#A83 INTEGERS 26 (2026)

ON EXTENSIONS OF TRIANGULAR ARRAYS TO NEGATIVE ROW INDICES AND THEIR GENERATING FUNCTIONS

Lars Palapies
Dr. Nagler & Company GmbH, Schnaittenbach, Germany
lars.palapies@nagler-company.com

Received: 12/12/25, Accepted: 6/3/26, Published: 6/26/26

Abstract

In this paper, we construct extensions of triangular arrays $(a_{n,k})_{n,k\geq 0}$, which are generated by a linear recurrence, to $n<0$, and generalize Knuth's relation between Stirling numbers of the first and second kind with $n,k\in\mathbb{Z}$. For two classes of such sequences with $n<0$, the exponential generating function $\bar{F}$ is derived, including a class where $\bar{F}$ is expressed as a Hankel transform involving the usual exponential generating function for the $a_{n,k}$ with $n\geq 0$.

1. Introduction

Let $\mathcal{L}$ denote the class of all sequences of numbers $a_{n,k}$, with $n,k\geq 0$, that satisfy a linear recurrence relation,

$$a_{n,k}=(\alpha n+\beta k+\gamma)a_{n-1,k}+(\alpha' n+\beta' k+\gamma')a_{n-1,k-1}, \tag{1}$$

for some $(\alpha,\beta,\gamma;\alpha',\beta',\gamma')\in\mathbb{Z}^6$, along with initial conditions $a_{n,-1}:=0$, for $n\geq 0$, and $a_{0,k}:=\delta_{k=0}$, for $k\geq 0$, where $\delta_{x=y}:=\begin{cases}1, \text{if } x=y,\\ 0, \text{else,}\end{cases}$ denotes the Kronecker delta. We note that $\mathcal{L}$ contains many sequences of combinatorial interest, namely binomial coefficients and various types of Stirling, Eulerian, and Lah numbers, among others; see Table 1 in [2] and Table 1 in [20]. The problem of developing a general theory of all solutions $a_{n,k}$ of this recurrence, and writing them in terms of certain "fundamental solutions", is the subject of Problem 89 in [13, Ch. 6]. Even though this problem still remains unsolved in full generality, explicit formulas of the $a_{n,k}$ as finite sums of binomial coefficients and Stirling numbers of the first and second kind, as well as combinatorial interpretations, are available for various subclasses of $\mathcal{L}$; see [16] and the references therein.

DOI: 10.5281/zenodo.20931322

A standard approach for determining the generating functions

$$F(u,x) := \sum_{n\geq 0}\sum_{k\geq 0} a_{n,k} u^k \frac{x^n}{n!}$$

of the $a_{n,k} \in \mathcal{L}$, is to translate the linear recurrence scheme into a class of partial differential equations (PDEs) solved by these F. More specifically, an equivalent way to formulate the above dependencies among the $a_{n,k}$ is to specify F as the unique solution of

$$\frac{\partial}{\partial x} f(u,x) = (\Omega_x + \Omega_u) f(u,x), \tag{2}$$

where $\Omega_x = (\alpha + \alpha' u)(1 + x\frac{\partial}{\partial x})$ and $\Omega_u = \beta u \frac{\partial}{\partial u} + \gamma + \beta' u(1 + u\frac{\partial}{\partial u}) + \gamma' u$, and with boundary condition $f(u,0) = 1$. By substituting a power series expansion for f into the PDE, applying the analytical operations term by term, and comparing coefficients, it follows that these two views are indeed equivalent; see [21, Ch. 2]. The solution F for arbitrary parameters is expressed as an implicitly defined function in [2] and [22].

In contrast to this general framework, extensions of combinatorial numbers $a_{n,k}$ to $n < 0$ are so far only available for specific examples, such as binomial coefficients [15] and Stirling numbers of various types ([3], [18], [4], [8]). In the present paper, we define extended initial conditions and continue the $a_{n,k}$ to $n < 0$ whenever this is uniquely possible. This generic approach subsumes the constructions in the current literature. For instance, it allows us to extend Knuth's relation between Stirling numbers of both kinds from [14],

$$\begin{bmatrix} -n \\ -k \end{bmatrix} = \begin{Bmatrix} k \\ n \end{Bmatrix},$$

where $n, k \in \mathbb{Z}$, to more general pairs of combinatorial numbers.

