ON ROBIN'S INEQUALITY AND THE KANEKO–LAGARIAS INEQUALITY

Idris Assani
Dept. of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina
assani@math.unc.edu

Aiden Chester
Dept. of Mathematics, North Carolina State University, Raleigh, North Carolina
azcheste@ncsu.edu

Alex Paschal
Dept. of Mathematics, University of North Carolina at Chapel Hill, North Carolina
ampasch@unc.edu

Received: 8/15/25, Accepted: 6/16/25, Published: 6/26/26

Abstract

We provide new, elementary proofs that Robin's inequality and the Lagarias inequality hold for almost every positive integer, including all positive integers not divisible by one of the prime numbers 2, 3, 5; all primorials; given k a positive integer, all sufficiently large positive integers of the form $2^k n$ for $n \geq 1$ odd; and all 21-free integers. Additionally, we prove that the Kaneko–Lagarias inequality holds for all positive integers if and only if it holds for all superabundant numbers.

1. Introduction

We define the *sum of divisors function* and *Euler's totient function* as

$$\sigma(n) = \sum_{d|n} d, \quad \phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

respectively, where the product is taken over primes p that divide n. It has long been known that these functions have connections to the Riemann hypothesis (RH). Robin's inequality [15] states that RH holds if and only if, for all $n > 5040$ we have

$$\sigma(n) < e^{\gamma} n \log(\log(n)), \tag{1}$$

where $\gamma \approx .57721\ldots$ denotes the Euler–Mascheroni constant.

DOI: 10.5281/zenodo.20931450

We briefly survey some known results concerning families of positive integers that satisfy Robin's inequality. A positive integer is *k-free* if all powers in its prime factorization are less than k. In [7], it is proven that all odd integers greater than 9 satisfy Robin's inequality, as do all 5-free integers. This was extended to 7-free positive integers in [17], 11-free positive integers in [6], 20-free positive integers in [14], and finally 21-free positive integers in [4]. We reprove the result of Axler [4, Theorem 1] using more elementary methods in Section 2.4: while Axler's proof relies on estimates from [17] and combinatorial prime counting algorithms, our proof uses only arithmetic manipulations and a sharper bound from [5] than the one used in [16, Theorem 15].

Similarly, Theorem 4, which states that Robin's inequality holds for positive integers not divisible by one of the primes 2, 3, 5, is implied by the results of [10], but is proven using more elementary methods. The proof in [10], which works with a positive integer's *p-adic order*, relies on an algorithm from [2], whereas ours is derived only from arithmetic manipulations.

The final class of results concerning Robin's inequality is density results. The first such result was by Robin [15], who proved that Robin's inequality holds for all square-free (that is, 2-free) positive integers. In [18, p. 46], it is shown that the logarithmic density of non-square-free integers is $\frac{1}{2} - \frac{2}{\pi^2} \approx .2973....$ Similarly, Theorem 4 shows that Robin's inequality holds for a set of logarithmic density $\frac{29}{30}$. Wójtowicz [20] was the first to show that Robin's inequality holds on a set of density 1. We prove this using different methods in Section 2.5. Our proof, again, relies mostly on arithmetic manipulations, as opposed to the "deep" results of Ford and Luca-Pomerance. It is worth noting that according to [12], for $x > 7!$ we have

$$\#\{n \le x \mid \sigma(n) \ge e^{\gamma} n \log(\log(n))\} = x^{O(\frac{1}{\log(\log(x))})}.$$

So, the density of counterexamples to Robin's inequality up to large but finite x is also quite small.

Another well-studied inequality which is equivalent to RH is the Lagarias inequality. Denote by H_n the n-th *harmonic number*; that is, $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. The Lagarias inequality [11] states that, for all $n \ge 1$, we have

$$\sigma(n) < H_n + \exp(H_n)\log(H_n).$$

It turns out that the H_n term in the right-hand side is negligible in the sense that the following inequality, which we name the Kaneko–Lagarias inequality (see the acknowledgments in [11]), is also equivalent to RH: for all $n > 60$, we have

$$\sigma(n) < \exp(H_n)\log(H_n).$$

We note that similar alternative inequalities have been introduced in [19].

