#A85 INTEGERS 26 (2026)

q-ANALOGUES OF SUMS OF CONSECUTIVE POWERS OF NATURAL NUMBERS AND EXTENDED CARLITZ q-BERNOULLI NUMBERS AND POLYNOMIALS

Bakir Farhi
National Higher School of Mathematics, Algiers, Algeria
bakir.farhi@nhsm.edu.dz

Received: 6/23/25, Revised: 2/25/26, Accepted: 6/14/26, Published: 6/26/26

Abstract

In this paper, we investigate a specific class of q-polynomial sequences that serves as a q-analogue of the classical Appell sequences. This framework offers a new approach to revisiting classical results by Carlitz and, more interestingly, to establishing an important extension of the Carlitz q-Bernoulli polynomials and numbers. In addition, we establish explicit series representations for our extended Carlitz q-Bernoulli numbers and express them in terms of q-Stirling numbers of the second kind. This leads to a new formula that explicitly connects the Carlitz q-Bernoulli numbers with the q-Stirling numbers of the second kind.

1. Introduction and Notation

Throughout this paper, we let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers, respectively. For $n \in \mathbb{N}_0$, we let $(X)_n$ denote the *falling factorial* of X to depth n, defined as

$$(X)_n := X(X-1)\cdots(X-n+1).$$

The *Stirling numbers of the second kind*, denoted $S(n,k)$ $(n,k \in \mathbb{N}_0,\ n \geq k)$, are then defined through the polynomial identity:

$$X^n = \sum_{k=0}^{n} S(n,k)(X)_k \qquad (n \in \mathbb{N}_0).$$

The *Bernoulli polynomials* are denoted, as usual, by $B_n(X)$ and the *Bernoulli numbers* by B_n $(n \in \mathbb{N}_0)$. A modern definition of the Bernoulli polynomials and numbers employs their exponential generating functions, given by

$$\frac{t}{e^t-1}e^{Xt} = \sum_{n=0}^{\infty} B_n(X)\frac{t^n}{n!} \quad , \quad \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}$$

DOI: 10.5281/zenodo.20931405

(so $B_n = B_n(0)$ for all $n \in \mathbb{N}_0$). As is well known, the Bernoulli polynomials and numbers play a central role in various branches of mathematics, including number theory, mathematical analysis, and algebraic geometry. For a modern and comprehensive overview, the reader is referred to the paper by Kouba [13]. An interesting and well-known formula expressing the Bernoulli numbers in terms of the Stirling numbers of the second kind is given by:

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} S(n,k) \qquad (n \in \mathbb{N}_0) \tag{1.1}$$

(see, e.g., [5, Corollary 9]).

It should be noted that the Bernoulli polynomial sequence $(B_n(X))_{n\in\mathbb{N}_0}$ belongs to the broader class of *Appell sequences* (see [3, 15]). These are sequences of polynomials $(P_n(X))_{n\in\mathbb{N}_0}$ characterized by the following properties: $P_0(X)$ is a nonzero constant polynomial, and for all $n \in \mathbb{N}$, the derivatives satisfy the relation $P_n'(X) = nP_{n-1}(X)$. Using the Taylor expansion, an Appell polynomial sequence $(P_n(X))_n$ can also be characterized by its general term, which takes the form

$$P_n(X) = \sum_{k=0}^{n} \binom{n}{k} p_k X^{n-k} \qquad (n \in \mathbb{N}_0),$$

where $(p_k)_{k\in\mathbb{N}_0}$ is a sequence of scalars with $p_0 \neq 0$.

Next, let q be a positive real parameter. The *q-analogue* of X, whether treated as an indeterminate or a number, is defined by:

$$[X] := \frac{q^X - 1}{q - 1}.$$

For $k \in \mathbb{N}_0$, the q-analogue $[X]_k$ of $(X)_k$ is defined as

$$[X]_k := [X] \cdot [X-1] \cdots [X-k+1].$$

For $n \in \mathbb{N}_0$, the q-analogue of $n!$ is given by

$$[n]! := [n]_n = [n] \cdot [n-1] \cdots [1].$$

Furthermore, for $n, k \in \mathbb{N}_0$ with $n \geq k$, the q-analogue $\begin{bmatrix} n \\ k \end{bmatrix}$ of the binomial coefficient $\binom{n}{k}$ is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!} = \frac{[n]_k}{[k]!}.$$

These numbers appear, in particular, in the famous *Gauss binomial formula*:

$$\prod_{k=0}^{n-1} \left(x + q^k y\right) = \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k \qquad (n \in \mathbb{N} \text{ and } x, y \in \mathbb{R}), \tag{1.2}$$

and, just like the classical binomial coefficients, they also have a combinatorial interpretation (see, e.g., [6]).

On the other hand, we let Δ denote the forward difference operator which acts linearly on $\mathbb{R}[X]$ by the formula:

$$(\Delta P)(X) := P(X+1) - P(X) \qquad (P \in \mathbb{R}[X]).$$

It is well known and easy to verify that the n-fold composition of Δ is given by the formula:

$$(\Delta^n P)(X) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} P(X+k) \qquad (P \in \mathbb{R}[X] \text{ and } n \in \mathbb{N}_0). \tag{1.3}$$

In this paper, we call any polynomial in $[X]$ with real coefficients a *q-polynomial*. Equivalently, a q-polynomial is a polynomial in q^X with real coefficients. The *degree* of a q-polynomial refers to its degree as a polynomial in $[X]$ (or equivalently in q^X). We denote by $\mathscr{E}$ the $\mathbb{R}$-vector space of all q-polynomials and by $\mathscr{E}_d$ ($d \in \mathbb{N}_0$) the $\mathbb{R}$-vector subspace of $\mathscr{E}$ consisting of q-polynomials of degree $\leq d$. An important example of q-polynomials arises naturally in the evaluation of sums of the form

$$S_{n,r}(N) := \sum_{k=0}^{N-1} q^{rk} [k]^n$$

($N, n \in \mathbb{N}_0$, $r \in \mathbb{N}$), which are q-analogues of the sum of powers $\sum_{k=0}^{N-1} k^n$. These sums of consecutive q-integers have been extensively investigated in the literature; see, for instance, the pioneering works of Kim [7, 8]. More interestingly, when expressed as a linear combination of the q-polynomials $q^{kN}[N]^{n-k}$ ($0 \leq k \leq n$), $S_{n,r}(N)$ leads to a q-analogue example of Appell polynomial sequences (see Theorem 2.3). Perhaps it was by exploring this fact that Carlitz [1] obtained important q-analogues of Bernoulli numbers and polynomials. By means of the symbolic calculus, Carlitz [1] defined two real sequences $(\eta_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ (depending on q) by:

$$\begin{cases} \eta_0 = 1, \eta_1 = 0, \\ (q\eta + 1)^n = \eta^n \quad (n \geq 2) \end{cases},$$

$$\begin{cases} \beta_0 = 1, \\ q(q\beta + 1)^n - \beta^n = \delta_{n,1} \quad (n \in \mathbb{N}) \end{cases},$$

where $\delta_{i,j}$ denotes the Kronecker delta. Furthermore, it is noted that for all $n \in \mathbb{N}_0$:

$$\beta_n = \eta_n + (q-1)\eta_{n+1}.$$

Although both sequences are intriguing, the sequence $(\beta_n)_n$ is particularly noteworthy because its terms are well-defined at $q = 1$ (unlike $(\eta_n)_n$, for which only the first two terms are defined at $q = 1$). Moreover, specializing $q = 1$ in β_n recovers the n^{th} Bernoulli number B_n. These facts are far from trivial. To prove them, Carlitz was led to introduce q-analogues of the Stirling numbers of the second kind, $S_q(n,k)(n,k \in \mathbb{N}_0, n \geq k)$, which he defined via the following identity of q-polynomials:

$$[X]^n = \sum_{k=0}^{n} q^{\frac{1}{2}k(k-1)} S_q(n,k)[X]_k \tag{1.4}$$

(for all $n \in \mathbb{N}_0$). Specializing $q = 1$ in $S_q(n,k)$ $(n,k \in \mathbb{N}_0,\ n \geq k)$ recovers the classical Stirling numbers of the second kind $S(n,k)$. Using the $S_q(n,k)$'s, Carlitz [1] established the following formula for β_n $(n \in \mathbb{N}_0)$:

$$\beta_n = \sum_{k=0}^{n} (-1)^k \frac{[k]!}{[k+1]} S_q(n,k), \tag{1.5}$$

which confirms that β_n is always well-defined at $q = 1$, with value equal to the Bernoulli number B_n (according to Equation (1.1)).

Carlitz also associated with the sequences $(\eta_n)_n$ and $(\beta_n)_n$ the q-polynomial sequences $(\eta_n(X))_n$ and $(\beta_n(X))_n$, given by:

$$\eta_n(X) = \sum_{k=0}^{n} \binom{n}{k} \eta_k q^{kX} [X]^{n-k},$$
$$\beta_n(X) = \sum_{k=0}^{n} \binom{n}{k} \beta_k q^{kX} [X]^{n-k}$$

(for all $n \in \mathbb{N}_0$). The original sequences $(\eta_n)_n$ and $(\beta_n)_n$ are then obtained as the values of $(\eta_n(X))_n$ and $(\beta_n(X))_n$ at $X = 0$, respectively. Especially, $(\beta_n(X))_n$ constitutes a q-analogue of the classical Bernoulli polynomial sequence $(B_n(X))_n$. That said, alternative definitions of the Carlitz q-Bernoulli numbers and polynomials exist, such as those based on generating functions or p-adic q-integrals on $\mathbb{Z}_p$. For example, Koblitz [12] constructed q-analogues of the p-adic Dirichlet L-series that interpolate the Carlitz q-Bernoulli numbers. Further developments involving p-adic q-integrals, generalized q-Bernoulli polynomials, and multiple q-zeta functions have been extensively explored by T. Kim and his co-authors (see [10, 9, 16]). Moreover, recent research by T. Kim, D. S. Kim, and others has introduced sums of consecutive powers of degenerate integers and linked them to probabilistic models, such as moment values of Bernoulli random variables [11, 4].