The majority of the paper is then dedicated to the discussion of the exponential generating functions

$$\bar{F}(u,x) := \sum_{n\geq 0}\sum_{k\geq 0} a_{-n,k} u^k \frac{x^n}{n!}.$$

First, we observe that $\bar{F}$ is determined as the solution of a second-order variant of (2). Since this PDE is in general difficult to solve explicitly, we focus on two important special cases known from the literature on linear recurrences for $n, k \geq 0$, namely $\beta = \beta' = 0$ and $\alpha' = 0$; see [2], [20], [22]. In the first case, $\bar{F}$ becomes a generalized hypergeometric function, while in the second case, under certain integrability and smoothness assumptions, and up to a change of variables, it is shown that $\bar{F}$ can be written as a Hankel transform of F, noting that for the latter an analytic expression of the general form is known; see Equation (7) below. This relation is reminiscent of a classical identity for univariate a_n, with $n \in \mathbb{Z}$, where

$\bar{F}(x) := \sum_{n\geq 1} a_{-n}x^n$ is also traced back to $F(x) := \sum_{n\geq 0} a_n x^n$, namely

$$\bar{F}(x) = -F(1/x),$$

which holds if the a_n are generated by a linear recurrence with constant coefficients; see [19, Proposition 4.2.3].

For convenience, we provide in Table 1 various definitions that will be used throughout this paper.

Name	$a_{n,k}$	$(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$
Binomial coefficient	$\binom{n}{k}$	$(0, 0, 1; 0, 0, 1)$
Stirling number of the first kind	$s(n, k)$	$(-1, 0, 1; 0, 0, 1)$
Unsigned Stirling number of the first kind	$\left[{n \atop k}\right]$	$(1, 0, -1; 0, 0, 1)$
Stirling number of the second kind	$\left\{{n \atop k}\right\}$	$(0, 1, 0; 0, 0, 1)$
r-Stirling number of the first kind	$\left[{n+r \atop k+r}\right]_r$	$(1, 0, r-1; 0, 0, 1)$
r-Stirling number of the second kind	$\left\{{n+r \atop k+r}\right\}_r$	$(0, 1, r; 0, 0, 1)$
r-Whitney number of the second kind	$W_{m,r}(n, k)$	$(0, m, r; 0, 0, 1)$

Table 1: Combinatorial numbers $a_{n,k}$ generated by (1) and the corresponding parameters.

2. Continuation of Combinatorial Numbers

As discussed in [4] and [18] in the case of the Stirling numbers, to uniquely determine the $a_{n,k}$ when $n < 0$, one has to continue the initial conditions. In this section, we propose two such continuations which are seen to conform with and generalize other constructions in the literature. We then observe that these approaches are in a sense equivalent.

Definition 1. Assume that $\alpha(n + 1) + \beta k + \gamma \neq 0$, when $n < 0$ and $k \geq 0$. The *negative-positive* numbers $a_{n,k} = a^{NP}_{n,k}$ are defined as the continuation of the $a_{n,k}$ to $n, k \in \mathbb{Z}$, when the initial conditions are extended to $a_{n,-1} := 0$, for all $n \in \mathbb{Z}$, and the $a_{n,k}$, with $n < 0$ and $k \geq 0$, are generated by (1) rewritten as $a_{n,k} = \frac{a_{n+1,k} - (\alpha'(n+1)+\beta' k+\gamma')a_{n,k-1}}{\alpha(n+1)+\beta k+\gamma}$. Further, $a_{n,k} := 0$, for $k < -1$.

Definition 2. Assume that $\alpha'(n + 1) + \beta'(k + 1) + \gamma' \neq 0$, when $k \leq n < 0$. The *negative-negative* numbers $a_{n,k} = a^{NN}_{n,k}$ are defined as the continuation of the $a_{n,k}$ to $n, k \in \mathbb{Z}$, when the initial conditions are extended to $a_{0,k} := \delta_{k=0}$, for all $k \in \mathbb{Z}$, and

$a_{n,0} := 0$, for $n < 0$, and the $a_{n,k}$, with $k \leq n < 0$, are generated by (1) rewritten as $a_{n,k} = \frac{a_{n+1,k+1} - (\alpha(n+1)+\beta(k+1)+\gamma)a_{n,k+1}}{\alpha'(n+1)+\beta'(k+1)+\gamma'}$. Further, $a_{n,k} := 0$, for $n > 0$ and $k < 0$, and $a_{n,k} := 0$, for $n < 0$ and $k > n$.

The nomenclature has been adopted from [3] and [4]. Our definitions recover the negative-positive and negative-negative Stirling numbers of the first kind constructed therein by continuing the generating function to negative n, and by extending the initial conditions and applying the recurrence scheme, respectively. Regarding Stirling numbers of the second kind, Definition 1 is not applicable due to the parameter restrictions dictated by the reversed recurrence[1], while Definition 2 yields the negative-negative numbers from the above references. On the other hand, the usual binomial coefficients $\binom{n}{k}$, for negative integers n, from e.g., [7] and [15] (see also Table 4), as well as the extended r-Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_r$, with $\Re(r) > 0$ and $r > 0$, from [8] and [17], respectively, are of negative-positive type.

To rule out the existence of the undesired zeros in Definition 1 or 2, it is clearly necessary that $\gamma \neq 0$ or $\gamma' \neq 0$, respectively. To find all eventual zeros in the given range of n and k, one may employ the following application of Bézout's identity.