A positive integer n is *superabundant* if $m < n$ implies $\sigma(m)/m < \sigma(n)/n$. By dividing both sides of Equation (1) by n and noting that the right-hand side is increasing, one immediately sees that Robin's inequality holds if and only if it holds for superabundant numbers (this observation was made in [1]). One would like to say the same for the Lagarias inequality and the Kaneko–Lagarias inequality, but the picture is more complicated since monotonicity is harder to prove. Nevertheless, in Theorem 9 we prove that the Kaneko–Lagarias inequality holds if and only if it holds for superabundant numbers. We would like to extend this result to the Lagarias inequality in future work. This extension has been announced recently in [13].

The layout of our paper is as follows. In Section 2, we consider Robin's inequality. Each subsection corresponds to a result which we prove, and is labeled as such. We note that we use a unified method throughout the section, reobtaining some results that were found using various methods. In Section 3, we focus on the Lagarias inequality. We introduce the Kaneko–Lagarias inequality and prove the content of the last paragraph.

2. Robin's Inequality

2.1. Sufficiently Large Positive Integers not Divisible by One of the Prime Numbers 2,3,5

Let $p_1 = 2$, $p_2 = 3$, etc. be the increasing enumeration of the prime numbers. We denote the set of prime numbers by $\mathbb{P}$. Fix $j \in \mathbb{N}$ and let $q_1 < q_2 < \cdots < q_k$ be some prime numbers distinct from p_j. Given $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$, let $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$.

Lemma 1. *For $k \geq j$, we have*

$$\frac{\sigma(n)}{n} < \prod_{\ell=1}^{k} \frac{q_\ell}{q_\ell - 1} = \frac{n}{\phi(n)} < \prod_{\substack{\ell=1,\ldots,j-1,\\ j+1,\ldots,k+1}} \frac{p_\ell}{p_\ell - 1}.$$

Proof. The first inequality follows from the fact that for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}$,

$$\frac{\sigma(p^\alpha)}{p^\alpha} = \frac{p - \frac{1}{p^\alpha}}{p-1} \nearrow \frac{p}{p-1} \text{ as } \alpha \to \infty.$$

The second inequality follows from the fact that $p_i \leq q_i$ for all $1 \leq i \leq k$. □

Note that

$$\frac{n}{\phi(n)} = \left(\prod_{\substack{\ell=1,\ldots,j-1,\\ j+1,\ldots,k+1}} \frac{p_\ell + 1}{p_\ell} \right) \left(\prod_{\substack{\ell=1,\ldots,j-1,\\ j+1,\ldots,k+1}} \frac{p_\ell^2}{p_\ell^2 - 1} \right) =: A(k)B(k).$$

We can bound $A(k)$ as follows:

$$\log(A(k)) = \sum_{\substack{\ell=1,\ldots,j-1,\\ j+1,\ldots,k+1}} \log\left(1+\frac{1}{p_\ell}\right) \leq \sum_{\substack{\ell=1,\ldots,j-1,\\ j+1,\ldots,k+1}} \frac{1}{p_\ell}$$
$$= \left(\sum_{\ell=1}^{k+1}\frac{1}{p_\ell}\right) - \frac{1}{p_j}$$

and

$$\sum_{\ell=1}^{k+1}\frac{1}{p_\ell} \leq \log(\log(p_{k+1})) + c_1 + \frac{5}{\log(p_{k+1})},$$

where $c_1 \approx .261497$ by Theorem 1.10 in [18]. Thus, we obtain the following result.

Lemma 2. *For all $k \in \mathbb{N}$, we have*

$$A(k) \leq \log(p_{k+1})\exp\left(c_1 - \frac{1}{p_j} + \frac{5}{\log(p_{k+1})}\right). \tag{2}$$

Combining [8] and Corollary 3 from [16], we obtain the following theorem. The first inequality holds for $k \geq 2$ as shown in [8], while the second given by Corollary 3 in [16] holds for $k \geq 6$.

Theorem 1. *For $k \geq 6$, we have*

$$k(\log(k) + \log(\log(k)) - 1) < p_k < k(\log(k) + \log(\log(k))). \tag{3}$$

Furthermore, combining Equation (2) and Equation (3), we see that the following holds.