In this work, we first incorporate the Carlitz q-polynomial sequences $(\eta_n(X))_n$ and $(\beta_n(X))_n$ into a broader class of q-polynomial sequences, which we call *Carlitz-type q-polynomial sequences*. We establish a fundamental theorem showing that this class of q-polynomial sequences serves as a q-analogue of the classical Appell polynomial sequences. Next, we verify that the closed forms of the q-analogues of power sums of consecutive natural numbers can essentially be expressed in terms of Carlitz-type q-polynomial sequences. Furthermore, we show that a specific subclass of Carlitz-type q-polynomial sequences (including $(\eta_n(X))_n$ and $(\beta_n(X))_n$) can be generated using simple symbolic formulas, such as those established by Carlitz in [1]. We then establish several properties of Carlitz-type q-polynomial sequences and leverage these properties to extend $(\eta_n(X))_n$ and $(\beta_n(X))_n$ into $(\beta_n^{(r)}(X))_n$ ($r \in \mathbb{N}_0$), which provide important q-analogues of the classical Bernoulli polynomials for $r \geq 1$. Additionally, we represent the numbers $\beta_n^{(r)} := \beta_n^{(r)}(0)$ as series and express them in terms of the q-Stirling numbers of the second kind. Finally, we derive a new formula connecting the Carlitz q-Bernoulli numbers with the q-Stirling numbers of the second kind.

2. The Results and the Proofs

2.1. Carlitz-Type q-Polynomial Sequences

We first introduce the following definition.

Definition 2.1. Let $(T_n(X))_{n\in\mathbb{N}_0}$ be a q-polynomial sequence. We say that $(T_n(X))_n$ is *Carlitz-type* if its general term $T_n(X)$ ($n \in \mathbb{N}_0$) has the form:

$$T_n(X) = (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k q^{kX},$$

where $(a_k)_{k\in\mathbb{N}_0}$ is a real sequence. In this case, $(a_k)_k$ is called the *associated sequence* of the Carlitz-type q-polynomial sequence $(T_n(X))_n$.

An important characterization of the Carlitz-type q-polynomial sequences is given by the following fundamental theorem.

Theorem 2.1. *Let $(T_n(X))_{n\in\mathbb{N}_0}$ be a q-polynomial sequence, and for all $n \in \mathbb{N}_0$, set $t_n := T_n(0)$. Then $(T_n(X))_n$ is Carlitz-type if and only if we have*

$$T_n(X) = \sum_{k=0}^{n} \binom{n}{k} t_k q^{kX} [X]^{n-k}$$

for all $n \in \mathbb{N}_0$, that is, symbolically,

$$T_n(X) = \left(q^X t + [X]\right)^n .$$

To prove this theorem, we need to use the inversion formula from the following lemma (for a proof, see, e.g., [2] or [14]).

Lemma 2.2. *Let $(u_n)_{n\in\mathbb{N}_0}$ and $(v_n)_{n\in\mathbb{N}_0}$ be two real sequences. Then the two following identities are equivalent:*

$$u_n = \sum_{k=0}^{n} \binom{n}{k} v_k \qquad (n \in \mathbb{N}_0),$$
$$v_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} u_k \qquad (n \in \mathbb{N}_0).$$

Proof of Theorem 2.1. Suppose that $(T_n(X))_n$ is Carlitz-type. So there exists a real sequence $(a_n)_{n\in\mathbb{N}_0}$ for which we have

$$T_n(X) = (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k q^{kX}$$

for all $n \in \mathbb{N}_0$. In particular, we have

$$t_n := T_n(0) = (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k$$

for all $n \in \mathbb{N}_0$, that is,

$$(q-1)^n t_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k.$$

By inverting this last formula (using Lemma 2.2), we derive that

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (q-1)^k t_k$$

for all $n \in \mathbb{N}_0$. Then, by inserting this into the expression of $T_n(X)$, we obtain that

$$\begin{aligned} T_n(X) &= (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left(\sum_{i=0}^{k} \binom{k}{i} (q-1)^i t_i \right) q^{kX} \\ &= (q-1)^{-n} \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^{n-k} \binom{n}{k} \binom{k}{i} (q-1)^i t_i q^{kX} \end{aligned}$$

for all $n \in \mathbb{N}_0$. By interchanging the order of summation and noting that $\binom{n}{k}\binom{k}{i} =$

$\binom{n}{i}\binom{n-i}{k-i}$ (for all $n \in \mathbb{N}_0$, $k \in \{0, 1, \ldots, n\}$, and $i \in \{0, 1, \ldots, k\}$), we derive that

$$\begin{aligned} T_n(X) &= (q-1)^{-n} \sum_{0 \le i \le n} \sum_{i \le k \le n} (-1)^{n-k} \binom{n}{i} \binom{n-i}{k-i} (q-1)^i t_i q^{kX} \\ &= (q-1)^{-n} \sum_{0 \le i \le n} \sum_{0 \le j \le n-i} (-1)^{n-i-j} \binom{n}{i} \binom{n-i}{j} (q-1)^i t_i q^{(i+j)X} \\ &\qquad \text{(where we set } j = k - i) \\ &= (q-1)^{-n} \sum_{0 \le i \le n} \binom{n}{i} (q-1)^i t_i q^{iX} \sum_{0 \le j \le n-i} \binom{n-i}{j} q^{jX} (-1)^{n-i-j} \end{aligned}$$

for all $n \in \mathbb{N}_0$. But according to the binomial formula, we have

$$\sum_{0 \le j \le n-i} \binom{n-i}{j} q^{jX} (-1)^{n-i-j} = \left(q^X - 1\right)^{n-i} = (q-1)^{n-i} [X]^{n-i}$$

for all $n \in \mathbb{N}_0$ and all $i \in \{0, 1, \ldots, n\}$. By inserting this into the last obtained expression for $T_n(X)$, we finally derive that

$$T_n(X) = \sum_{0 \le i \le n} \binom{n}{i} t_i q^{iX} [X]^{n-i}$$

for all $n \in \mathbb{N}_0$, as required.

Conversely, suppose that we have

$$T_n(X) = \sum_{k=0}^{n} \binom{n}{k} t_k q^{kX} [X]^{n-k}$$

for all $n \in \mathbb{N}_0$, and define:

$$a_n := \sum_{k=0}^{n} \binom{n}{k} (q-1)^k t_k \tag{2.1}$$

and

$$S_n(X) := (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k q^{kX}$$

for all $n \in \mathbb{N}_0$. Equivalently, $(S_n(X))_{n \in \mathbb{N}_0}$ is a Carlitz-type q-polynomial sequence, with associated sequence $(a_n)_{n \in \mathbb{N}_0}$. So, according to the first part of this proof, we have

$$S_n(X) = \sum_{k=0}^{n} \binom{n}{k} s_k q^{kX} [X]^{n-k} \tag{2.2}$$

for all $n \in \mathbb{N}_0$, where

$$s_n := S_n(0) = (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k. \tag{2.3}$$

But the inversion of Equation (2.1) (using Lemma 2.2) gives:

$$(q-1)^n t_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k$$

for all $n \in \mathbb{N}_0$, that is

$$t_n = (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k = s_n$$

(according to Equation (2.3)). Hence, we have

$$T_n(X) = \sum_{k=0}^{n} \binom{n}{k} t_k q^{kX} [X]^{n-k} = \sum_{k=0}^{n} \binom{n}{k} s_k q^{kX} [X]^{n-k} = S_n(X)$$

for all $n \in \mathbb{N}_0$ (according to Equation (2.2)). Consequently $\left(T_n(X)\right)_n = \left(S_n(X)\right)_n$ is Carlitz-type, as required. This completes this proof. $\square$

Remark 2.2. From the perspective of Theorem 2.1, the Carlitz-type q-polynomial sequences can be regarded as a q-analogue of the Appell polynomial sequences.

Examples 2.3. In [1], Carlitz introduced two important q-polynomial sequences, which are $\left(\eta_n(X)\right)_{n \in \mathbb{N}_0}$ and $\left(\beta_n(X)\right)_{n \in \mathbb{N}_0}$, defined for all $n \in \mathbb{N}_0$ by:

$$\eta_n(X) := (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{k}{[k]} q^{kX},$$

$$\beta_n(X) := (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{k+1}{[k+1]} q^{kX}$$

(where, by convention, $\frac{k}{[k]} = 1$ for $k = 0$). From their definitions, it is clear that $\left(\eta_n(X)\right)_n$ and $\left(\beta_n(X)\right)_n$ are both Carlitz-type with associated real sequences $\left(\frac{k}{[k]}\right)_k$ and $\left(\frac{k+1}{[k+1]}\right)_k$, respectively. In addition, Carlitz [1] explored the real sequences consisting of the values of $\left(\eta_n(X)\right)_n$ and $\left(\beta_n(X)\right)_n$ at $X = 0$; that is, the sequences $\left(\eta_n\right)_{n \in \mathbb{N}_0}$ and $\left(\beta_n\right)_{n \in \mathbb{N}_0}$, defined by:

$$\eta_n := \eta_n(0) \quad \text{and} \quad \beta_n := \beta_n(0) \qquad (n \in \mathbb{N}_0).$$

So, according to Theorem 2.1, we have important alternative expressions for $\eta_n(X)$ and $\beta_n(X)$ $(n \in \mathbb{N}_0)$, which are

$$\eta_n(X) = \sum_{k=0}^{n} \binom{n}{k} \eta_k q^{kX} [X]^{n-k}, \tag{2.4}$$

$$\beta_n(X) = \sum_{k=0}^{n} \binom{n}{k} \beta_k q^{kX} [X]^{n-k}. \tag{2.5}$$

Using the symbolic calculus, Equations (2.4) and (2.5) can be written more simply as:

$$\eta_n(X) = \left(q^X\eta + [X]\right)^n,$$
$$\beta_n(X) = \left(q^X\beta + [X]\right)^n$$

(for all $n \in \mathbb{N}_0$).