Proposition 1. *Let $a, b \in \mathbb{Z}$, not both zero, with common divisor d, and $c \in \mathbb{Z}$ be a multiple of d. Then, all integral solutions (x, y) of $ax + by + c = 0$ are given as $x = -\frac{c}{d}u + m\frac{b}{d}$ and $y = -\frac{c}{d}v - m\frac{a}{d}$, where d is the greatest common divisor of a and b, (u, v) is any solution of $au + bv = d$, which can be obtained by the extended Euclidean algorithm, and m is an arbitrary integer.*

If instead $c \in \mathbb{Z}$ is not a multiple of d, then the equation has no integral solution.

The famous duality relation between extended Stirling numbers of both kinds,

$$\left[{-n \atop -k} \right] = \left\{ {k \atop n} \right\}, \tag{3}$$

for all $n, k \in \mathbb{Z}$, from [14], holds true if both sides are taken in the negative-negative sense. This identity is generalized in the next proposition.

Proposition 2. *Let $a_{n,k}$ and $b_{n,k}$ have the recurrence schemes $(\alpha, \beta, \gamma; 0, 0, 1)$, and $(\beta, \alpha, -\gamma; 0, 0, 1)$, respectively. When continued as in Definition 2, the numbers are related via*

$$a^{NN}_{-n,-k} = b^{NN}_{k-1,n-1},$$

for all $n, k \in \mathbb{Z}$.

[1] In [4], Table 6, this issue on the vertical line $k = 0$ is avoided by setting the initial condition $\left\{ {n \atop 0} \right\} = 0$, for all $n \in \mathbb{Z}$, contrary to the usual assumption $a_{0,0} = 1$ for linear recurrence schemes.

Proof. The statement clearly holds when $n \geq 0$ and $k \leq 0$, and when $n \leq 0$ and $k > 0$. For an inductive proof of the claim for $n, k > 0$, Definition 2 yields

$$\begin{aligned} a^{NN}_{-n,-k} &= a^{NN}_{-n+1,-k+1} - (\alpha(1-n) + \beta(1-k) + \gamma) a^{NN}_{-n,-k+1} \\ &= b^{NN}_{k-2,n-2} - (\alpha(1-n) + \beta(1-k) + \gamma) b^{NN}_{k-2,n-1} \\ &= b^{NN}_{k-1,n-1}. \end{aligned}$$

The case $n < 0$ and $k \leq 0$ is similarly seen by induction. ∎

As an application of the last proposition, consider the r-Stirling numbers, where the identity becomes

$$\begin{bmatrix} -n+r \\ -k+r \end{bmatrix}_r = \begin{Bmatrix} k-r \\ n-r \end{Bmatrix}_{1-r},$$

which indeed generalizes (3), since $\left\{ {n \atop k} \right\}_1 = \left\{ {n \atop k} \right\}_0$, for all $n, k \in \mathbb{Z}$.

The next proposition shows that both types of extensions are in fact dual to each other.

Proposition 3. *Let $\tilde{a}_{n,k} := a_{n,n-k}$, which satisfy a linear recurrence with parameters $(\alpha' + \beta', -\beta', \gamma'; \alpha + \beta, -\beta, \gamma)$. Then, we have*

$$\tilde{a}^{NP}_{n,k} = a^{NN}_{n,n-k},$$

for all $n, k \in \mathbb{Z}$. Conversely,

$$\tilde{a}^{NN}_{n,n-k} = a^{NP}_{n,k},$$

for all $n, k \in \mathbb{Z}$. In both equations, the left-hand side exists exactly when the right-hand side does, according to Definitions 1 and 2.

Proof. For the first claim, see Table 2 in [2]. Now, the first equality holds by definition when $n, k \geq 0$. Assuming that the parameters allow applying Definitions 1 and 2, both sides are seen to be zero when $k \leq -1$. For an inductive proof of the claim for arbitrary indices, it remains to consider $n < 0$ and $k \geq 0$. For these indices,

$$\begin{aligned} \tilde{a}^{NP}_{n,k} &= \frac{\tilde{a}^{NP}_{n+1,k} - ((\alpha+\beta)(n+1) - \beta k + \gamma)\tilde{a}^{NP}_{n,k-1}}{(\alpha'+\beta')(n+1) - \beta' k + \gamma'} \\ &= \frac{a^{NN}_{n+1,n-k+1} - ((\alpha+\beta)(n+1) - \beta k + \gamma) a^{NN}_{n,n-k+1}}{(\alpha'+\beta')(n+1) - \beta' k + \gamma'} \\ &= a^{NN}_{n,n-k}. \end{aligned}$$

The second equality of the proposition follows after interchanging the roles of $a_{n,k}$ and $\tilde{a}_{n,k}$, and due to the fact that $\tilde{\tilde{a}}_{n,k} = a_{n,k}$. The existence claims are evident from the above derivation. ∎

