces Euler type sums which lie in larger subrings of the ring of periods. For example, when $a \equiv \frac{1}{4}, \frac{3}{4} \pmod{\mathbb{Z}}$ the evaluation of $S(k,r;p,a)$ also requires values of the Dirichlet beta function

$$\beta(s) := \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s} \qquad (\Re(s)>1).$$

It is known that odd beta values $\beta(2j-1)$ are rational multiples of π^{2j-1}, while the algebraic nature of even beta values $\beta(2j)$ are unknown.

Corollary 6. *If $a \in \frac{1}{4}+\mathbb{Z}$ or $a \in \frac{3}{4}+\mathbb{Z}$, then for integers r,k,p with $k \geq 0$ and $0 \leq r \leq p$ we have*

$$\sum_{n=k}^{\infty}\frac{H_n^{[r]}(k)}{(n+a)^p} \in \mathbb{Q}[\log 2, \pi, \zeta(2j+1)_{j\geq 1}, \beta(2j)_{j\geq 1}]$$

where $\beta(s)$ is the Dirichlet beta function.

Proof. If $b \in \mathbb{N}$, then $B(a,b) \in \mathbb{Q}$, and we evaluate the sum using (3.2) and (3.3) as in Theorem 2. We need the identity $\psi(\frac{3}{4}), \psi(\frac{1}{4}) = \pm\frac{\pi}{2} - 3\log 2 - \gamma$ and Kölbig's evaluation [15] of $\psi^{(j)}(\frac{1}{4})$ and $\psi^{(j)}(\frac{3}{4})$ as $\mathbb{Q}$-linear combinations of Riemann zeta and Dirichlet beta values. □

Note that by duality (Corollary 1), the above result holds as well in the case where $a \in \mathbb{Z}$ and $r \in \frac{1}{4} + \mathbb{Z}$ or $r \in \frac{3}{4} + \mathbb{Z}$.

Example 3. These few examples involve the Catalan constant $G := \beta(2)$, whose arithmetic nature is unknown:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^{[1/4]}}{(n+1)^2} &= \sum_{n=1}^{\infty} \frac{1}{n(n+\frac{3}{4})^2} \\
&= \tfrac{128}{27} + \tfrac{32}{3}G - \tfrac{4}{3}\pi^2 - \tfrac{16}{3}\log 2 + \tfrac{8}{9}\pi; \\
\sum_{n=2}^{\infty} \frac{H_n^{[1/4]}(2)}{(n+1)^2} &= \sum_{n=1}^{\infty} \frac{1}{n(n+\frac{3}{4})^3} \\
&= \tfrac{4}{3}\pi^3 - \tfrac{16}{9}\pi^2 + \tfrac{32}{27}\pi - \tfrac{112}{3}\zeta(3) - \tfrac{64}{9}\log 2 + \tfrac{128}{9}G + \tfrac{256}{27}; \\
\sum_{n=1}^{\infty} \frac{H_n^{[1/4]}}{(n+1)^3} &= \sum_{n=2}^{\infty} \frac{H_{n-1}}{n(n+\frac{3}{4})^2} \\
&= \tfrac{2}{3}\pi^3 + (4\log 2 - \tfrac{62}{27})\pi^2 + (\tfrac{16}{3}G - \tfrac{8}{3}\log 2 + \tfrac{64}{27})\pi - \tfrac{112}{3}\zeta(3) \\
&\quad\ +8\log^2 2 - (32G + \tfrac{128}{9})\log 2 + \tfrac{64}{3}G + \tfrac{256}{27}.
\end{aligned}$$

Sums of the form $\sum_{n=0}^{\infty} \frac{H_n}{(n+a)^p(n+b)^q}$ similar to the dual of the last example were treated in [19, 25, 26, 27]. The hyperharmonic numbers $H_n^{[1/4]}(k)$ appearing above have the form

$$H_n^{[1/4]} = (-1)^n \binom{-1/4}{n} \sum_{j=1}^{n} \frac{4}{4j-3},$$

$$H_n^{[1/4]}(2) = (-1)^n \binom{-1/4}{n} \sum_{0<i<j\le n} \frac{16}{(4i-3)(4j-3)}.$$

Similar sums involving binomial coefficients $(-1)^n\binom{p/q}{n}$ and harmonic numbers and alternating harmonic numbers were evaluated in [3].