We now see that Carlitz-type q-polynomial sequences naturally emerge when deriving closed-form expressions for q-analogues of power sums of consecutive positive integers. We begin with the following theorem.

Theorem 2.3. *Let r be a positive integer. For every nonnegative integer n and every positive integer N, we have*

$$\sum_{k=0}^{N-1} q^{rk}[k]^n = q^{rN} S_n(N) - S_n(0),$$

where $(S_k(X))_{k\in\mathbb{N}_0}$ is the Carlitz-type q-polynomial sequence associated with the real sequence of general term $a_k = \frac{1}{q-1}\frac{1}{[k+r]}$.

Proof. Let $n \in \mathbb{N}_0$ and $N \in \mathbb{N}$ be fixed. We have

$$\begin{aligned}
\sum_{k=0}^{N-1} q^{rk}[k]^n &= \sum_{k=0}^{N-1} q^{rk}\left(\frac{q^k-1}{q-1}\right)^n \\
&= (q-1)^{-n}\sum_{k=0}^{N-1} q^{rk}\left(q^k-1\right)^n \\
&= (q-1)^{-n}\sum_{k=0}^{N-1} q^{rk}\sum_{i=0}^{n}\binom{n}{i}q^{ki}(-1)^{n-i} \qquad \text{(by the binomial formula)} \\
&= (q-1)^{-n}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\sum_{k=0}^{N-1} q^{(i+r)k} \\
&= (q-1)^{-n}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{q^{(i+r)N}-1}{q^{i+r}-1} \\
&= (q-1)^{-n}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{1}{(q-1)[i+r]}\left(q^{(i+r)N}-1\right).
\end{aligned}$$

Setting for all $n \in \mathbb{N}_0$:

$$S_n(X) := (q-1)^{-n}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{1}{(q-1)[i+r]}q^{iX}$$

(so $(S_n(X))_n$ is the Carlitz-type q-polynomial sequence associated with the real sequence of general term $a_i = \frac{1}{(q-1)[i+r]}$), we conclude that

$$\sum_{n=0}^{N-1} q^{rk}[k]^n = q^{rN} S_n(N) - S_n(0),$$

as required. ☐

The following generalization of Theorem 2.3 aims to provide a larger class of real sequences, for which the associated Carlitz-type q-polynomial sequences naturally arise when evaluating q-analogues of power sums of consecutive positive integers.

Theorem 2.4. *Let r be a positive integer and d be a nonnegative integer such that $r \geq d$. For every integer $n \geq d$ and every positive integer N, we have*

$$\sum_{k=0}^{N-1} q^{rk}[k]^{n-d} = \frac{q^{(r-d)N} T_n(N) - T_n(0)}{n(n-1)\cdots(n-d+1)},$$

where $(T_k(X))_{k\in\mathbb{N}_0}$ is the Carlitz-type q-polynomial sequence associated with the real sequence of general term

$$b_k = (q-1)^{d-1} \frac{k(k-1)\cdots(k-d+1)}{[k+r-d]} \qquad (k \in \mathbb{N}_0).$$

Proof. Let n and N be fixed integers such that $n \geq d$ and $N \geq 1$. According to Theorem 2.3, we have that

$$\sum_{k=0}^{N-1} q^{rk}[k]^{n-d} = q^{rN} S_{n-d}(N) - S_{n-d}(0),$$

where $(S_k(X))_{k\in\mathbb{N}_0}$ is the Carlitz-type q-polynomial sequence associated with the real sequence of general term $a_k = \frac{1}{(q-1)[k+r]}$ $(\forall k \in \mathbb{N}_0)$. By introducing the q-polynomial sequence $(T_k(X))_{k\in\mathbb{N}_0}$ defined by:

$$T_k(X) = \begin{cases} 0 & \text{if } k < d \\ k(k-1)\cdots(k-d+1) q^{dX} S_{k-d}(X) & \text{if } k \geq d \end{cases} \qquad (k \in \mathbb{N}_0),$$

the closed form of the sum of q-powers above is transformed to

$$\sum_{k=0}^{N-1} q^{rk}[k]^{n-d} = \frac{q^{(r-d)N} T_n(N) - T_n(0)}{n(n-1)\cdots(n-d+1)}.$$

So it remains to verify that $(T_k(X))_k$ is Carlitz-type with the associated sequence given by the theorem. For an integer $k \geq d$, we have

$$\begin{aligned}
T_k(X) &:= k(k-1)\cdots(k-d+1)q^{dX}S_{k-d}(X) \\
&= k(k-1)\cdots(k-d+1)q^{dX}\cdot(q-1)^{-k+d}\sum_{i=0}^{k-d}(-1)^{k-d-i}\binom{k-d}{i}\frac{1}{(q-1)[i+r]}q^{iX} \\
&= k(k-1)\cdots(k-d+1)q^{dX}(q-1)^{-k+d} \\
&\qquad\times\sum_{j=d}^{k}(-1)^{k-j}\binom{k-d}{j-d}\frac{1}{(q-1)[j+r-d]}q^{(j-d)X}
\end{aligned}$$

(by setting $j = i+d$). But since $k(k-1)\cdots(k-d+1)\binom{k-d}{j-d} = j(j-1)\cdots(j-d+1)\binom{k}{j}$ (for all integers $k, j \geq d$), it follows that for every integer $k \geq d$:

$$\begin{aligned}
T_k(X) &= (q-1)^{-k}\sum_{j=d}^{k}(-1)^{k-j}\binom{k}{j}(q-1)^{d-1}\frac{j(j-1)\cdots(j-d+1)}{[j+r-d]}q^{jX} \\
&= (q-1)^{-k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(q-1)^{d-1}\frac{j(j-1)\cdots(j-d+1)}{[j+r-d]}q^{jX}.
\end{aligned}$$

By observing that the last formula for $T_k(X)$ is obviously valid for $0 \leq k < d$, we conclude that $(T_k(X))_k$ is Carlitz-type and its associated real sequence has the general term $b_j = (q-1)^{d-1}\frac{j(j-1)\cdots(j-d+1)}{[j+r-d]}$. The theorem is thus proved. □

Remark 2.4. It is important to note that the general term of the real sequence $(b_k)_{k\in\mathbb{N}_0}$ associated with the q-polynomial sequence in Theorem 2.4 has the form $\frac{P(k)}{[k+s]}$, with P being a real polynomial and s being a nonnegative integer. This fact will be used below for establishing some important symbolic formulas.

In the following theorem, we will establish symbolic formulas generating the values at 0 of some particular Carlitz-type q-polynomial sequences.

Theorem 2.5. *Let r and d be two nonnegative integers and P be a real polynomial of degree d. Also, let $(T_n(X))_{n\in\mathbb{N}_0}$ be the Carlitz-type q-polynomial sequence whose associated real sequence is $\left(\frac{P(k)}{[k+r]}\right)_{k\in\mathbb{N}_0}$ and $(t_n)_{n\in\mathbb{N}_0}$ be the real sequence defined by $t_n := T_n(0)$ (for all $n \in \mathbb{N}_0$). Then $(t_n)_n$ satisfies the symbolic formula:*

$$q^r(qt+1)^n = t^n \qquad (n > d).$$

Proof. Let $n > d$ be a fixed integer. From the definition of $T_n(X)$, we have that

$$T_n(X) = (q-1)^{-n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{P(k)}{[k+r]}q^{kX}.$$

Thus,

$$\begin{aligned} q^r T_n(1) - T_n(0) &= (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{P(k)}{[k+r]} \left(q^{k+r} - 1\right) \\ &= (q-1)^{-n+1} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} P(k) \quad (\text{since } q^{k+r} - 1 = (q-1)[k+r]) \\ &= (q-1)^{-n+1} \left(\Delta^n P\right)(0) \qquad (\text{according to Equation (1.3)}) \\ &= 0 \qquad (\text{since } n > \deg P = d). \end{aligned}$$

Consequently, we have

$$q^r T_n(1) = T_n(0) = t_n. \tag{2.6}$$

On the other hand, we have (according to Theorem 2.1) the symbolic formula

$$T_n(X) = \left(q^X t + [X]\right)^n.$$

In particular, we have, symbolically,

$$T_n(1) = (qt+1)^n.$$

By inserting this into Equation (2.6), it finally follows that we have, symbolically,

$$q^r (qt+1)^n = t^n,$$

as required. □

Examples 2.5.