Lemma 3. *For $k \geq 6$, we have $A(k) < C(k)$ where*

$$\begin{aligned} C(k) &= \log((k+1)(\log(k+1) + \log(\log(k+1)))) \\ &\quad \times \exp\left(c_1 - \frac{1}{p_j} + \frac{5}{\log((k+1)(\log(k+1) + \log(\log(k+1)) - 1))}\right). \end{aligned} \tag{4}$$

Now, put $m = p_{k+1}\#/p_j$. Our goal is to show the following, since it implies that Robin's holds for n as above.

Theorem 2. *For any $j \in \{1,2,3\}$, there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies*

$$C(k)B(k) < e^\gamma \log(\log(m)).$$

Corollary 1. *Suppose Theorem 2 holds. Then Robin's inequality holds for n as above.*

Proof. We have

$$\frac{\sigma(n)}{n} < \frac{n}{\phi(n)} = A(k)B(k) < C(k)B(k) < e^{\gamma}\log(\log(m)) \leq e^{\gamma}\log(\log(n)),$$

where the last inequality follows from the fact that $m \leq n$. □

Definition 1. The *Chebyshev function* is defined as follows:

$$\theta(x) = \sum_{p\in\mathbb{P}, p\leq x} \log(p) = \log\left(\prod_{p\in\mathbb{P}, p\leq x} p\right).$$

Theorem 3. *For $x \geq 529$, we have*

$$\prod_{\substack{p\in\mathbb{P}\\ p\leq x}} p = e^{\theta(x)} > e^{x\left(1-\frac{1}{2\log x}\right)} \geq (2.51)^x.$$

Proof. The first inequality is given by (3.14) in [16] and the second follows from computations since the function $f(x) = 1 - \frac{1}{2\log x}$ increases for $x > 1$. □

Lemma 4. *For $k \geq 99$, we have*

$$\log(\log(m)) > \log((k+1)(\log(k+1)+\log(\log(k+1))-1)\log(2.51)-\log(p_j)) =: D(k).$$

Proof. Noting that $k \geq 99$ implies $p_{k+1} > 529$, we have

$$\begin{aligned}\log(\log(m)) &= \log\left(\log\left(\frac{p_{k+1}\#}{p_j}\right)\right) > \log\left(\log\left(\frac{(2.51)^{p_{k+1}}}{p_j}\right)\right)\\ &= \log(p_{k+1}\log(2.51) - \log(p_j))\\ &> \log((k+1)(\log(k+1) + \log(\log(k+1)) - 1)\log(2.51) - \log(p_j)),\end{aligned}$$

where the last inequality uses Theorem 1. □

The following implies Theorem 2.

Proposition 1. *For $j \in \{1, 2, 3\}$, there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies*

$$C(k)B(k) < e^{\gamma}D(k). \tag{5}$$

Proof. Denote

$$\begin{aligned}\widetilde{C}(k) = e^{-c_1+\frac{1}{p_j}}C(k) &= \log((k+1)(\log(k+1) + \log(\log(k+1))))\\ &\quad\times \exp\left(\frac{5}{\log((k+1)(\log(k+1) + \log(\log(k+1)) - 1))}\right)\end{aligned}$$

and

$$\widehat{C}(k) = \exp\left(\frac{5}{\log((k+1)(\log(k+1)+\log(\log(k+1))-1))}\right).$$

Multiplying both sides of Equation (5) by $e^{-c_1+\frac{1}{p_j}} p_j^2/(p_j^2-1)$, we obtain

$$\widetilde{C}(k)\prod_{\ell=1}^{k+1}\frac{p_\ell^2}{p_\ell^2-1} < \frac{e^{\gamma-c_1+\frac{1}{p_j}}}{p_j^2-1}D(k). \tag{6}$$

Noting that

$$\prod_{\ell=1}^{k+1}\frac{p_\ell^2}{p_\ell^2-1} \nearrow \frac{\pi^2}{6} \text{ as } k\to\infty,$$

we see that Equation (6) is implied by

$$\widetilde{C}(k) < \frac{6p_j^2 e^{\gamma-c_1+\frac{1}{p_j}}}{\pi^2(p_j^2-1)}D(k) =: E_j D(k). \tag{7}$$