3.2. Two Non-integer Parameters

The evaluation of $S(k, r; p, a)$ by Theorem 1 is often simpler when both a and r are outside $\mathbb{Z}$, where $B(a, 1-r)$ is analytic. Here we give a general description of these values in the case of rational $a \equiv r \pmod{\mathbb{Z}}$.

Theorem 3. *Suppose $a, r \in \mathbb{Q} \setminus \mathbb{Z}$ with $Na \in \mathbb{Z}$ and $a \equiv r \pmod{\mathbb{Z}}$. Then for integers k, p with $p > r$ we have*

$$\sum_{n=k}^{\infty} \frac{H_n^{[r]}(k)}{(n+a)^p} \in \pi\mathbb{Q}(\zeta+\zeta^{-1})[\psi(a)+\gamma, \psi^{(j)}(a)_{j\ge1}, \pi^2, \zeta(2j+1)_{j\ge1}],$$

where ζ *is a primitive root of unity of order* $\operatorname{lcm}(2N, 4)$.

Proof. If $Na \in \mathbb{Z}$, then $2\sin a\pi = (i\omega) + (i\omega)^{-1}$, where ω is a primitive $2N$-th root of unity, and $\zeta = i\omega$ is a root of unity of order $\operatorname{lcm}(2N, 4)$. In the case where $a = r$, the reflection formula $\Gamma(a)\Gamma(1-a) = \pi \csc \pi a$ shows that $B(a, 1-r) \in \pi\mathbb{Q}(\zeta + \zeta^{-1})$. The proof in the case where $a + (1-r)$ is a general integer then follows from (3.2) and (3.3) as in the proof of Theorem 2. □

Example 4. For general rational arguments, Gauß [13] showed that

$$\psi\left(\tfrac{m}{N}\right) + \gamma = -\log 2N - \tfrac{\pi}{2}\cot\left(\tfrac{m\pi}{N}\right) + 2\sum_{j=1}^{[(N-1)/2]} \cos\left(\tfrac{2\pi jm}{N}\right)\log\sin\left(\tfrac{j\pi}{N}\right),$$

expressing $\psi(a)+\gamma$ as a linear combination of π and logarithms of algebraic numbers, with algebraic coefficients in $\mathbb{Q}(\zeta + \zeta^{-1})$. We give here only a few simple examples of this theorem with $N = 3, 4$:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^{[1/3]}}{(n+\frac{1}{3})^2} &= \sum_{n=1}^{\infty} \frac{H_n^{[2/3]}}{(n+\frac{2}{3})^2} = \frac{\pi}{\sqrt{3}}\left(\frac{9}{2}\log^2 3 - \frac{\pi^2}{2}\right);\\
\sum_{n=1}^{\infty} \frac{H_n^{[1/4]}}{(n+\frac{1}{4})^2} &= \sum_{n=1}^{\infty} \frac{H_n^{[3/4]}}{(n+\frac{3}{4})^2} = \pi\sqrt{2}\left(9\log^2 2 - \frac{5\pi^2}{12}\right);\\
\sum_{n=1}^{\infty} \frac{H_n^{[1/4]}}{(n+\frac{1}{4})^3} &= \sum_{n=2}^{\infty} \frac{H_n^{[3/4]}(2)}{(n+\frac{3}{4})^2}\\
&= \pi\sqrt{2}\big(12G\log 2 - 2\pi G + \tfrac{27}{2}\log^3 2 + \tfrac{9}{4}\pi\log^2 2\\
&\qquad + \tfrac{3}{8}\pi^2\log 2 - \zeta(3) - \tfrac{17}{48}\pi^3\big).
\end{aligned}$$