1. Since the general term of the associated real sequences of the Carlitz-type q-polynomial sequences $(\eta_n(X))_{n\in\mathbb{N}_0}$ and $(\beta_n(X))_{n\in\mathbb{N}_0}$ are $\frac{k}{[k]}$ and $\frac{k+1}{[k+1]}$, respectively, it follows (according to Theorem 2.5) that the real sequences $(\eta_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ satisfy the symbolic formulas:

$$\left.\begin{aligned} (q\eta+1)^n &= \eta^n \\ q\,(q\beta+1)^n &= \beta^n \end{aligned}\right. \qquad (n > 1).$$

2. Let us put ourselves in the situation of Theorem 2.4 and set $t_k := T_k(0)$ (for all $k \in \mathbb{N}_0$). Since the general term of the associated real sequence of the Carlitz-type q-polynomial sequence $(T_k(X))_k$ has the form $\frac{P(k)}{[k+r-d]}$, where P is a real polynomial of degree d, it follows (according to Theorem 2.5) that the sequence $(t_k)_k$ satisfies the symbolic formula

$$q^{r-d}\,(qt+1)^k = t^k \qquad (k > d).$$

In the following proposition, we provide some important properties of Carlitz-type q-polynomial sequences. For simplicity, given a real or functional sequence $(f_n)_{n\in\mathbb{N}_0}$, we denote (with a slight abuse of notation) by $(nf_{n-1})_{n\in\mathbb{N}_0}$ the sequence whose terms are defined as

$$\begin{cases} 0 & \text{if } n=0, \\ nf_{n-1} & \text{otherwise.} \end{cases}$$

Proposition 2.6. *Let $(T_n(X))_{n\in\mathbb{N}_0}$ be a Carlitz-type q-polynomial sequence and $(a_k)_{k\in\mathbb{N}_0}$ be its associated real sequence. Then the following properties hold.*

(i) The q-polynomial sequence $\left(q^X nT_{n-1}(X)\right)_{n\in\mathbb{N}_0}$ is Carlitz-type with associated real sequence $((q-1)ka_{k-1})_{k\in\mathbb{N}_0}$.

(ii) If $T_0(X)=0_{\mathscr{E}}$ (i.e., $a_0=0$) then the q-polynomial sequence $\left(\frac{q^{-X}T_{n+1}(X)}{n+1}\right)_{n\in\mathbb{N}_0}$ is Carlitz-type with associated real sequence $\left(\frac{1}{q-1}\frac{a_{k+1}}{k+1}\right)_{k\in\mathbb{N}_0}$.

(iii) The sequence $\left(q^{-X}\left(T_n(X)+(q-1)T_{n+1}(X)\right)\right)_{n\in\mathbb{N}_0}$ is a Carlitz-type q-polynomial sequence with associated real sequence $(a_{k+1})_{k\in\mathbb{N}_0}$.

(iv) The q-polynomial sequence $\left(n\left(T_{n-1}(X)+(q-1)T_n(X)\right)\right)_{n\in\mathbb{N}_0}$ is Carlitz-type with associated real sequence $((q-1)ka_k)_{k\in\mathbb{N}_0}$.

Proof. By definition, we have

$$T_n(X)=(q-1)^{-n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}a_kq^{kX} \tag{2.7}$$

for all $n\in\mathbb{N}_0$.

For statement (i), using Equation (2.7), we have

$$\begin{aligned} q^X nT_{n-1}(X) &= (q-1)^{-n+1}\sum_{k=0}^{n-1}(-1)^{n-1-k}n\binom{n-1}{k}a_kq^{(k+1)X} \\ &= (q-1)^{-n}\sum_{k=1}^{n}(-1)^{n-k}n\binom{n-1}{k-1}(q-1)a_{k-1}q^{kX} \end{aligned}$$

for all $n\in\mathbb{N}_0$. Noting that $n\binom{n-1}{k-1}=k\binom{n}{k}$ (for all $1\le k\le n$), it follows that

$$q^X nT_{n-1}(X)=(q-1)^{-n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\left((q-1)ka_{k-1}\right)q^{kX}$$

for all $n\in\mathbb{N}_0$, showing that the q-polynomial sequence $\left(q^X nT_{n-1}(X)\right)_{n\in\mathbb{N}_0}$ is Carlitz-type with associated real sequence $((q-1)ka_{k-1})_{k\in\mathbb{N}_0}$. The first statement of the proposition is proved.

For statement (ii), suppose that $T_0(X) = 0_{\mathscr{E}}$, that is, $a_0 = 0$. Using Equation (2.7), we have

$$\begin{aligned} q^{-X}\frac{T_{n+1}(X)}{n+1} &= (q-1)^{-n-1}\sum_{k=1}^{n+1}(-1)^{n+1-k}\frac{1}{n+1}\binom{n+1}{k}a_k q^{(k-1)X} \\ &= (q-1)^{-n}\sum_{k=0}^{n}(-1)^{n-k}\frac{1}{n+1}\binom{n+1}{k+1}\frac{1}{q-1}a_{k+1}q^{kX} \end{aligned}$$

for all $n \in \mathbb{N}_0$. Noting that $\frac{1}{n+1}\binom{n+1}{k+1} = \frac{1}{k+1}\binom{n}{k}$ (for all $0 \leq k \leq n$), it follows that

$$q^{-X}\frac{T_{n+1}(X)}{n+1} = (q-1)^{-n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\left(\frac{1}{q-1}\frac{a_{k+1}}{k+1}\right)q^{kX}$$

for all $n \in \mathbb{N}_0$, showing that the q-polynomial sequence $\left(q^{-X}\frac{T_{n+1}(X)}{n+1}\right)_{n\in\mathbb{N}_0}$ is Carlitz-type with associated real sequence $\left(\frac{1}{q-1}\frac{a_{k+1}}{k+1}\right)_{k\in\mathbb{N}_0}$. The second statement of the proposition is proved.

For statement (iii), using Equation (2.7), we have

$$\begin{aligned} &q^{-X}\left(T_n(X) + (q-1)T_{n+1}(X)\right) \\ &= (q-1)^{-n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}a_k q^{(k-1)X} + (q-1)^{-n}\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}a_k q^{(k-1)X} \\ &= (q-1)^{-n}\sum_{k=0}^{n+1}(-1)^{n-k+1}\left(\binom{n+1}{k} - \binom{n}{k}\right)a_k q^{(k-1)X} \end{aligned}$$

for all $n \in \mathbb{N}_0$. But since $\binom{n+1}{k} - \binom{n}{k} = \begin{cases} 0 & \text{if } k = 0 \\ \binom{n}{k-1} & \text{if } k \geq 1 \end{cases}$ (for all $0 \leq k \leq n+1$), it follows that

$$\begin{aligned} q^{-X}\left(T_n(X) + (q-1)T_{n+1}(X)\right) &= (q-1)^{-n}\sum_{k=1}^{n+1}(-1)^{n-k+1}\binom{n}{k-1}a_k q^{(k-1)X} \\ &= (q-1)^{-n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}a_{k+1}q^{kX}, \end{aligned}$$

showing that the sequence $\left(q^{-X}\left(T_n(X) + (q-1)T_{n+1}(X)\right)\right)_{n\in\mathbb{N}_0}$ is q-polynomial of Carlitz-type with associated real sequence $(a_{k+1})_{k\in\mathbb{N}_0}$. The third statement of the proposition is proved.

Finally, the statement (iv) of the proposition follows by successively applying its statements (iii) and (i), which have already been proven. This completes the proof. □

Example 2.6. Since the real sequence associated with $(\eta_n(X))_{n\in\mathbb{N}_0}$ (which is a Carlitz-type q-polynomial sequence) is $(\frac{k}{[k]})_{k\in\mathbb{N}_0}$, it follows from Proposition 2.6 (iii) that the sequence $\left(q^{-X}\left(\eta_n(X)+(q-1)\eta_{n+1}(X)\right)\right)_{n\in\mathbb{N}_0}$ is a Carlitz-type q-polynomial sequence with associated real sequence $\left(\frac{k+1}{[k+1]}\right)_{k\in\mathbb{N}_0}$. Hence, $\left(q^{-X}\left(\eta_n(X)+(q-1)\eta_{n+1}(X)\right)\right)$ coincides with $(\beta_n(X))_n$; that is,

$$\beta_n(X) = q^{-X}\left(\eta_n(X)+(q-1)\eta_{n+1}(X)\right) \qquad (n\in\mathbb{N}_0). \tag{2.8}$$

We now derive from Theorem 2.4 formulas for q-analogues of sums of powers of positive integers, which involve the q-polynomial sequences $(\eta_n(X))_{n\in\mathbb{N}_0}$ and $(\beta_n(X))_{n\in\mathbb{N}_0}$ introduced by Carlitz in [1]. Namely, we obtain the following corollaries, the first of which recovers an identity previously established by T. Kim in [7] and [8].