From now on, we focus on negative-positive extensions, and simply write $a_{n,k}$ instead of $a_{n,k}^{NP}$. Due to Proposition 3, this amounts to no loss of generality. In this sense, we define the exponential generating function,

$$\bar{F}(u,x) := \sum_{n\geq 0}\sum_{k\geq 0} a_{-n,k}u^k \frac{x^n}{n!}, \tag{4}$$

where the term $a_{0,0} = 1$ has been included to facilitate identifying this function by its analytical properties. Indeed, by definition and due to (1), $\bar{F}$ is determined as the unique solution of the PDE

$$f(u,x) = (\bar{\Omega}_x + \Omega_u)\frac{\partial}{\partial x} f(u,x), \tag{5}$$

with $\bar{\Omega}_x = -(\alpha + \alpha' u)x\frac{\partial}{\partial x} = \alpha + \alpha' u - \Omega_x$ and Ω_u, Ω_x as in (2), and boundary condition $f(u,0) = 1$. Since (5) constitutes a second-order PDE involving mixed derivatives, it is considerably more difficult to solve than (2). Also, if the double power series $\bar{F}$ does not converge on any bidisc $\{(u,x) \in \mathbb{C}^2 : |u| < \rho_1, |x| < \rho_2\}$, with $\rho_1, \rho_2 > 0$, then it satisfies (5) when the analytical operations are applied formally [21, Ch. 2.1], and the boundary condition is interpreted as $[x^0]f(u,x) = 1$.

In the following sections, we first discuss the easily solvable case $\beta = \beta' = 0$, and then the case $\alpha' = 0$, where $\bar{F}$ can be traced back to F, subject to some technical assumptions.

3. The Case $\beta = \beta' = 0$

Under the assumption $\beta = \beta' = 0$, the PDE (2) for F reduces to an ordinary differential equation (ODE), which is solved by

$$F(u,x) = (1-(\alpha+\alpha' u)x)^{-\frac{\alpha+\gamma+(\alpha'+\gamma')u}{\alpha+\alpha' u}} = {}_1F_0\left(1+\frac{\gamma+\gamma' u}{\alpha+\alpha' u}; -; (\alpha+\alpha' u)x\right),$$

with the generalized hypergeometric function ${}_1F_0\,(a; -; z) := \sum_{n\geq 0} a^{\bar{n}} \frac{z^n}{n!}$, and the rising factorial $x^{\bar{n}} := x(x+1)\cdots(x+n-1)$. Further, if $\alpha = \alpha' = 0$, one has the limiting case $F(u,x) = e^{(\gamma+\gamma' u)x}$.

Similarly, for the following theorems, we need ${}_0F_1\,(-; b; z) := \sum_{n\geq 0} \frac{1}{b^{\bar{n}}}\frac{z^n}{n!}$. The function is defined as long as $b \notin \mathbb{Z}^{\leq 0}$, and solves the generalized hypergeometric differential equation

$$f(z) = z\frac{d^2}{dz^2}f(z) + b\frac{d}{dz}f(z). \tag{6}$$

Theorem 1. *Assume that $\beta = \beta' = 0$ and that the $a_{-n,k}$, with $n, k \geq 0$, exist. If $\alpha = \alpha' = 0$, then we have*

$$\bar{F}(u,x) = e^{\frac{x}{\gamma+\gamma' u}}.$$

Otherwise, the generating function in (4) is given as

$$\bar{F}(u,x) = {}_0F_1\left(-; -\frac{\gamma+\gamma' u}{\alpha+\alpha' u}; -\frac{x}{\alpha+\alpha' u}\right).$$

If $\alpha = 0$, the representation in terms of ${}_0F_1$ has to be taken as an identity of formal power series. Else, both sides are locally holomorphic around $(u,x) = (0,0)$.

Proof. By assumption, the PDE for $\bar{F}$ in (5) becomes

$$f(u,x) = -(\alpha+\alpha' u)x\frac{\partial^2}{\partial x^2}f(u,x) + (\gamma+\gamma' u)\frac{\partial}{\partial x}f(u,x).$$

If $\alpha = \alpha' = 0$, the resulting first-order ODE is clearly solved by the claimed $\bar{F}(u,x)$. Else, holding the variable u fixed, and introducing $g(z)$ via $f(x) = g(-x/(\alpha+\alpha' u))$, we find that g solves (6), with $b = -\frac{\gamma+\gamma' u}{\alpha+\alpha' u}$. Taking $g(z) = {}_0F_1(-; b; z)$, we get the asserted expression, which indeed satisfies the boundary condition $f(u,0) = 1$. If $\alpha \neq 0$, the parameter restrictions from Definition 1 guarantee that $b \notin \mathbb{Z}^{\leq 0}$, for $|u|$ small enough. □