For general rational parameters with $a \equiv r \pmod{\mathbb{Z}}$, Theorem 3 typically expresses the series $S(k, r; p, a)$ as rather complicated combinations of less familiar constants. However, in the case $N = 2$, the preceding theorem takes a much simpler form. For example, in a recent article [30], it was shown using contour integration that

$$\sum_{n=0}^{\infty} \frac{n\binom{2n}{n}}{4^n(n-\frac{1}{2})^p} \in \mathbb{Q}[\log 2, \pi, \zeta(2j+1)_{j\geq 1}]$$

for any integer $p > 1$. By (3.11), this is the sum $\frac{1}{2}S(0, \frac{3}{2}; p, \frac{1}{2})$, whose dual is

$$\tfrac{1}{2}S(p-1, \tfrac{1}{2}; 1, -\tfrac{1}{2}) = \frac{1}{2}\sum_{n=p-1}^{\infty} \frac{\binom{2n}{n}O_n(p-1)}{4^n(n-\frac{1}{2})}.$$

These sums are actually part of a large family of Euler-Apéry sums which lie in this ring, as we now show.

Corollary 7. *If $a, r \in \frac{1}{2} + \mathbb{Z}$, then for integers $k \geq 0$ and $p > r$ we have*

$$\sum_{n=k}^{\infty} \frac{H_n^{[r]}(k)}{(n+a)^p} \in \pi\mathbb{Q}[\log 2, \pi^2, \zeta(2j+1)_{j\geq 1}].$$

In particular, if $a \in \frac{1}{2} + \mathbb{Z}$, then for integers $k \geq 0$ and $p > m \geq 0$ we have

$$\sum_{n=k}^{\infty} \frac{\binom{n}{m}\binom{2n}{n} O_n(k)}{4^n (n+a)^p} \in \pi\mathbb{Q}[\log 2, \pi^2, \zeta(2j+1)_{j\geq 1}],$$

and for integers $k \geq 0$ and $p, m > 0$, we have

$$\sum_{n=k}^{\infty} \frac{\binom{2n}{n} O_n(k)}{4^n \binom{n+m}{m} (n+a)^p} \in \mathbb{Q} + \pi\mathbb{Q}[\log 2, \pi^2, \zeta(2j+1)_{j\geq 1}].$$

Proof. When $a, r \in \frac{1}{2} + \mathbb{Z}$ we have $B(a, b) \in \pi\mathbb{Q}$, so the first statement follows from Theorem 3 in the case $N = 2$. The remaining statements follow from the reduction formulas for orders $r = \frac{1}{2} \pm m$ as in the proof of Corollary 4; in the last statement, the reduction formula (3.13) expresses $S(k, \frac{1}{2} - m, p, a - m)$ recursively in terms of the stated sums, except for the terms $0 \leq n < m$, whose sum lies in $\mathbb{Q}$. □

Example 5. We first record a few examples with $a = r = \frac{1}{2}$, emphasizing the duality in this family; the third of these is self-dual:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\binom{2n}{n} O_n}{4^n (n+\frac{1}{2})} &= 2\pi \log 2 = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (n+\frac{1}{2})^2}; \\
\sum_{n=2}^{\infty} \frac{\binom{2n}{n} O_n(2)}{4^n (n+\frac{1}{2})} &= \pi(\zeta(2) + 2\log^2(2)) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (n+\frac{1}{2})^3}; \\
\sum_{n=1}^{\infty} \frac{\binom{2n}{n} O_n}{4^n (n+\frac{1}{2})^2} &= \pi\left(4\log^2 2 - \zeta(2)\right); \\
\sum_{n=2}^{\infty} \frac{\binom{2n}{n} O_n(2)}{4^n (n+\frac{1}{2})^2} &= \pi\left(4\log^3 2 - \zeta(3)\right) = \sum_{n=1}^{\infty} \frac{\binom{2n}{n} O_n}{4^n (n+\frac{1}{2})^3}; \\
\sum_{n=3}^{\infty} \frac{\binom{2n}{n} O_n(3)}{4^n (n+\frac{1}{2})^2} &= \sum_{n=1}^{\infty} \frac{\binom{2n}{n} O_n}{4^n (n+\frac{1}{2})^4} \\
&= \pi\left(2\zeta(3)\log 2 + 2\zeta(2)\log^2 2 + \tfrac{8}{3}\log^4 2 - \zeta(4) - \zeta(2)^2\right).
\end{aligned}$$