Corollary 2.7. *For all positive integers n and N, we have*

$$\sum_{k=0}^{N-1} q^k[k]^{n-1} = \frac{\eta_n(N)-\eta_n}{n}.$$

Proof. Take $r=d=1$ in Theorem 2.4. □

Corollary 2.8. *For all positive integers n and N, we have*

$$n\sum_{k=0}^{N-1} q^{2k}[k]^{n-1} + (q-1)\sum_{k=0}^{N-1} q^k[k]^n = q^N\beta_n(N)-\beta_n.$$

Proof. Let n and N be fixed positive integers. By applying Theorem 2.4 with the pairs $(r,d)=(2,1)$ and $(r,d)=(1,0)$, we obtain the two following formulas:

$$\sum_{k=0}^{N-1} q^{2k}[k]^{n-1} = \frac{q^N T_n(N)-T_n(0)}{n},$$
$$\sum_{k=0}^{N-1} q^k[k]^n = q^N S_n(N)-S_n(0),$$

respectively, where $(T_k(X))_{k\in\mathbb{N}_0}$ and $(S_k(X))_{k\in\mathbb{N}_0}$ represent the Carlitz-type q-polynomial sequences, associated with the real sequences $\left(\frac{k}{[k+1]}\right)_{k\in\mathbb{N}_0}$ and $\left(\frac{1}{q-1}\frac{1}{[k+1]}\right)_{k\in\mathbb{N}_0}$, respectively. By combining the two above formulas, we derive that

$$n\sum_{k=0}^{N-1} q^{2k}[k]^{n-1} + (q-1)\sum_{k=0}^{N-1} q^k[k]^n = q^N U_n(N)-U_n(0),$$

where

$$U_k(X) := T_k(X) + (q-1)S_k(X) \qquad (k \in \mathbb{N}_0).$$

Next, from the fact that $(T_k(X))_k$ and $(S_k(X))_k$ are Carlitz-type q-polynomial sequences whose associated real sequences are $\left(\frac{k}{[k+1]}\right)_k$ and $\left(\frac{1}{q-1}\frac{1}{[k+1]}\right)_k$, respectively, we derive that $(U_k(X))_k$ is also a Carlitz-type q-polynomial sequence, with an associated real sequence having the general term:

$$\frac{k}{[k+1]} + (q-1)\left(\frac{1}{q-1}\frac{1}{[k+1]}\right) = \frac{k+1}{[k+1]} \qquad (k \in \mathbb{N}_0).$$

Consequently, $(U_k(X))_k$ is nothing other than the q-polynomial sequence $(\beta_k(X))_k$ of Carlitz q-Bernoulli polynomials. This completes the proof. □

2.2. Extended Carlitz q-Bernoulli Numbers and Polynomials

2.2.1. Definitions and Basic Results

We begin by introducing the following definition.

Definition 2.7. We define $\left(\beta_n^{(r)}(X)\right)_{r,n\in\mathbb{N}_0}$ recursively by

$$\beta_n^{(0)}(X) := \eta_n(X) \qquad (n \in \mathbb{N}_0), \tag{2.9}$$

$$\beta_n^{(r+1)}(X) := q^{-X}\left(\beta_n^{(r)}(X) + (q-1)\beta_{n+1}^{(r)}(X)\right) \qquad (r, n \in \mathbb{N}_0). \tag{2.10}$$

We also define $\left(\beta_n^{(r)}\right)_{r,n\in\mathbb{N}_0}$ by

$$\beta_n^{(r)} := \beta_n^{(r)}(0) \qquad (r, n \in \mathbb{N}_0). \tag{2.11}$$

It follows from Equations (2.9), (2.10), and (2.11) that

$$\beta_n^{(0)} = \eta_n \qquad (n \in \mathbb{N}_0),$$

$$\beta_n^{(r+1)} = \beta_n^{(r)} + (q-1)\beta_{n+1}^{(r)} \qquad (r, n \in \mathbb{N}_0). \tag{2.12}$$

Since $\left(\beta_n^{(0)}(X)\right)_{n\in\mathbb{N}_0} = (\eta_n(X))_{n\in\mathbb{N}_0}$ is a Carlitz-type q-polynomial sequence (see Examples 2.3), it follows from Proposition 2.6 (iii) and a simple induction on r that all the sequences $\left(\beta_n^{(r)}(X)\right)_{n\in\mathbb{N}_0}$ $(r \in \mathbb{N}_0)$ are Carlitz-type q-polynomial sequences.

Definition 2.8. Given $r \in \mathbb{N}_0$, the q-polynomials $\beta_n^{(r)}(X)$ $(n \in \mathbb{N}_0)$ are called the *extended Carlitz q-Bernoulli polynomials of order r*, and the numbers $\beta_n^{(r)}$ $(n \in \mathbb{N}_0)$ are called the *extended Carlitz q-Bernoulli numbers of order r*.

Taking $r = 0$ in Equation (2.10), we find in particular that

$$\beta_n^{(1)}(X) = q^{-X}\left(\eta_n(X) + (q-1)\eta_{n+1}(X)\right) = \beta_n(X)$$

for all $n \in \mathbb{N}_0$ (according to Equation (2.8)). So, the Carlitz q-Bernoulli polynomials of order 1 are nothing other than the Carlitz q-Bernoulli polynomials $\beta_n(X)$ (introduced by Carlitz in [1]). Consequently, the Carlitz q-Bernoulli numbers of order 1 are nothing other than the Carlitz q-Bernoulli numbers β_n of [1]. Furthermore, Carlitz [1] showed that the q-polynomials $\beta_n(X) = \beta_n^{(1)}(X)$ $(n \in \mathbb{N}_0)$ are well-defined for $q = 1$ and coincide with the Bernoulli polynomials when specializing q to 1; that is, $\beta_n^{(1)}(X)|_{q=1} = B_n(X)$ for all $n \in \mathbb{N}_0$. Consequently, in view of Equation (2.10), the q-polynomials $\beta_n^{(r)}(X)$ $(r \in \mathbb{N},\ n \in \mathbb{N}_0)$ are also well-defined for $q = 1$, and for all $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we have

$$\beta_n^{(r+1)}(X)|_{q=1} = \beta_n^{(r)}(X)|_{q=1}.$$

By induction on r, it follows that

$$\beta_n^{(r)}(X)|_{q=1} = \beta_n^{(1)}(X)|_{q=1} = B_n(X)$$

for all $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. This demonstrates that for all $r \in \mathbb{N}$, the q-polynomials $\beta_n^{(r)}(X)$ $(n \in \mathbb{N}_0)$ can serve as a q-analogue of the classical Bernoulli polynomials. In view of Equation (2.11), it follows that for all $r \in \mathbb{N}$, the numbers $\beta_n^{(r)}$ $(n \in \mathbb{N}_0)$ can also serve as a q-analogue of the classical Bernoulli numbers. This justifies the terminology introduced in Definition 2.8.

The following proposition gathers several formulas for the Carlitz q-Bernoulli polynomials and numbers of a given order, which can be directly derived from the results of §2.1.

Proposition 2.9. *For all $r, n \in \mathbb{N}_0$, we have that*

$$\beta_n^{(r)}(X) = (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{k+r}{[k+r]} q^{kX}, \tag{2.13}$$

$$\beta_n^{(r)} = (q-1)^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{k+r}{[k+r]}, \tag{2.14}$$

$$\beta_n^{(r)}(X) = \sum_{k=0}^{n} \binom{n}{k} \beta_k^{(r)} q^{kX} [X]^{n-k}. \tag{2.15}$$

In addition, for a given $r \in \mathbb{N}_0$, the Carlitz q-Bernoulli polynomials and numbers of order r satisfy the following symbolic formulas:

$$q^r \left(q\beta^{(r)} + 1\right)^n = \left(\beta^{(r)}\right)^n \qquad (n \geq 2), \tag{2.16}$$

$$\beta_n^{(r)}(X) = \left(q^X \beta^{(r)} + [X]\right)^n \qquad (n \in \mathbb{N}_0). \tag{2.17}$$

Proof. Using Proposition 2.6 (iii) together with Equation (2.10), we immediately show by induction that for all $r \in \mathbb{N}_0$, the associated real sequence of the Carlitz-type q-polynomial sequence $\left(\beta_n^{(r)}(X)\right)_{n\in\mathbb{N}_0}$ is $\left(\frac{k+r}{[k+r]}\right)_{k\in\mathbb{N}_0}$. Equation (2.13) of the proposition then follows. Next, Equation (2.14) of the proposition follows by taking $X = 0$ in Equation (2.13), which we have just proven. Equations (2.15) and (2.17) of the proposition immediately follow from Theorem 2.1. Finally, Equation (2.16) of the proposition immediately follows from Theorem 2.5. This completes this proof. □

We now see, for a general $r \in \mathbb{N}_0$, how the q-polynomials $\beta_n^{(r)}(X)$ arise in q-analogues of sums of powers of consecutive positive integers. We have the following theorem providing a generalization of Corollaries 2.7 and 2.8.

Theorem 2.10. *Let r be a nonnegative integer. Then we have*

$$n\sum_{k=0}^{N-1} q^{(r+1)k}[k]^{n-1} + (q-1)r\sum_{k=0}^{N-1} q^{rk}[k]^n = q^{rN}\beta_n^{(r)}(N) - \beta_n^{(r)}$$

for all positive integers n and N.