As a first example, let us consider $a_{n,k} = \left[{n+r \atop k+r}\right]_r$, with $r \leq 0$, which has parameters $(1, 0, r-1; 0, 0, 1)$. Theorem 1 yields

$$\sum_{n\geq 0}\sum_{k\geq 0}\left[{-n+r \atop k+r}\right]_r u^k \frac{x^n}{n!} = {}_0F_1(-; 1-r-u; -x).$$

Indeed, from $\frac{(-1)^n}{(1-r-u)^{\bar{n}}} = (-n+r-1)\frac{(-1)^{n+1}}{(1-r-u)^{\overline{n+1}}} + u\frac{(-1)^{n+1}}{(1-r-u)^{\overline{n+1}}}$, for $n \geq 0$, we see that the $a_{-n,k} = [u^k]\frac{(-1)^n}{(1-r-u)^{\bar{n}}}$ satisfy the recurrence of r-Stirling numbers of the first kind,

$$a_{-n,k} = (-n+r-1)a_{-n-1,k} + a_{-n-1,k-1}.$$

By taking $r = 0$, we get the generating function of the $s(-n,k) = (-1)^{n-k}\left[{-n \atop k}\right]$, $n \geq 0$, from [3]. The resulting $a_{n,k}$ are shown in Table 2. Further, for $n, r \geq 0$, we know from [5] that

$$\sum_{k\geq 0}\left[{n+r \atop k+r}\right]_r = (r+1)^{\bar{n}}.$$

By calculating the derivatives $\frac{\partial^n}{\partial x^n}F(u,x)|_{(u,x)=(1,0)} = \sum_{k\geq 0}\left[{n+r \atop k+r}\right]_r$, the identity is seen to hold for all $n \geq 0$ and $r \in \mathbb{Z}$. In addition, when the rising factorials are continued to $n < 0$ via $x^{\overline{n+1}} = (x+n)x^{\bar{n}}$, evaluating

$$\sum_{k\geq 0}\left[{-n+r \atop k+r}\right]_r u^k = \frac{(-1)^n}{(1-r-u)^{\bar{n}}} = (r+u)^{\overline{-n}}$$

at $u = 1$ proves the identity for $n, r < 0$.

Next, let $a_{n,k} = \left[{n \atop n-k}\right]$, with parameters $(0,0,1;1,0,-1)$; see Table 3. Now, applying (3) and Proposition 3, and then Theorem 1 yields the formal power series

$$\sum_{n\geq 0}\sum_{k\geq 0}\left[{-n \atop -n-k}\right]u^k\frac{x^n}{n!} = \sum_{n\geq 0}\sum_{k\geq 0}\left\{{n+k \atop n}\right\}u^k\frac{x^n}{n!} = {}_0F_1(-;1-1/u;-x/u).$$

Alternatively, this exponential generating function of the $\left\{{n \atop k}\right\}$ along the columns follows from the known ordinary power series $\sum_{k\geq 0}\left\{{n+k \atop n}\right\}u^k = \frac{1}{(-u)^n(1-1/u)^{\bar{n}}}$.

4. The Case $\alpha' = 0$

For combinatorial numbers $a_{n,k}$, with $n \geq 0$, the case $\alpha' = 0$ has been studied in [20] and [22]. From these references, we know that

$$F(u,x) = \sum_{n\geq 0}\sum_{k\geq 0} a_{n,k}u^k\frac{x^n}{n!} = \frac{(1-\alpha x)^{-1-\frac{\gamma}{\alpha}}}{\left(1+\frac{\beta'}{\beta}u(1-(1-\alpha x)^{-\frac{\beta}{\alpha}})\right)^{1+\frac{\gamma'}{\beta'}}}, \tag{7}$$

with 7 limiting cases, depending on which ones out of α, β, β' are zero.

Theorem 2. *Assume that $\alpha' = 0$ and that the $a_{-n,k}$, with $n, k \geq 0$, exist. When $\alpha \neq 0$, let $\eta = \frac{1}{\alpha}$; otherwise, let $\eta = \pm\infty$, with the sign of $-\gamma$. Then, for $(u,x) \in \mathcal{D}$, with $\mathcal{D}$ specified below, the generating function in (4) is given as*

$$\bar{F}(u,x) = 1 - x\int_0^\eta F(u,y)_0F_1\left(-;2;-xy\right)dy.$$

Let $\mathcal{I}$ denote the integration range and $\mathcal{D} \subseteq \mathbb{C}^2$ be an open, connected neighborhood of $(0,0)$. Also, for all $(u,x) \in \mathcal{D}$, the following conditions hold:

(a) There exists a $\delta > 0$ such that $y \mapsto e^{2|\Im(\sqrt{yz})|}F(u,y)$ has an integrable majorant on $\mathcal{I}$, valid for all $z \in B_\delta(x) = \{z : |z-x| < \delta\}$.