Remark. In the case $a = r = \frac{1}{2}$, $m = 0$, the sums of this corollary were recently shown [38, Corollary 9.8] to lie in the $\mathbb{Q}$-vector space spanned by imaginary parts of colored multiple zeta values of level 4. That space contains elements, such as Dirichlet beta values $\beta(2j)$, which the ring of this corollary ostensibly does not; thus Corollary 7 includes a refinement of that result for these sums. For the first example above, we prefer our value $2\pi \log 2$ to the value

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n} O_n}{4^n (n + \frac{1}{2})} = 16 \left(G + \Im \left(\sum_{n>m>0} \frac{i^{m-n}}{nm} \right) \right)$$

which was given in [38, Example 9.9], not only for its simplicity and explicity, but for its clarity emphasizing that the sum lies in a $\mathbb{Q}$-vector space presumably not containing the Catalan constant $G := \beta(2)$. Similarly, we prefer the value $4\pi \log^2 2 - \pi\zeta(2)$ for the third example above to the definite integral given in [38, Example 9.9] for the value of that sum.

Some of these examples may be combined with other known evaluations; for example, in [7] and in [25] we find the values

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n} O_n^{(2)}}{4^n (n + \frac{1}{2})} = \frac{\pi^3}{6} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n} O_n^{(2)}}{4^n (n + \frac{1}{2})^2} = \pi \left(\frac{\pi^2 \log 2}{3} - \frac{3\zeta(3)}{2} \right),$$

respectively, where $O_n^{(2)}$ denotes the generalized harmonic number (1.5). Since $O_n(2) = \frac{1}{2}(O_n^2 - O_n^{(2)})$, we may combine these with the second and fourth examples above to get

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n} O_n^2}{4^n (n + \frac{1}{2})} = \frac{\pi^3}{2} + 4\pi \log^2 2$$

and

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n} O_n^2}{4^n (n + \frac{1}{2})^2} = \pi \left(\frac{\pi^2 \log 2}{3} - \frac{7}{2}\zeta(3) + 8 \log^3 2 \right),$$

which also lie in the ring indicated in this corollary.

Taking $(r, a) = (-\frac{1}{2}, \frac{1}{2})$ gives another family of series, with Catalan numbers and odd harmonic numbers, having simple expression in this same ring.

Corollary 8. *For any integers $k \geq 0$, $p > 0$, we have*

$$\sum_{n=k}^{\infty} \frac{C_n O_n(k)}{4^n (n + \frac{1}{2})^p} - (-2)^p 2^{k+1} \in \pi\mathbb{Q}[\log 2, \pi^2, \zeta(2j+1)_{j \geq 1}].$$

Proof. Taking $a = r = \frac{1}{2}$, we have $B(\frac{1}{2}, \frac{1}{2}) = \pi$, and we evaluate the sums $S(k, -\frac{1}{2}, p, -\frac{1}{2}) \in \pi\mathbb{Q}[\log 2, \pi^2, \zeta(2j+1)_{j \geq 1}]$ using Corollary 7. Iterating the reduction formula (3.15) gives

$$4^{-n} C_n O_n(k) = -2 \sum_{j=0}^{k} 2^j H_{n+1}^{[-1/2]}(k - j),$$

so dividing by $(n+\frac{1}{2})^p$ and summing over $n \ge 0$ yields

$$\sum_{n=k}^{\infty} \frac{C_n O_n(k)}{4^n (n+\frac{1}{2})^p} = 2^{k+1}(-2)^{-p} - 2\sum_{j=0}^{k} 2^j S(k-j, -\tfrac{1}{2}; p, -\tfrac{1}{2}),$$

since the $n=0$ term of $S(0,-\frac{1}{2};p,-\frac{1}{2})$ is $(-2)^p$. The sum on the right lies in the indicated ideal by the first statement of Corollary 7. □