Proof. Let n and N be fixed positive integers. By applying Theorem 2.4 with the pairs $(r, 0)$ and $(r+1, 1)$ (instead of (r, d)), we respectively obtain the two following formulas:

$$\sum_{k=0}^{N-1} q^{rk}[k]^n = q^{rN}T_n(N) - T_n(0),$$
$$\sum_{k=0}^{N-1} q^{(r+1)k}[k]^{n-1} = \frac{q^{rN}S_n(N) - S_n(0)}{n},$$

where $(T_k(X))_{k\in\mathbb{N}_0}$ and $(S_k(X))_{k\in\mathbb{N}_0}$ are the Carlitz-type q-polynomial sequences associated with the real sequences of general term:

$$a_k = \frac{1}{q-1}\frac{1}{[k+r]} \quad \text{and} \quad b_k = \frac{k}{[k+r]} \qquad (k \in \mathbb{N}_0),$$

respectively. By combining the above formulas, we derive that

$$n\sum_{k=0}^{N-1} q^{(r+1)k}[k]^{n-1} + (q-1)r\sum_{k=0}^{N-1} q^{rk}[k]^n = q^{rN}U_n(N) - U_n(0),$$

where

$$U_k(X) := S_k(X) + (q-1)rT_k(X) \qquad (k \in \mathbb{N}_0).$$

On the other hand, from the fact that $(T_k(X))_k$ and $(S_k(X))_k$ are Carlitz-type q-polynomial sequences whose associated real sequences are $\left(\frac{1}{q-1}\frac{1}{[k+r]}\right)_k$ and $\left(\frac{k}{[k+r]}\right)_k$, respectively, we derive that $(U_k(X))_k$ is also a Carlitz-type q-polynomial sequence, with an associated real sequence having the general term

$$\frac{k}{[k+r]} + (q-1)r\left(\frac{1}{q-1}\frac{1}{[k+r]}\right) = \frac{k+r}{[k+r]} \qquad (k \in \mathbb{N}_0).$$

Hence, $(U_k(X))_k$ coincides with $\left(\beta_k^{(r)}(X)\right)_k$, leading to the required result. $\square$

2.2.2. Representation of $\beta_n^{(r)}$ as a Series

We now turn our attention to the representation of the extended Carlitz q-Bernoulli numbers $\beta_n^{(r)}$ as a numerical series. We have the following theorem.

Theorem 2.11. *Suppose that $|q| < 1$. Then we have*

$$\beta_n^{(r)} = \begin{cases} -n\sum_{k=0}^{+\infty} q^k [k]^{n-1} + (1-q)^{-n} & \text{if } r = 0 \\ -n\sum_{k=0}^{+\infty} q^{(r+1)k}[k]^{n-1} + r(1-q)\sum_{k=0}^{+\infty} q^{rk}[k]^n & \text{if } r \neq 0 \end{cases}$$

for all $r \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

Proof. The result follows by letting $N \to +\infty$ in the formula of Theorem 2.10. However, before doing so, we must evaluate $\lim_{N\to+\infty} q^{rN}\beta_n^{(r)}(N)$ (for given $r \in \mathbb{N}_0$, $n \in \mathbb{N}$). To achieve this, we use Equation (2.13) of Proposition 2.9. Let $r \in \mathbb{N}_0$ and $n, N \in \mathbb{N}$ be fixed. According to Equation (2.13) of Proposition 2.9, we have that

$$\begin{aligned}\beta_n^{(r)}(N) &= (q-1)^{-n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{k+r}{[k+r]}q^{kN} \\ &= (q-1)^{-n}\left[(-1)^n\frac{r}{[r]} + \sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}\frac{k+r}{[k+r]}q^{kN}\right].\end{aligned}$$

Since $|q| < 1$ by hypothesis, we derive that

$$\lim_{N\to+\infty}\beta_n^{(r)}(N) = (1-q)^{-n}\frac{r}{[r]}.$$

Thus,

$$\lim_{N\to+\infty} q^{rN}\beta_n^{(r)}(N) = \begin{cases}\lim_{N\to+\infty}\beta_n^{(r)}(N) = (1-q)^{-n} & \text{if } r = 0 \\ 0 & \text{if } r \neq 0.\end{cases}$$

Finally, by letting $N \to +\infty$ in the formula of Theorem 2.10 and incorporating the result we just obtained, we get

$$n \sum_{k=0}^{+\infty} q^{(r+1)k} [k]^{n-1} + (q-1) r \sum_{k=0}^{+\infty} q^{rk} [k]^n = \begin{cases} (1-q)^{-n} - \beta_n^{(r)} & \text{if } r = 0 \\ -\beta_n^{(r)} & \text{if } r \neq 0, \end{cases}$$

implying the desired result of the theorem and completing the proof. □

Remark 2.9. For $r \in \mathbb{N}$, the formula of Theorem 2.11 can be written more simply as:

$$\beta_n^{(r)} = \sum_{k=0}^{+\infty} \left(r q^{rk} - (n+r) q^{(r+1)k} \right) [k]^{n-1}.$$

Taking in particular $r = 1$, we derive the following important representation of the Carlitz q-Bernoulli number β_n as a series

$$\beta_n = \sum_{k=0}^{+\infty} \left(q^k - (n+1) q^{2k} \right) [k]^{n-1}$$

(valid for $|q| < 1$ and $n \in \mathbb{N}$).

From Remark 2.9 and the fact that the extended Carlitz q-Bernoulli numbers of any positive order are q-analogues of the classical Bernoulli numbers, we immediately derive the following corollary.

Corollary 2.12. *For all $r, n \in \mathbb{N}$, we have that*

$$B_n = \lim_{q \to 1^-} \sum_{k=0}^{+\infty} \left(r q^{rk} - (n+r) q^{(r+1)k} \right) [k]^{n-1}.$$

Remark 2.10. If, in the formula of Corollary 2.12, we allow the limit to be taken inside the sum and also permit the manipulation of a divergent series, we obtain

$$B_n = \sum_{k=0}^{+\infty} -n k^{n-1} = -n \sum_{k=1}^{+\infty} k^{n-1} = -n \zeta(1-n)$$

for all $n \in \mathbb{N}$. This provides a well-known and valid formula for the classical Bernoulli numbers, where ζ denotes the Riemann zeta function.

2.2.3. Expression of the $\beta_n^{(r)}$'s in Terms of q-Stirling Numbers of the Second Kind

The following theorem provides a generalization of Equation (1.5) of Carlitz, which expresses the Carlitz q-Bernoulli numbers in terms of q-Stirling numbers of the second kind.

Theorem 2.13. *For all $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we have*

$$\beta_n^{(r)} = \sum_{k=0}^{n} \varphi_r(q,k) S_q(n,k),$$

where

$$\begin{aligned}\varphi_r(q,k) &= (-1)^k [k]! \sum_{i=0}^{r-1} q^{ik} \frac{[r-1]_i}{[k+r]_{i+1}} \\ &= (-1)^k [k]! [r-1]! q^{-r+1} \sum_{i=0}^{r-1} \frac{(q-1)^i q^{\frac{1}{2}i(i+1)-i(r-1)}}{[r-1-i]!} \frac{1}{[k+i+1]}\end{aligned}$$

(for all $k \in \mathbb{N}$).

The proof of this theorem needs some preparations which are presented below. We begin with the following proposition.

Proposition 2.14. *For all $r, N \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we have*

$$\sum_{i=0}^{N-1} q^{ir} [i]_k = \left(\sum_{\ell=0}^{r-1} q^{(r-\ell-1)N+(\ell+1)k} \frac{[r-1]_\ell}{[k+r]_{\ell+1}} \right) [N]_{k+1}.$$

Proof. Let $k \in \mathbb{N}_0$ and $N \in \mathbb{N}$ be fixed. For all $r \in \mathbb{N}$, we have

$$\begin{aligned}\Delta \left(q^{(r-1)X} [X]_{k+1} \right) &= q^{(r-1)(X+1)} [X+1]_{k+1} - q^{(r-1)X} [X]_{k+1} \\ &= q^{(r-1)(X+1)} [X+1][X] \cdots [X-k+1] \\ &\quad - q^{(r-1)X} [X][X-1] \cdots [X-k] \\ &= \left(q^{(r-1)(X+1)} [X+1] - q^{(r-1)X} [X-k] \right) [X]_k.\end{aligned}$$

But since

$$\begin{aligned}q^{(r-1)(X+1)} [X+1] - q^{(r-1)X} [X-k] &= q^{(r-1)(X+1)} \frac{q^{X+1}-1}{q-1} - q^{(r-1)X} \frac{q^{X-k}-1}{q-1} \\ &= \frac{q^{r(X+1)} - q^{(r-1)(X+1)} - q^{rX-k} + q^{(r-1)X}}{q-1} \\ &= \frac{q^{rX-k} \left(q^{k+r} - 1 \right) - q^{(r-1)X} \left(q^{r-1} - 1 \right)}{q-1} \\ &= q^{rX-k} [k+r] - q^{(r-1)X} [r-1],\end{aligned}$$

it follows that

$$\Delta \left(q^{(r-1)X} [X]_{k+1} \right) = \left(q^{rX-k} [k+r] - q^{(r-1)X} [r-1] \right) [X]_k.$$

By multiplying both sides of this last equality by $\frac{q^k}{[k+r]}$ and then summing both sides of the resulting equality from $X = 0$ to $N-1$, we get (after simplifying and rearranging)

$$\sum_{X=0}^{N-1} q^{rX}[X]_k = \frac{q^{(r-1)N+k}}{[k+r]}[N]_{k+1} + q^k \frac{[r-1]}{[k+r]} \sum_{X=0}^{N-1} q^{(r-1)X}[X]_k.$$

By reiterating this last formula r times, we get

$$\begin{aligned}\sum_{X=0}^{N-1} q^{rX}[X]_k &= q^{(r-1)N+k}\frac{1}{[k+r]}[N]_{k+1} + q^{(r-2)N+2k}\frac{[r-1]}{[k+r][k+r-1]}[N]_{k+1} \\ &\quad + q^{(r-3)N+3k}\frac{[r-1]_2}{[k+r]_3}[N]_{k+1} + \cdots + q^{rk}\frac{[r-1]_{r-1}}{[k+r]_r}[N]_{k+1} \\ &\quad + q^{rk}\frac{[r-1]_r}{[k+r]_r}\sum_{X=0}^{N-1}[X]_k \\ &= \left(\sum_{\ell=0}^{r-1} q^{(r-\ell-1)N+(\ell+1)k}\frac{[r-1]_\ell}{[k+r]_{\ell+1}}\right)[N]_{k+1} \qquad (\text{since } [r-1]_r = 0).\end{aligned}$$

The proposition is proved. □

In what follows, for $r, N \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we set

$$f_r(q,k,N) := q^{\frac{k(k-1)}{2}} \sum_{i=0}^{N-1} q^{ir}[i]_k,$$

which is a q-polynomial in N (according to Proposition 2.14). We also let $f_r(q,k,X)$ denote the unique q-polynomial interpolating $f_r(q,k,N)$ at the positive integers $X = N$, and $g_r(q,k)$ denote the limit:

$$g_r(q,k) := \lim_{X\to 0} \frac{1}{[X]} f_r(q,k,X).$$

A first useful expression for $g_r(q,k)$ is directly derived from Proposition 2.14. We have the following corollary.