Raising both sides to the power of e, we see that Equation (7) is implied by

$$\begin{aligned}&[(k+1)(\log(k+1)+\log(\log(k+1))]^{\widehat{C}(k)}\\ &< [(k+1)(\log(k+1)+\log(\log(k+1))-1)\log(2.51)-\log(p_j)]^{E_j}.\end{aligned} \tag{8}$$

Equation (8) is equivalent to

$$\begin{aligned}1 &< [(k+1)(\log(k+1)+\log(\log(k+1)))]^{-\widehat{C}(k)+E_j}\\ &\left[1-\frac{(k+1)\log(2.51)+\log(p_j)}{(k+1)(\log(k+1)+\log(\log(k+1)))}\right]^{E_j}.\end{aligned} \tag{9}$$

Noting that $E_j > 1$ for $j \in \{1,2,3\}$, we see that there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies $-\widehat{C}(k)+E_j > \epsilon$ for some $\epsilon \in (0,1)$. If needed, we can increase K_j so that $k \geq K_j$ implies

$$\left[1-\frac{(k+1)\log(2.51)+\log(p_j)}{(k+1)(\log(k+1)+\log(\log(k+1)))}\right]^{E_j} > \epsilon,$$

and also so that $k \geq K_j$ implies

$$1 < \epsilon[(k+1)(\log(k+1)+\log(\log(k+1))]^{\epsilon},$$

which implies Equation (9). □

Remark 1. As suggested by the referee, in Equation (4) the constant 5 can be replaced by $2(1+\log 4)$, which is sharper. However, computations with this constant do not improve any of the results.

2.2. All Positive Integers not Divisible by One of the Prime Numbers 2, 3, 5.

Leaving $j = 1$ in Equation (7), we seek to show that

$$\widetilde{C}(k) < \frac{8e^{\gamma - c_1 + .5}}{\pi^2} D(k). \tag{10}$$

Lemma 5. *For $k \geq 13042$, we have $\widehat{C}(k) < 1.525$.*

Proof. The sequence $\widehat{C}(k)$ decreases with k, so the result follows from the calculation. □

Denote $f(k) = (k+1)(\log(k+1) + \log(\log(k+1))$. Applying Lemma 5 to Equation (10) and performing some algebraic manipulations, our goal is to show that

$$\log(f(k)) < \frac{8e^{\gamma - c_1 + .5}}{\pi^2(1.525)} \log((f(k) - 1)\log(2.51) - \log(2)).$$

Raising both sides to the power of e, this becomes

$$1 < f(k)^{0.20166} \left[1 - \frac{(k+1)\log(2.51) + \log(2)}{f(k)}\right]^{1.20166}. \tag{11}$$

The right-hand side of Equation (11) is increasing, and a calculation reveals that it holds for $k \geq 13042$. Additionally, using the lemma 1, one can check that

$$\frac{\sigma(n)}{n} < \frac{n}{\phi(n)} < e^{\gamma} \log(\log(m))$$

for $k \geq 3$. Finally, when $k \in \{1, 2\}$, we check that

$$\frac{\sigma(n)}{n} < \frac{n}{\phi(n)} \leq \frac{15}{8} < e^{\gamma} \log(\log(n))$$

for $n \geq 680$. This confirms the following for $j = 1$.

Theorem 4. *For $j \in \{1, 2, 3\}$, Robin's inequality holds for every positive integer greater than 5040 that is not divisible by p_j.*

To confirm Theorem 4 when $j \in \{2, 3\}$, one can repeat the above process to see that, for sufficiently large k, Equation (7) is satisfied. The cases with smaller k have been verified in [14].

2.3. Primorials and Sufficiently Large Even Positive Integers

We consider positive integers of the form $2^k n$ for odd n.