Example 6.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{C_n}{4^n (n+\frac{1}{2})^2} - 8 &= 4\pi(\log 2 - 1);\\
\sum_{n=0}^{\infty} \frac{C_n}{4^n (n+\frac{1}{2})^3} + 16 &= 2\pi(2\log^2 2 - 4\log 2 + 4 + \zeta(2));\\
\sum_{n=1}^{\infty} \frac{C_n O_n}{4^n (n+\frac{1}{2})} + 8 &= 4\pi \log 2;\\
\sum_{n=1}^{\infty} \frac{C_n O_n}{4^n (n+\frac{1}{2})^2} - 16 &= 2\pi(4\log^2 2 - 4\log 2 - \zeta(2));\\
\sum_{n=2}^{\infty} \frac{C_n O_n(2)}{4^n (n+\frac{1}{2})} + 16 &= 2\pi(\zeta(2) + 2\log^2 2);\\
\sum_{n=2}^{\infty} \frac{C_n O_n(2)}{4^n (n+\frac{1}{2})^2} - 32 &= 2\pi(4\log^3 2 - 4\log^2 2 - \zeta(3) - 2\zeta(2)).
\end{aligned}$$

The next corollary gives an explicit version of the last statement of Corollary 7 for general $m \in \mathbb{N}$, in the special case $p=2$ and $k=0,1$.

Corollary 9. *For every positive integer m we have*

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n \binom{n+m}{m} (n+\frac{1}{2})^2} = \frac{4^m \pi}{\binom{2m}{m}}(2\log 2 - O_m) - \sum_{q=1}^{m} \frac{(-4)^q \binom{m}{q}}{\binom{2q}{q}(q-\frac{1}{2})^2}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\binom{2n}{n}(O_n - O_m)}{4^n \binom{n+m}{m}(n+\frac{1}{2})^2} &= \frac{4^m \pi}{\binom{2m}{m}}((O_m - 2\log 2)^2 - \zeta(2))\\
&\quad + \sum_{q=1}^{m} \frac{(-4)^q \binom{m}{q}(O_m - O_q)}{\binom{2q}{q}(q-\frac{1}{2})^2}.
\end{aligned}$$

Proof. These express the values $S(k, \frac{1}{2}-m, 2, \frac{1}{2}-m)$ for $k=0,1$. Here we have $B(a, 1-r) = (-1)^m \pi$, $P_{1,0}(a,b) = O_m - 2\log 2$, and $P_{1,1}(a,b) = (O_m - 2\log 2)^2 - \zeta(2)$ in Theorem 2, giving the evaluations. In each statement above, (2.12) and the reduction formula (3.13) expresses $S(k, \frac{1}{2}-m, 2, \frac{1}{2}-m)$ as $(-1)^m\binom{2m}{m}4^{-m}$ times the infinite series on the left minus the finite sum on the right. □

When none of a, r, $a-r$ are integers, the sum $S(k, r; p, a)$ may also be evaluated directly from (3.2). For example, for $a \in \{\frac{1}{4}, \frac{3}{4}\}$ and $b = \frac{1}{2}$, we have $B(\frac{1}{4}, \frac{1}{2}) = 2\varpi$ and $B(\frac{3}{4}, \frac{1}{2}) = \frac{\pi}{2\varpi}$, where ϖ denotes the elliptic integral

$$\varpi := 2\int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

known as the lemniscate constant, thus producing the following corollary.

Corollary 10. *For integers $k \geq 0$ and $p > m \geq 0$ we have*

$$\sum_{n=k}^{\infty} \frac{\binom{n}{m}\binom{2n}{n} O_n(k)}{4^n (n+\frac{1}{4})^p} \in \varpi\mathbb{Q}[\log 2, \pi, \zeta(2j+1)_{j\geq 1}, \beta(2j)_{j\geq 1}],$$

$$\sum_{n=k}^{\infty} \frac{\binom{n}{m}\binom{2n}{n} O_n(k)}{4^n (n+\frac{3}{4})^p} \in \frac{\pi}{\varpi}\mathbb{Q}[\log 2, \pi, \zeta(2j+1)_{j\geq 1}, \beta(2j)_{j\geq 1}],$$

and for integers $k \geq 0$ and $p, m > 0$ we have

$$\sum_{n=k}^{\infty} \frac{\binom{2n}{n} O_n(k)}{4^n \binom{n+m}{m} (n+\frac{1}{4})^p} \in \mathbb{Q} + \varpi\mathbb{Q}[\log 2, \pi, \zeta(2j+1)_{j\geq 1}, \beta(2j)_{j\geq 1}],$$

$$\sum_{n=k}^{\infty} \frac{\binom{2n}{n} O_n(k)}{4^n \binom{n+m}{m} (n+\frac{3}{4})^p} \in \mathbb{Q} + \frac{\pi}{\varpi}\mathbb{Q}[\log 2, \pi, \zeta(2j+1)_{j\geq 1}, \beta(2j)_{j\geq 1}].$$