(b) There exists a $\delta > 0$ such that

  (i) $y \mapsto e^{2|\Im(\sqrt{yx})|}F(v,y)$ has an integrable majorant on $\mathcal{I}$, valid for all $v \in B_\delta(u) = \{v : |v-u| < \delta\}$;

  (ii) $v \mapsto F(v,y)$ is analytic in $B_\delta(u)$ for almost all $y \in \mathcal{I}$.

(c) When $y \in \mathcal{I}$ approaches η, we have $\lim_{y\to\eta}(1-y\alpha)F(u,y) = 0$.

(d) $y \mapsto F(u,y)$ is continuously differentiable on $\mathcal{I}$.

Proof. Let $f(u,x) := 1 - x\int_0^\eta F(u,y)_0F_1(-;2;-xy)dy$. As the boundary condition $f(u,0) = 1$ of (5) clearly holds, it remains to validate that f satisfies the PDE.

Since our conditions do not rule out that $\lim_{y\to\eta} F(u,y) = \infty$, we first localize f by integrating up to some fixed ξ between 0 and η, and consider

$$1 - x\int_0^\xi F(u,y)_0F_1(-;2;-xy)dy = \chi(\xi) - \int_0^\xi {}_0F_1(-;1;-xy)\frac{\partial}{\partial y}F(u,y)dy, \quad (8)$$

where $\chi(y) := F(u,y)_0F_1(-;1;-xy)$. The equality follows by integration by parts using (d), and $\frac{d}{dz}{}_0F_1(-;1;z) = {}_0F_1(-;2;z)$.

By (2), with y instead of x and $\Omega_y = \alpha(1 + y\frac{\partial}{\partial y})$, and upon applying another integration by parts, (8) equals

$$\begin{aligned}\chi(\xi) - \int_0^\xi {}_0F_1(-;1;-xy)(\Omega_y + \Omega_u)F(u,y)dy \\ = \chi(\xi) - \psi(\xi) + \alpha\int_0^\xi F(u,y)\frac{\partial}{\partial y}y_0F_1(-;1;-xy)dy \\ - \int_0^\xi {}_0F_1(-;1;-xy)(\Omega_u + \alpha)F(u,y)dy,\end{aligned}$$

where $\psi(y) := \alpha yF(u,y)_0F_1(-;1;-xy)$.

Taking the limit $\xi \to \eta$, the right-hand side of the above expression equals $f(u,x)$. Therefore, since we also have $\chi(\xi) - \psi(\xi) \to 0$, due to (c), and by the product rule, the difference becomes

$$\alpha\int_0^\eta F(u,y)y\frac{\partial}{\partial y}{}_0F_1(-;1;-xy)dy - \int_0^\eta {}_0F_1(-;1;-xy)\Omega_uF(u,y)dy.$$

Replacing $\alpha y\frac{\partial}{\partial y}$ with $\alpha x\frac{\partial}{\partial x} = -\overline{\Omega}_x$ in the first term and pulling the differential operators out of the integrals yields the equivalent expression

$$-(\overline{\Omega}_x + \Omega_u)\int_0^\eta F(u,y)_0F_1(-;1;-xy)dy.$$

The interchanges of integration and differentiation are justified by the Leibniz integral rule (see, for example, [11, (13.8.6) (iii)]). For the integrability conditions, we utilize (a), (b), and the fact that $|{}_0F_1(-;b;z)| \le e^{2|\Im(\sqrt{-z})|}$, for $b \ge \frac{1}{2}$, from [1, Equation (9.1.62)].

Now, by means of

$${}_0F_1(-;1;z) = \frac{d}{dz}z\frac{d}{dz}{}_0F_1(-;1;z) = \frac{d}{dz}z_0F_1(-;2;z),$$

and then interchanging operations as before, we obtain

$$-(\overline{\Omega}_x + \Omega_u)\int_0^\eta F(u,y)\frac{\partial}{\partial x}x_0F_1(-;2;-xy)dy = (\overline{\Omega}_x + \Omega_u)\frac{\partial}{\partial x}f(u,x). \qquad \square$$

Note that the assumption that $\eta = \pm\infty$ has the sign of $-\gamma$, if $\alpha = 0$, was not used explicitly in the proof of Theorem 2, but is required for the finiteness of the integral around $(u, x) = (0, 0)$, since $F(0, y) = e^{\gamma y}$, when $\alpha = 0$; see Appendix and Equation (2.15) in [2].

The determination of a suitable $\mathcal{D}$ depending on the parameters is slightly subtle but straightforward. For instance, for the general F in (7), where only $\alpha\beta\beta' \neq 0$, we need $\frac{\beta}{\alpha}, \frac{\gamma}{\alpha} < 0$, and may take $(u, x) \in \mathcal{D} = (\mathbb{C}\backslash\mathcal{B})\times\mathbb{C}$, where $\mathcal{B} := \{-\frac{\beta}{\beta'}t | t \geq 1\} \subseteq \mathbb{R}$.