Fix $k \in \mathbb{N}$ and let n be odd. We have

$$\frac{\sigma(2^k n)}{2^k n} = \frac{\sigma(2^k)}{2^k}\frac{\sigma(n)}{n} < \frac{\sigma(2^k)}{2^k}\frac{n}{\phi(n)} = \frac{\sigma(2^k)}{2^k}\frac{\phi(2^k)}{2^k}\frac{2^k n}{\phi(2^k n)} = \left(1 - \frac{1}{2^{k+1}}\right)\frac{2^k n}{\phi(2^k n)}.$$

Applying Theorem 15 from [16], we know

$$\left(1 - \frac{1}{2^{k+1}}\right)\frac{2^k n}{\phi(2^k n)} < \left(1 - \frac{1}{2^{k+1}}\right)\left(e^\gamma \log(\log(2^k n)) + \frac{2.51}{\log(\log(2^k n))}\right).$$

We ask which n satisfy

$$\left(1 - \frac{1}{2^{k+1}}\right)\left(e^\gamma \log(\log(2^k n)) + \frac{2.51}{\log(\log(2^k n))}\right) < e^\gamma(\log(\log(2^k n))).$$

This is equivalent to asking when

$$\frac{2.51(2^{k+1}-1)}{e^\gamma} < (\log(\log(2^k n)))^2$$

holds, which is when

$$n > \frac{e^{e^{\sqrt{\frac{2.51(2^{k+1}-1)}{e^\gamma}}}}}{2^k} =: b(k). \tag{12}$$

Thus, we obtain the following result.

Theorem 5. *Given any $k \in \mathbb{N}$, Robin's inequality holds for all positive integers of the form $2^k n$ when n is odd and satisfies Equation* (12).

In particular, we have the following corollary.

Corollary 2. *If $n \geq 620$ is odd, then Robin's inequality holds for $2n$. Furthermore, Robin's inequality holds for all primorials strictly greater than* 30.

Proof. The first statement follows immediately from Theorem 5 and the second follows from the computation of primorials strictly less than 1240. □

2.4. All 21-Free Positive Integers

The results of the previous subsection are based on the inequality in Theorem 15 from [16]. This inequality can be improved by using a stricter bound, such as the following from [5]. We have

$$\frac{m}{\phi(m)} < e^\gamma \log(\log(m)) + \frac{.0168}{(\log(\log(m)))^2},$$

for $m \geq 10^{10^{13.11485}} = C$. Using the same reasoning as before, we derive the following result.

Theorem 6. *Given any k positive integer, Robin's inequality holds for all positive integers of the form $2^k n$ when n is odd and satisfies*

$$2^k n > e^{e^{\left(\frac{.0168(2^{k+1}-1)}{e^{\gamma}}\right)^{\frac{1}{3}}}} =: 2^k \tilde{b}(k).$$

Furthermore, it was shown in [14] that Robin's inequality holds for all positive integers $5041 < m \le C$. We can thus conclude the following.

Theorem 7. *Robin's inequality holds for all positive integers of the form $2^k n$ with n odd as long as $2^k \tilde{b}(k) < C$. In particular, Robin's inequality holds for all 21-free positive integers.*

Proof. Let k be a positive integer and n be an odd positive integer.

- if $5041 < 2^k n \le 2^k \tilde{b}(k) < C$ then $2^k n$ satisfies Robin's inequality by [14].
- Alternatively, if $2^k n > 2^k \tilde{b}(k)$ then $2^k n$ satisfies Robin's inequality by Theorem 6.

Recalling that a ℓ-free positive integer is a positive integer not divisible by any ℓ power of a prime number greater than or equal to 2, we can see that if $2^k \tilde{b}(k) < C$ then all $(k+1)$-free numbers satisfy Robin's inequality. Since $\log(2^{20}\tilde{b}(20)) < 6(10^{11}) < 2.3(10^{13.11485}) < \log(C)$, we can conclude that Robin's inequality holds for all 21-free positive integers. □

Remark 2. The validity of Robin's inequality for ℓ-free positive integers was proved for $\ell = 7$ in [17], for $\ell = 11$ in [6], for $\ell = 20$ in [14] and (as pointed out by one of the referees) for $\ell = 21$ in [4].

2.5. Almost Every Positive Integer

Definition 2. The *natural density* of a set E is

$$d(E) = \lim_{n\to\infty} \frac{\#E \cap \{1, 2, \ldots, n\}}{n}$$

when the limit exists.

Theorem 8. *Denote by $\mathcal{R}$ the set of positive integers that satisfies Robin's inequality. Then the natural density of $\mathcal{R}$ is* 1.