Proof. The first statement is immediate from Theorem 1, Equation (3.2), and Kölbig's evaluation [15] of the polygamma values $\psi^{(n)}(\frac{1}{4})$ and $\psi^{(n)}(\frac{3}{4})$ as $\mathbb{Q}$-linear combinations of $\zeta(n)$ and $\beta(n)$. The remaining statements follow from the reduction formulas (3.13) and (3.14) as in Corollary 4. As in Corollary 7, in the last two statements above the reduction formula (3.13) expresses $S(k, \frac{1}{2}-m, p, a-m)$ recursively in terms of the stated sums, except for the terms $0 \leq n < m$, whose sum lies in $\mathbb{Q}$. □

Example 7.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\binom{2n}{n}O_n}{4^n(n+\frac14)^2} &= \sum_{n=1}^{\infty}\frac{H_n^{[3/4]}}{(n+\frac12)^2}=2\varpi\left(8G-\frac{\pi^2}{2}-\pi\log 2\right);\\
\sum_{n=2}^{\infty}\frac{\binom{2n}{n}O_n(2)}{4^n(n+\frac14)^2} &= \sum_{n=1}^{\infty}\frac{H_n^{[3/4]}}{(n+\frac12)^3}=\varpi\big(16\pi G-16G\log 2\\
&\quad +\pi\log^2 2+\pi^2\log 2+\tfrac34\pi^3-56\zeta(3)\big);\\
\sum_{n=1}^{\infty}\frac{\binom{2n}{n}O_n}{4^n(n+\frac34)^2} &= \sum_{n=1}^{\infty}\frac{H_n^{[1/4]}}{(n+\frac12)^2}\\
&= \frac{\pi}{2\varpi}\left(32-4\log 2+\pi\log 2-6\pi-\frac{\pi^2}{2}-8G\right).\\
\sum_{n=1}^{\infty}\frac{\binom{2n}{n}O_n}{4^n(n+\frac14)^4} &= \sum_{n=3}^{\infty}\frac{H_n^{[3/4]}(3)}{(n+\frac12)^2}=\varpi\big(128G^2-16\pi G\log 2-\tfrac53\pi^3\log 2\\
&\quad -56\pi\zeta(3)-\tfrac56\pi^4+256\beta(4)\big).
\end{aligned}$$

As a consequence of the previous corollary, we observe that certain products of these series with parameters $\frac14$, $\frac34$ lie in the ideal (π) of the ring containing the sums of Corollary 6.

Corollary 11. *For any $r_1, r_2 \in \frac12+\mathbb{Z}$, $a \in \frac14+\mathbb{Z}$, $b \in \frac34+\mathbb{Z}$, and integers $k_1, k_2 \geq 0$, $p_1 > r_1$, $p_2 > r_2$, we have*

$$S(k_1, r_1; p_1, a)S(k_2, r_2; p_2, b) \in \pi\mathbb{Q}[\log 2, \pi, \zeta(2j+1)_{j\geq 1}, \beta(2j)_{j\geq 1}].$$

In particular, for integers $k_1, k_2 \geq 0$ and $p_1, p_2 > 0$ we have

$$\left(\sum_{n=k_1}^{\infty}\frac{\binom{2n}{n}O_n(k_1)}{4^n(n+\frac14)^{p_1}}\right)\left(\sum_{n=k_2}^{\infty}\frac{\binom{2n}{n}O_n(k_2)}{4^n(n+\frac34)^{p_2}}\right) \in \pi\mathbb{Q}[\log 2, \pi, \zeta(2j+1)_{j\geq 1}, \beta(2j)_{j\geq 1}].$$

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