Corollary 2.15. *For all $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we have*

$$g_r(q,k) = (-1)^k [k]! \left(\sum_{i=0}^{r-1} q^{ik}\frac{[r-1]_i}{[k+r]_{i+1}}\right).$$

Proof. Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$ be fixed. According to Proposition 2.14, we have

$$f_r(q,k,N) = q^{\frac{k(k-1)}{2}}\left(\sum_{\ell=0}^{r-1} q^{(r-\ell-1)N+(\ell+1)k}\frac{[r-1]_\ell}{[k+r]_{\ell+1}}\right)[N]_{k+1}$$

for all $N \in \mathbb{N}$. Hence,

$$f_r(q,k,X) = q^{\frac{k(k-1)}{2}} \left(\sum_{\ell=0}^{r-1} q^{(r-\ell-1)X+(\ell+1)k} \frac{[r-1]_\ell}{[k+r]_{\ell+1}} \right) [X]_{k+1}.$$

Consequently, we have

$$\begin{aligned} g_r(q,k) &:= \lim_{X\to 0} \frac{1}{[X]} f_r(q,k,X) \\ &= \lim_{X\to 0} q^{\frac{k(k-1)}{2}} \left(\sum_{\ell=0}^{r-1} q^{(r-\ell-1)X+(\ell+1)k} \frac{[r-1]_\ell}{[k+r]_{\ell+1}} \right) [X-1]_k \\ &= q^{\frac{k(k+1)}{2}} \left(\sum_{\ell=0}^{r-1} q^{\ell k} \frac{[r-1]_\ell}{[k+r]_{\ell+1}} \right) [-1]_k \\ &= (-1)^k [k]! \sum_{\ell=0}^{r-1} q^{\ell k} \frac{[r-1]_\ell}{[k+r]_{\ell+1}} \end{aligned}$$

(since $[-1]_k = (-1)^k q^{-\frac{k(k+1)}{2}} [k]!$). The corollary is proved. □

An equally interesting formula for $g_r(q,k)$ ($r \in \mathbb{N}, k \in \mathbb{N}_0$) can be derived by decomposing the q-rational function $\frac{q^{iX}}{[X]_{i+1}}$ ($0 \leq i \leq r-1$) into q-partial fractions. This decomposition is given by the following proposition.

Proposition 2.16. *For all $k \in \mathbb{N}_0$, we have*

$$\frac{q^{kX}}{[X]_{k+1}} = \frac{1}{[k]!} \sum_{i=0}^{k} \left((-1)^{k-i} \begin{bmatrix} k \\ i \end{bmatrix} q^{\frac{k^2+k}{2} + \frac{i^2-i}{2}} \frac{1}{[X-i]} \right).$$

Proof. Let $k \in \mathbb{N}_0$ be fixed. We will first establish the existence and uniqueness of real numbers $\alpha_{k,i}(q)$ ($0 \leq i \leq k$) such that the following holds:

$$\frac{q^{kX}}{[X]_{k+1}} = \sum_{i=0}^{k} \frac{\alpha_{k,i}(q)}{[X-i]}. \tag{2.18}$$

We have that Equation (2.18) is equivalent to

$$q^{kX} = \sum_{i=0}^{k} \alpha_{k,i}(q) \frac{[X]_{k+1}}{[X-i]},$$

which is equivalent to

$$q^{kX} = \sum_{i=0}^{k} \alpha_{k,i}(q) \left(\prod_{\substack{0 \leq j \leq k \\ j \neq i}} [X-j] \right). \tag{2.19}$$

Consider the family $\mathscr{B}$ of $(k+1)$ q-polynomials of degree k given by:

$$\mathscr{B} := \left\{ \prod_{\substack{0 \leq j \leq k \\ j \neq i}} [X - j] \right\}_{0 \leq i \leq k}.$$

It can be observed that any k arbitrary q-polynomials from $\mathscr{B}$ always share a common root that is not a root of the remaining q-polynomial in $\mathscr{B}$. Therefore, no q-polynomial in $\mathscr{B}$ can be expressed as a linear combination of the others, which shows that $\mathscr{B}$ is a linearly independent family in $\mathscr{E}_k$. Consequently, since $\mathrm{Card}\mathscr{B} = \dim \mathscr{E}_k = k+1$, $\mathscr{B}$ is a basis of $\mathscr{E}_k$. It follows that the q-polynomial q^{kX} (belonging to $\mathscr{E}_k$) can be uniquely expressed as a linear combination of the q-polynomials in $\mathscr{B}$. This establishes the existence and uniqueness of real numbers $\alpha_{k,i}(q)$ $(0 \leq i \leq k)$ satisfying Equation (2.19) (equivalently, Equation (2.18)).

Let us now determine the real numbers $\alpha_{k,i}(q)$ $(0 \leq i \leq k)$. For a given $\ell \in \{0, 1, \ldots, k\}$, by multiplying both sides of Equation (2.18) by $[X - \ell]$ and then substituting $X = \ell$ into the resulting identity, we obtain:

$$\frac{q^{k\ell}}{[\ell][\ell-1]\cdots[1]\cdot[-1][-2]\cdots[\ell-k]} = \alpha_{k,\ell}(q).$$

But since $[\ell][\ell-1]\cdots[1] = [\ell]!$ and $[-1][-2]\cdots[\ell-k] = (-1)^{k-\ell} q^{-\frac{(k-\ell)(k-\ell+1)}{2}}[k-\ell]!$ (using the property $[-X] = -q^{-X}[X]$), it follows that

$$\alpha_{k,\ell}(q) = \frac{(-1)^{k-\ell} q^{\frac{(k-\ell)(k-\ell+1)}{2}+k\ell}}{[\ell]![k-\ell]!} = \frac{1}{[k]!}(-1)^{k-\ell} \begin{bmatrix} k \\ \ell \end{bmatrix} q^{\frac{k^2+k}{2}+\frac{\ell^2-\ell}{2}},$$

as required. This completes the proof. ∎

Using the formula from Proposition 2.16, the formula of Corollary 2.15 for $g_r(q,k)$ can be transformed into another, more interesting form for studying the limiting case as $q \to 1$. We have the following proposition.

Proposition 2.17. *For all $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we have*

$$g_r(q,k) = (-1)^k [k]![r-1]! q^{-r+1} \sum_{i=0}^{r-1} \frac{(q-1)^i q^{\frac{1}{2}i(i+1)-i(r-1)}}{[r-1-i]!} \frac{1}{[k+i+1]}.$$

Proof. Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$ be fixed. By successively applying Corollary 2.15 and

Proposition 2.16, we obtain

$$\begin{aligned}
g_r(q,k) &= (-1)^k[k]!\sum_{i=0}^{r-1} q^{-ir}[r-1]_i \left\{\frac{q^{i(k+r)}}{[k+r]_{i+1}}\right\} \\
&= (-1)^k[k]!\sum_{i=0}^{r-1} q^{-ir}[r-1]_i \frac{1}{[i]!}\sum_{j=0}^{i}(-1)^{i-j}\begin{bmatrix} i \\ j\end{bmatrix} q^{\frac{i^2+i}{2}+\frac{j^2-j}{2}}\frac{1}{[k+r-j]} \\
&= (-1)^k[k]!\sum_{\substack{0\le i\le r-1\\ 0\le j\le i}} q^{-ir}\frac{[r-1]_i}{[i]!}\begin{bmatrix} i \\ j\end{bmatrix}(-1)^{i-j} q^{\frac{i^2+i}{2}+\frac{j^2-j}{2}}\frac{1}{[k+r-j]}.
\end{aligned}$$

By inverting the summations over i and j and noting that $\frac{[r-1]_i}{[i]!}\begin{bmatrix} i \\ j\end{bmatrix} = \begin{bmatrix} r-1 \\ j\end{bmatrix}\begin{bmatrix} r-1-j \\ i-j\end{bmatrix}$ and that $q^{-ir}\cdot q^{\frac{i^2+i}{2}+\frac{j^2-j}{2}} = q^{\frac{(i-j)^2+(i-j)}{2}-i(r-j)}$ (for all $0\le i\le r-1$ and $0\le j\le i$), we derive that

$$g_r(q,k) = (-1)^k[k]!\sum_{0\le j\le r-1}\left(\sum_{i=j}^{r-1}\begin{bmatrix} r-1-j \\ i-j\end{bmatrix}(-1)^{i-j}q^{\frac{(i-j)^2+(i-j)}{2}-i(r-j)}\right) \times \begin{bmatrix} r-1 \\ j\end{bmatrix}\frac{1}{[k+r-j]}.$$