Further, we remark that due to the identity

$$ {}_0F_1\left(-; b; z\right) = \Gamma(b)\sqrt{-z}^{\,1-b} J_{b-1}(2\sqrt{-z}), $$

where J_ν denotes a Bessel function of the first kind and order ν, Theorem 2 allows us to write $\bar{F}$ as a Hankel transform, which is defined for functions f on $(0, \infty)$ as

$$ \mathcal{H}_n\{f(y)\}(\kappa) := \int_0^\infty y J_n(\kappa y) f(y) dy; $$

see [10]. Let now $\sigma = \pm 1$ have the sign of η. Then,

$$ \bar{F}(u, x) = 1 - \sqrt{\sigma x}\mathcal{H}_1\left\{\frac{F(u, \sigma y^2/4)}{y}\chi_{(0,2\sqrt{|\eta|})}(y)\right\}(\sqrt{\sigma x}), $$

with the indicator function $\chi_A(y) := \begin{cases} 1, & \text{if } y \in A, \\ 0, & \text{else.} \end{cases}$

In some cases, Theorem 2 yields an expression for $\bar{F}$ in analytical form or at least as a single series, by consulting an integral table. For example, from [12, Equation 6.643.1], we extract the special case

$$ \int_0^\infty e^{-\alpha y} \frac{J_1(2\sqrt{-xy})}{\sqrt{y}} dy = \frac{1 - e^{\frac{x}{\alpha}}}{\sqrt{-x}}. $$

When $a_{n,k} = \binom{n}{k}$ (see Table 4), we have $F(u, x) = e^{(1+u)x}$, such that the last formula implies

$$ \bar{F}(u, x) = 1 - \sqrt{-x} \int_0^\infty e^{-(1+u)y} \frac{J_1(2\sqrt{-xy})}{\sqrt{y}} dy = e^{\frac{x}{1+u}}, $$

which is also contained in Theorem 1.

Next, for $a_{n,k} = W_{m,r}(n, k)$ (see Table 5), with non-zero integers m, r having the same sign, we know from [6] that $F(u, x) = e^{rx + u\frac{e^{mx}-1}{m}}$. Now,

$$ \begin{aligned} \bar{F}(u, x) &= 1 - \sqrt{\sigma x} e^{-\frac{u}{m}} \sum_{j\geq 0} \frac{(\frac{u}{m})^j}{j!} \int_0^\infty e^{\sigma(jm+r)y} \frac{J_1(2\sqrt{\sigma xy})}{\sqrt{y}} dy \\ &= e^{-\frac{u}{m}} \sum_{j\geq 0} \frac{(\frac{u}{m})^j}{j!} e^{\frac{x}{jm+r}}. \end{aligned} $$

By expanding, it is readily verified that the explicit formula for $n, k \geq 0$ from [6],

$$W_{m,r}(n,k) = \frac{1}{m^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (mj+r)^n,$$

still holds for $n < 0$.

By evaluating the row sums $\sum_{k\geq 0} W_{m,r}(n,k)u^k$ at $u = 1$, we recover the formula for the r-Dowling numbers from [9],

$$\mathcal{D}_{m,r}(n) = \sum_{k\geq 0} W_{m,r}(n,k) = e^{-\frac{1}{m}} \sum_{j\geq 0} \frac{(mj+r)^n}{m^j j!},$$

which may be taken as a natural extension of the $\mathcal{D}_{m,r}(n)$ to all $n \in \mathbb{Z}$.

5. Summary

We have presented a general approach to systematically continue combinatorial numbers $a_{n,k}$ generated by linear recurrences to $n < 0$. In the cases $\beta = \beta' = 0$ and $\alpha' = 0$, the corresponding exponential generating function can be obtained in analytical form or as an integral transform of the generating function for the $a_{n,k}$, with $n \geq 0$, respectively.

For future research, it might be worthwhile to investigate the generating functions for further parameter ranges, to capture for instance Eulerian numbers, where β, α', β' are all non-zero. Moreover, by lifting the restriction that the initial condition and recurrence must uniquely determine the $a_{n,k}$ with $n < 0$ for all $k \geq 0$, which is implicit in Definition 1, it might be possible to define additional extensions, e.g., for Lah numbers.

n \ k	0	1	2	3	4	5
−4	$\frac{1}{24}$	$\frac{25}{288}$	$\frac{415}{3456}$	$\frac{5845}{41472}$	$\frac{76111}{497664}$	$\frac{952525}{5971968}$
−3	$-\frac{1}{6}$	$-\frac{11}{36}$	$-\frac{85}{216}$	$-\frac{575}{1296}$	$-\frac{3661}{7776}$	$-\frac{22631}{46656}$
−2	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{31}{32}$	$\frac{63}{64}$
−1	−1	−1	−1	−1	−1	−1
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	2	3	1	0	0
4	0	6	11	6	1	0