Proof. We will prove that the natural density of $\mathcal{R}^c$ is 0. Fix $\epsilon > 0$. Let $E_k = \{2^k n : n \in \mathbb{N}_{\text{odd}}, n \le b(k)\}$ and note that $\mathcal{R}^c \subseteq \bigcup_{k\ge 1} E_k$ by Theorem 4 and Theorem 5.[1]

[1] Here $\mathbb{N}_{\text{odd}} := \{1, 3, \ldots\}$.

Pick M so that $\sum_{k=M+1}^{\infty} \frac{1}{2^k} < \frac{\epsilon}{2}$. For $s \in \mathbb{N}$ we calculate

$$\frac{\#\mathcal{R}^c \cap \{1,2,\ldots,s\}}{s} \leq \frac{\#\bigcup_{k\geq 1} E_k \cap \{1,2,\ldots,s\}}{s} = \frac{\sum_{k\geq 1} \#E_k \cap \{1,2,\ldots,s\}}{s}$$
$$= \frac{\sum_{k=1}^{M} \#E_k \cap \{1,2,\ldots,s\} + \sum_{k=M+1}^{\infty} \#E_k \cap \{1,2,\ldots,s\}}{s},$$

where the first equality follows from the fact that the E_k's are disjoint. Noting that if $\sum_{k=1}^{M} \#E_k \cap \{1,2,\ldots,s\}$ is finite for all $m \in \mathbb{N}$ then we can choose S so that s greater than or equal to S implies that the right-hand side of Equation (13) is strictly less than ϵ, which completes our proof. □

Remark 3. Using less elementary results from number theory, Wójtowicz [20] proves the following stronger result: for every $0 < C \leq 1$, there exists a subset $W_C \subseteq \mathcal{R}$ such that $d(W_C) = 1$ and for all $n \in W_C$, we have

$$\frac{\sigma(n)}{n} < Ce^{\gamma} \log(\log(n)).$$

3. The Lagarias and Kaneko-Lagarias Inequalities

3.1. Superabundant Numbers

Let $\Gamma(x)$ denote the gamma function. We define two functions:

$$H(x) = \int_0^1 \frac{t^x - 1}{t-1} dt,$$
$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

ψ is known as the *digamma function*. It is known that H is smooth for $x \geq 1$ and that $H(n) = H_n$ for all $n \in \mathbb{N}$. Additionally, ψ, known as the digamma function, satisfies

$$H(x) = \psi(x+1) + \gamma. \tag{13}$$

Lemma 6. *For all $x \geq 1$, we have*

$$H(x) < \log(x) + \gamma + \frac{1}{2x}.$$

Proof. By (2.2) from [3], we get

$$\psi(x) < \log(x) - \frac{1}{2x}.$$

for all $x \geq 1$. Then we use Equation (13) and $\psi(x+1) = \psi(x) + \frac{1}{x}$ to finish. □

Lemma 7. *For all $x \geq 4$, we have*

$$H(x) < \frac{2\log(x)}{1 + \frac{6}{\pi^2 x}}.$$

Proof. By Lemma 6, it suffices to show that

$$\log(x) + \gamma + \frac{1}{2x} < \frac{2\log(x)}{1 + \frac{6}{\pi^2 x}} \tag{14}$$

for $x \geq 4$. By arithmetic manipulations, Equation (14) becomes

$$\frac{1}{\pi^2 x - 6}\left(\gamma \pi^2 x + \frac{\pi^2}{2} + 6\gamma + \frac{3}{x}\right) < \log(x). \tag{15}$$

Computation reveals that Equation (15) holds for $x = 4$, and the left hand side of Equation (15) is decreasing while the right-hand side is increasing, so we obtain the result. □

Lemma 8. *The following hold.*

1. *For all $n > 1$, we have $H_{n+1} \leq \frac{n}{\log(n)}$.*
2. *For all $x \geq 4$, we have $\log(H(x)) \leq \frac{x}{2\log(x)}$.*