Then, performing the change of index $\ell = i-j$ in the inner summation gives

$$g_r(q,k) = (-1)^k[k]!\sum_{0\le j\le r-1}\left(\sum_{\ell=0}^{r-1-j} q^{\frac{\ell(\ell-1)}{2}}\begin{bmatrix} r-1-j \\ \ell\end{bmatrix}\left(-q^{1-r+j}\right)^{\ell}\right)q^{j^2-jr} \times \begin{bmatrix} r-1 \\ j\end{bmatrix}\frac{1}{[k+r-j]}.$$

But according to the Gauss binomial formula (see Equation (1.2)), we have

$$\begin{aligned}
\sum_{\ell=0}^{r-1-j} q^{\frac{\ell(\ell-1)}{2}}\begin{bmatrix} r-1-j \\ \ell\end{bmatrix}\left(-q^{1-r+j}\right)^{\ell} &= \left(1-q^{1-r+j}\right)\left(1-q^{2-r+j}\right)\cdots\left(1-q^{-1}\right) \\
&= q^{1-r+j}\left(q^{r-1-j}-1\right)\cdot q^{2-r+j}\left(q^{r-2-j}-1\right) \\
&\quad \times\cdots q^{-1}(q-1) \\
&= q^{-\frac{(r-1-j)(r-j)}{2}}(q-1)^{r-1-j}[r-1-j]!
\end{aligned}$$

for all $j\in\{0,1,\ldots,r-1\}$. Substituting this into the last obtained expression for $g_r(q,k)$, we get

$$g_r(q,k) = (-1)^k[k]!\sum_{j=0}^{r-1} q^{-\frac{(r-1-j)(r-j)}{2}}(q-1)^{r-1-j}[r-1-j]!q^{j^2-jr}\begin{bmatrix} r-1 \\ j\end{bmatrix}\frac{1}{[k+r-j]}.$$

Finally, changing the index $i = r-1-j$ and rearranging yields

$$g_r(q,k) = (-1)^k [k]![r-1]! q^{-r+1} \sum_{i=0}^{r-1} \frac{(q-1)^i q^{\frac{1}{2}i(i+1)-i(r-1)}}{[r-1-i]!} \frac{1}{[k+i+1]},$$

as required. □

We are now ready to prove Theorem 2.13.

Proof of Theorem 2.13. Let $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$ be fixed. In view of Corollary 2.15 and Proposition 2.17, we need to show that

$$\beta_n^{(r)} = \sum_{k=0}^{n} g_r(q,k) S_q(n,k).$$

To achieve this, consider $N \in \mathbb{N}$ and evaluate the sum

$$\sum_{i=0}^{N-1} q^{ri}[i]^n$$

in two different ways. On the one hand, from Theorem 2.3, we have

$$\sum_{i=0}^{N-1} q^{ri}[i]^n = q^{rN} S_n(N) - S_n(0),$$

where $(S_k(X))_{k \in \mathbb{N}_0}$ is the Carlitz-type q-polynomial sequence associated with the real sequence of general term $a_k := \frac{1}{q-1}\frac{1}{[k+r]}$ $(k \in \mathbb{N}_0)$. So, putting $s_k := S_k(0)$ $(k \in \mathbb{N}_0)$ and using the formula of Theorem 2.1 for $S_k(X)$, we derive that

$$\begin{aligned} \sum_{i=0}^{N-1} q^{ri}[i]^n &= q^{rN} \sum_{k=0}^{n} \binom{n}{k} s_k q^{kN} [N]^{n-k} - s_n \\ &= \sum_{k=0}^{n-1} \binom{n}{k} s_k q^{(k+r)N} [N]^{n-k} + \left(q^{(n+r)N} - 1\right) s_n. \end{aligned} \tag{2.20}$$

On the other hand, using Equation (1.4), we have that

$$\begin{aligned} \sum_{i=0}^{N-1} q^{ri}[i]^n &= \sum_{i=0}^{N-1} q^{ri} \left(\sum_{k=0}^{n} q^{\frac{1}{2}k(k-1)} S_q(n,k) [i]_k \right) \\ &= \sum_{k=0}^{n} S_q(n,k) \left(q^{\frac{1}{2}k(k-1)} \sum_{i=0}^{N-1} q^{ri} [i]_k \right) \\ &= \sum_{k=0}^{n} S_q(n,k) f_r(q,k,N). \end{aligned} \tag{2.21}$$

Comparing Equations (2.20) and (2.21), we derive the identity:

$$\sum_{k=0}^{n-1}\binom{n}{k}s_k q^{(k+r)N}[N]^{n-k}+\left(q^{(n+r)N}-1\right)s_n=\sum_{k=0}^{n}S_q(n,k)f_r(q,k,N).$$

Since this identity consists of q-polynomials in N and it is valid for all $N\in\mathbb{N}$, it can be extended to a q-polynomial identity. Namely, we have

$$\sum_{k=0}^{n-1}\binom{n}{k}s_k q^{(k+r)X}[X]^{n-k}+\left(q^{(n+r)X}-1\right)s_n=\sum_{k=0}^{n}S_q(n,k)f_r(q,k,X).$$

Dividing through by $[X]$ and letting $X\to 0$, we derive

$$ns_{n-1}+(q-1)(n+r)s_n=\sum_{k=0}^{n}g_r(q,k)S_q(n,k). \tag{2.22}$$

Now, setting

$$\begin{aligned}T_k(X)&:=kS_{k-1}(X)+(q-1)(k+r)S_k(X)\\&=(q-1)rS_k(X)+k\left(S_{k-1}(X)+(q-1)S_k(X)\right)\end{aligned}$$

for all $k\in\mathbb{N}_0$, it is clear that $ns_{n-1}+(q-1)(n+r)s_n=T_n(0)$. Next, the q-polynomial sequence $((q-1)rS_k(X))_{k\in\mathbb{N}_0}$ is clearly Carlitz-type with associated real sequence $((q-1)ra_k)_k=\left(\frac{r}{[k+r]}\right)_k$. Moreover, according to Proposition 2.6 (iv), the q-polynomial sequence $\left(k\left(S_{k-1}(X)+(q-1)S_k(X)\right)\right)_{k\in\mathbb{N}_0}$ is also Carlitz-type with associated real sequence $((q-1)ka_k)_k=\left(\frac{k}{[k+r]}\right)_k$. Consequently, the q-polynomial sequence $(T_k(X))_{k\in\mathbb{N}_0}$ is Carlitz-type with associated real sequence $\left(\frac{r}{[k+r]}\right)_k+\left(\frac{k}{[k+r]}\right)_k=\left(\frac{k+r}{[k+r]}\right)_k$. Thus, $(T_k(X))_k$ coincides with $\left(\beta_k^{(r)}(X)\right)_k$, that is, $T_k(X)=\beta_k^{(r)}(X)$ (for all $k\in\mathbb{N}_0$). In particular, we have $ns_{n-1}+(q-1)(n+r)s_n=T_n(0)=\beta_n^{(r)}(0)=\beta_n^{(r)}$. Substituting this into Equation (2.22) yields the required formula. This completes the proof. □

From Theorem 2.13, we derive the following curious corollary.

Corollary 2.18. *For all $n\in\mathbb{N}_0$, we have*

$$\beta_n+q\beta_{n+1}=\sum_{k=0}^{n}(-1)^k\frac{[k]!}{[k+2]}S_q(n,k).$$

Proof. Let $n\in\mathbb{N}_0$ be fixed. According to Equation (2.12), we have

$$\beta_n^{(2)}=\beta_n^{(1)}+(q-1)\beta_{n+1}^{(1)}=\beta_n+(q-1)\beta_{n+1},$$

which gives $\beta_{n+1} = \frac{\beta_n^{(2)} - \beta_n}{q-1}$, and then

$$\beta_n + q\beta_{n+1} = \beta_n + q\left(\frac{\beta_n^{(2)} - \beta_n}{q-1}\right) = \frac{q\beta_n^{(2)} - \beta_n^{(1)}}{q-1}.$$

It follows by using Theorem 2.13 that

$$\beta_n + q\beta_{n+1} = \sum_{k=0}^{n} \frac{q\varphi_2(q,k) - \varphi_1(q,k)}{q-1} S_q(n,k). \tag{2.23}$$

Further, from the second expression of $\varphi_r(q,k)$ in Theorem 2.13, we have that

$$\varphi_1(q,k) = (-1)^k \frac{[k]!}{[k+1]} \quad \text{and} \quad \varphi_2(q,k) = (-1)^k [k]! q^{-1} \left(\frac{1}{[k+1]} + \frac{q-1}{[k+2]}\right).$$

Thus,

$$\frac{q\varphi_2(q,k) - \varphi_1(q,k)}{q-1} = (-1)^k \frac{[k]!}{[k+2]}.$$

Substituting this into Equation (2.23) yields the required formula. This achieves the proof. □

Letting $q \to 1$ in Corollary 2.18 gives the following result on the classical Bernoulli numbers, which was very recently pointed out by Farhi [5, Page 13, Formula (21)].

Corollary 2.19. *For all $n \in \mathbb{N}_0$, we have*

$$B_n + B_{n+1} = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+2} S(n,k).$$

Acknowledgements. The author would like to express his sincere gratitude to the anonymous reviewer for their careful reading of the manuscript and for their highly valuable comments and suggestions. In particular, the reviewer's recommendations to include the pioneering works on sums of powers of consecutive q-integers and to mention the related probabilistic models have significantly enriched the historical context and the overall quality of this paper.

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