Table 2: Unsigned Stirling numbers of the first kind: $a_{n,k} = \left[{n \atop k}\right]$

n \ k	0	1	2	3	4	5
-4	1	10	65	350	1701	7770
-3	1	6	25	90	301	966
-2	1	3	7	15	31	63
-1	1	1	1	1	1	1
0	1	0	0	0	0	0
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	3	2	0	0	0
4	1	6	11	6	0	0

Table 3: Unsigned Stirling numbers of the first kind: $a_{n,k} = \left[{n \atop n-k}\right]$

n \ k	0	1	2	3	4	5
-4	1	-4	10	-20	35	-56
-3	1	-3	6	-10	15	-21
-2	1	-2	3	-4	5	-6
-1	1	-1	1	-1	1	-1
0	1	0	0	0	0	0
1	1	1	0	0	0	0
2	1	2	1	0	0	0
3	1	3	3	1	0	0
4	1	4	6	4	1	0

Table 4: Binomial coefficients: $a_{n,k} = \binom{n}{k}$

$n \backslash k$	0	1	2	3	4	5
-4	$\frac{1}{16}$	$-\frac{15}{512}$	$\frac{575}{82944}$	$-\frac{5845}{5308416}$	$\frac{874853}{6635520000}$	$-\frac{336581}{26542080000}$
-3	$\frac{1}{8}$	$-\frac{7}{128}$	$\frac{85}{6912}$	$-\frac{415}{221184}$	$\frac{12019}{55296000}$	$-\frac{13489}{663552000}$
-2	$\frac{1}{4}$	$-\frac{3}{32}$	$\frac{11}{576}$	$-\frac{25}{9216}$	$\frac{137}{460800}$	$-\frac{49}{1843200}$
-1	$\frac{1}{2}$	$-\frac{1}{8}$	$\frac{1}{48}$	$-\frac{1}{384}$	$\frac{1}{3840}$	$-\frac{1}{46080}$
0	1	0	0	0	0	0
1	2	1	0	0	0	0
2	4	6	1	0	0	0
3	8	28	12	1	0	0
4	16	120	100	20	1	0

Table 5: r-Whitney numbers: $a_{n,k} = W_{2,2}(n,k)$

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Washington, D.C., 1972.

[2] J. F. Barbero G., J. Salas, and E. J. S. Villaseñor, Bivariate generating functions for a class of linear recurrences: General structure, *J. Combin. Theory Ser. A* **125** (2014), 146–165.

[3] D. Branson, An extension of Stirling numbers, *Fibonacci Quart.* **34** (3) (1996), 212–223.

[4] D. Branson, Stirling number representations, *Discrete Math.* **306** (5) (2006), 478–494.

[5] A. Z. Broder, The r-Stirling numbers, *Discrete Math.* **49** (3) (1984), 241–259.

[6] G.-S. Cheon and J.-H. Jung, r-Whitney numbers of Dowling lattices, *Discrete Math.* **312** (15) (2012), 2337–2348.

[7] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel Publishing Company, Dordrecht, 1974.

[8] C. B. Corcino and R. B. Corcino, An asymptotic formula for r-Bell numbers with real arguments, *Int. Sch. Res. Notices* **2013** (1), 7 pp.

[9] C. B. Corcino, R. B. Corcino, I. Mező, and J. L. Ramírez, Some polynomials associated with the r-Whitney numbers, *Proc. Indian Acad. Sci. Math. Sci.* **128** (2018), 25 pp.

[10] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications, Second Edition*, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[11] J. Dieudonné, *Treatise on Analysis, Volume II*, Academic Press, Cambridge, MA, 1976.

[12] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products, Seventh Edition*, Elsevier, Amsterdam, 2007.

[13] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, MA, 1989.

[14] D. E. Knuth, Two notes on notation, *Amer. Math. Monthly* **99** (5) (1992), 403–422.

[15] M. J. Kronenburg, The binomial coefficient for negative arguments, preprint, `arXiv:1105.3689`.

[16] T. Mansour and M. Shattuck, A combinatorial approach to a general two-term recurrence, *Discrete Appl. Math.* **161** (2013), 2084–2094.

[17] L. Palapies, All iterated antiderivatives of the Lambert W function, *Integral Transforms Spec. Funct.* **37** (5) (2026), 374–388.

[18] R. Scurr and G. Olive, Stirling numbers revisited, *Discrete Math.* **189** (1998), 209–219.

[19] R. P. Stanley, *Enumerative Combinatorics, Volume 1. Second Edition*, Cambridge University Press, Cambridge, 2012.

[20] P. Théorêt, Fonctions génératrices pour une classe d'équations aux différences partielles, *Ann. Sci. Math. Québec* **19** (1) (1995), 91–105.

[21] H. S. Wilf, *generatingfunctionology. Second Edition*, Academic Press, Cambridge, MA, 1994.

[22] H. S. Wilf, The method of characteristics, and "problem 89" of Graham, Knuth and Patashnik, preprint, `arXiv:math/0406620`.