Proof. (1.) We can manually verify the inequality for $n \leq 6$. Because

$$H_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k} \leq 1 + \int_1^{n+1} \frac{dt}{t} = 1 + \log(n+1),$$

it suffices to show that

$$\log(x)(\log(x+1) + 1) \leq x.$$

Put $g(t) = e^t - t^2 - t - 1$. We see that $g(2) > 0$ and that $g'(t) = e^t - 2t - 1 > 0$ for $t \geq 2$, so $g(t) > 0$ for $t \geq 2$. For $x \geq e^2 - 1$ we have

$$0 < g(\log(x+1)) = x + 1 - (\log(x+1))^2 - \log(x+1) - 1 < x - \log(x)(\log(x+1) + 1).$$

(2.) For $x \geq 4$, note that the function mapping $x \mapsto \frac{x}{\log(x)}$ is increasing. If $n \leq x < n+1$, then

$$H_n \leq H(x) < H_{n+1} \leq \frac{n}{\log(n)} \leq \frac{x}{\log(x)}. \tag{16}$$

For $y > 2$ we see that $\log(y) < \frac{y}{2}$, so let $y = H(x)$ and apply Equation (16). □

Lemma 9. *For $x \geq 4$, we obtain*

$$H(x)\log(H(x)) < \frac{x^2}{x + \frac{6}{\pi^2}}.$$

Proof. Apply Lemma 6 and Lemma 8. □

Lemma 10. *For $x \geq 4$, we have*

$$H'(x) > \frac{H(x)\log(H(x))}{x^2}.$$

Proof. We will use (51) from [9] which states that

$$\frac{1}{\psi'(x)} \leq x + \frac{6}{\pi^2} - 1$$

for $x \geq 1$. We calculate

$$H'(x) = \psi'(x+1) \geq \frac{1}{x + 6\pi^2} > \frac{H(x)\log(H(x))}{x^2},$$

where the equality follows taking the derivative of Equation (13) and the second inequality follows from Lemma 9. □

Proposition 2. *The function*

$$g(x) = \frac{\exp(H(x))\log(H(x))}{x}$$

is increasing for $x \geq 4$.

Proof. We start with (3.5) from [11]:

$$H_n = \log(n) + \gamma + \int_n^\infty \frac{x - \lfloor x \rfloor}{x^2} dx$$

which implies

$$exp(H_n) = e^\gamma n \exp\left(\int_n^\infty \frac{x - \lfloor x \rfloor}{x^2} dx\right)$$

which gives

$$\frac{\exp(H_n)\log(H_n)}{n} = e^\gamma \log(H_n) \exp\left(\int_n^\infty \frac{x - \lfloor x \rfloor}{x^2} dx\right).$$

Given $k \in \mathbb{N}$, put

$$g_k(x) = e^\gamma \log(H(x)) \exp\left(\int_x^k \frac{t - \lfloor t \rfloor}{t^2} dt\right)$$

so that $\lim_{k\to\infty} g_k(x) = g(x)$. We compute

$$g_k'(x) = e^\gamma \exp\left(\int_x^k \frac{t - \lfloor t \rfloor}{t^2} dt\right) \left(\frac{H'(x)}{H(x)} + \log(H(x))\left(-\frac{x - \lfloor x \rfloor}{x^2}\right)\right),$$

so $g_k'(x) > 0$ if and only if

$$\frac{H'(x)}{H(x)} + \log(H(x))\left(-\frac{x-\lfloor x\rfloor}{x^2}\right) \geq \frac{H'(x)}{H(x)} - \frac{\log(H(x))}{x^2} > 0,$$

which is the content of Lemma 10. Thus, $g(x)$ is the limit of monotonically increasing functions and is therefore monotonically increasing. □

Corollary 3. *The sequence*

$$\left\{\frac{\exp(H_n)\log(H_n)}{n}\right\}_{n=1}^{\infty}$$

is monotonically increasing.

Proof. Proposition 2 gives the result for $n \geq 4$ and we can manually check the smaller cases. □

Definition 3. A positive integer n is *superabundant* if $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.

Theorem 9. *If there are counterexamples to the Kaneko–Lagarias inequality, the smallest such counterexample is a superabundant number.*

Proof. Suppose, for the sake of contradiction, that m is the smallest counterexample to the Kaneko–Lagarias inequality and that m is not superabundant. Let n be the greatest superabundant number strictly less than m. We calculate

$$\frac{\sigma(n)}{n} > \frac{\sigma(m)}{n} \geq \frac{\exp(H_m)\log(H_m)}{m} > \frac{\exp(H_n)\log(H_n)}{n},$$

so $n < m$ violates the Kaneko–Lagarias inequality: a contradiction. □

3.2. Connection to Robin's Inequality

Theorem 10. *If Robin's inequality holds for some $n \in \mathbb{N}$, then the Kaneko–Lagarias inequality holds for n.*

Proof. We use the approximation

$$H_n \geq \log(n) + \gamma + \frac{1}{2n+1}.$$

to calculate

$$\frac{\exp(H_n)\log(H_n)}{n} \geq \frac{e^{\gamma+\frac{1}{2n+1}}n\log\left(\log(n)+\gamma+\frac{1}{2n+1}\right)}{n} > e^{\gamma}\log(\log(n)),$$

which implies the result. □

Note that we obtain the same result for the Lagarias inequality.

Acknowledgments. The first author thanks Jeff Lagarias for his comments and the references he provided. The third author thanks Keith Briggs for providing his code used to compute superabundant numbers, Perry Thompson and Owen McAllister for their help in implementing it, and Jean-Louis Nicolas for sharing the paper [5]. We also thank anonymous referees for their helpful comments, which certainly improved the quality of this paper.

References

[1] A. Akbary and Z. Friggstad, Superabundant Numbers and the Riemann Hypothesis, *Amer. Math. Monthly* **116** (3) (2009), 273–275.

[2] A. Akbary, Z. Friggstad, and R. Juricevic, Explicit upper bounds for $\prod_{p_{\omega(n)}} \frac{p}{p-1}$, *Contrib. Discrete Math.* **2** (2) (2007), 153–160.

[3] H. Alzer, On Some Inequalities for the Gamma and Psi Functions, *Math. Comp.* **66** (217) (1997), 373–389.

[4] C. Axler, On Robin's inequality, *Ramanujan J.* **61** (3) (2023), 909–919.

[5] C. Axler and J.L. Nicolas, Large values of $\frac{n}{\phi(n)}$, *Acta Arith.* **209** (2022), 357–383.

[6] K. Boughan and T. Trudgian, Robin's inequality for 11-free integers, *Integers* **15** (2015), #A12, 5 pp.

[7] Y. Choie, N. Lichiardopol, P. Moree, and P. Solé, On Robin's criterion for the Riemann hypothesis, *J. Théor. Nombres Bordeaux* **19** (2) (2007), 357–372.

[8] P. Dusart, The k-th Prime is Greater than $k(\log(k) + \log(\log(k)) - 1)$ for $k \geq 2$, *Math. Comp.* **68** (225) (1999), 411–415.

[9] M. R. Farhangdoost and M. K. Dolatabadi, New Inequalities for gamma and digamma functions, *J. Appl. Math.* (2014), 1–7.

[10] A. Hertlein, Robin's inequality for new families of integers, preprint, `arXiv:1612.05186`.

[11] J.C. Lagarias, An Elementary Problem Equivalent to the Riemann Hypothesis, *Amer. Math. Monthly* **109** (6) (2002), 534–543.

[12] F. Luca, C. Pomerance, and P. Solé, Corrigendum to #A8 of Volume 25, On Robin's Inequality, *Integers*, **25** (2025), #A25, 4 pp.

[13] A. MacArevey, On the Lagarias Inequality and Superabundant Numbers, preprint, `arXiv:2602.15905`.

[14] T. Morrill and D. Platt, Robin's inequality for 20-free integers, *Integers* **21** (2021), #A28, 7 pp.

[15] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, *J. Math. Pures Appl.* **63** (1984), 187–213.

[16] J. Barkley Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962), 64–94.

[17] P. Sole and M. Planat, The Robin inequality for 7-free integers, *Integers* **12** (2012), 301–309.

[18] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge University Press, Cambridge, 1995.

[19] L.C. Washington and A. Yang, Analogues of the Robin–Lagarias criteria for the Riemann hypothesis, *Int. J. Number Theory* **17** (4) (2021), 843–870.

[20] M. Wójtowicz, Robin's inequality and the Riemann hypothesis, *Proc. Japan Acad. Ser. A Math. Sci.* **83** (4) (2007), 47–49.