ON THE DIOPHANTINE EQUATIONS $(p+a)^x + p^y = z^2$ AND $(p+a)^x - p^y = z^2$ WHERE p AND $p+a$ ARE ODD PRIMES

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Abstract

In this paper, we consider the exponential Diophantine equations $(p+a)^x + p^y = z^2$ and $(p+a)^x - p^y = z^2$ under suitable conditions and find all solutions under these conditions.

1. Introduction

In 1844, Catalan [3] proposed his classical conjecture that the only solution of the exponential Diophantine equation $a^x - b^y = 1$, where $\min\{a,b,x,y\} > 1$, is $(a,b,x,y) = (3,2,2,3)$, and this was later proved by Preda Mihăilescu [10] in 2004. A detailed exposition of the history of this conjecture, and its progress before Mihăilescu's proof, can be found in [12]. Later, Sroysang's [13] study revealed that the equation $3^x + 5^y = z^2$ has the unique solution $(1, 0, 2)$. He also established that the equation $7^x + 8^y = z^2$ has the unique solution $(0, 1, 3)$, and that the equation $5^x + 7^y = z^2$ admits no non-negative integer solution [14].

Suvarnamani [15] showed that the equation $p^x + (p+1)^y = z^2$, where p is an odd prime, has only one possible solution, namely $(p,x,y,z) = (3,1,0,2)$. Subsequently, Kumar et al. [8] investigated the Diophantine equation $p^x + (p+6)^y = z^2$, where p and $p+6$ are primes, and found that the equation has no solution when $p \equiv 1 \pmod 6$. Following this, Burshtein [1] found that the Diophantine equation $p^x + (p+4)^y = z^2$, when $p > 3$ and $p+4$ are primes, has no positive integer solution. In the same year, Fernando [5] showed that $p^x + (p+8)^y = z^2$, when $p > 3$ and $(p+8)$ are primes, has no positive integer solution. Gupta et al. [6] proved that $p^x + (p+12)^y = z^2$ has no non-negative integer solution when p and $p+12$ are

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primes, and $p = 6n + 1$ for some natural number n. In the same vein, Kumar et al. [9] investigated the equation $p^x + (p+2)^y = z^2$, and found all the possible solutions. In a further development, Dockchan and Pakapongpun [4] showed that the equation $p^x + (p + 20)^y = z^2$ has no positive integer solution when p and $p + 20$ are primes. Later, Thamgnauk and Kowang [17] found that the equation $7^x - 5^y = z^2$ has no non-trivial solution.

In the subsequent years, Tadee and Laomalaw [16] found that the Diophantine equation $(p+2)^x - p^y = z^2$ admits no non-trivial solution when p is prime and $p \equiv 5 \pmod{24}$. In the same year, Orosram [11] studied the equation $(p + a)^x + p^y = z^2$, where p and $p + a$ are primes, under the conditions $a \equiv 0 \pmod 4$, $p \equiv 3 \pmod 4$, and $p > 3$, and found the solution set with some additional conditions. Building on these studies, Tadee and Laomalaw [18] explored the Diophantine equations $(p + a)^x - p^y = z^2$ and $p^x - (p + a)^y = z^2$. They showed that the trivial solution is the only possible solution for the first equation under the conditions $a \equiv 2 \pmod 4$, $p \equiv 1 \pmod 4$, and $a \equiv \pm 1 \pmod p$. Moreover, they also showed that the same conclusion holds for the second equation when $a \equiv 2 \pmod 4, p \equiv 3 \pmod 4$, $\gcd(p,a + 1) = 1$, and $ord_p a = p - 1$, where p and $p + a$ are odd primes, and a is a positive integer.

In this paper, we determine the solution set of the equation $(p + a)^x + p^y = z^2$ under the conditions $a \equiv 0 \pmod 4$ and $a \equiv -1 \pmod p$, and also under the conditions $a \equiv 2 \pmod 4$, $a \equiv -1 \pmod p$, and $\left(\frac{a}{p}\right) = -1$, where $p \neq 3$ and $p + a$ are odd primes. Furthermore, we also prove that the equation $(p+a)^x - p^y = z^2$ has no non-trivial solution under the set of conditions $a \equiv 0 \pmod 4$, $a \equiv \pm 1 \pmod p$, and $\left(\frac{a}{p}\right) = -1$, where $p \neq 3$ and $p + a$ are odd primes.

2. Preliminaries

We list below some important definitions and results that will be used in the proof of our main results in Section 3.

Theorem 1 (Mihăilescu's Theorem [3], [10])**.** *The Diophantine equation $a^x - b^y = 1$ has the unique solution $(a,b,x,y) = (3,2,2,3)$, where $a,b,x,y \in \mathbb{N}$, and $\min\{a,b,x,y\} > 1$.*

Theorem 2 (Fermat's Little Theorem [2])**.** *Let p be any prime, and a be any integer such that a and p are co-prime. Then $a^{p-1} \equiv 1 \pmod p$.*

Definition 1. Let p be an odd prime, and $\gcd(a,p) = 1$. If the quadratic congruence $x^2 \equiv a \pmod p$ has a solution, then a is said to be a *quadratic residue* of p. Otherwise, a is called a *quadratic non-residue* of p.

Definition 2. Let a be any integer, and p be any odd prime number such that $p \nmid a$. The *Legendre symbol*, denoted by $\left(\frac{a}{p}\right)$, is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of p,} \\ -1 & \text{if } a \text{ is a quadratic non-residue of } p. \end{cases}$$

Theorem 3 ([2])**.** *Let a and b be integers, and p be any prime with* $\gcd(a,p) = \gcd(b,p) = 1$. *Then the following statements hold.*

(a) If $a \equiv b \pmod{n}$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

(b) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

(c) $\left(\frac{a^2}{p}\right) = 1$.

Theorem 4 (Law of Quadratic Reciprocity [2])**.** *If p and q are distinct odd primes, then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(\frac{p-1}{2})(\frac{q-1}{2})}.$$

So, by Theorem 4, we find that

$$\left(\frac{p}{q}\right) = \begin{cases} \left(\frac{q}{p}\right) & \text{if } p \equiv 1 \pmod 4 \text{ or } q \equiv 1 \pmod 4, \\ -\left(\frac{q}{p}\right) & \text{if } p \equiv q \equiv 3 \pmod 4. \end{cases}$$

whenever p and q are distinct odd primes.

Theorem 5 ([2])**.** *If p is any odd prime number, then*

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod 8, \\ -1 & \text{if } p \equiv \pm 3 \pmod 8. \end{cases}$$

3. Main Results

In this section, we state and prove the main results of this paper.

Theorem 6. *The Diophantine equation $p^x + (p+a)^y = z^2$, where $a \equiv 0 \pmod 4$ and $a \equiv -1 \pmod p$, has no solution other than $(x,y,z) = (0,1,\sqrt{p+a+1})$ when $p+a+1$ is a perfect square, provided $p \neq 3$ and $p+a$ are both odd primes.*

Proof. Let x, y, and z be non-negative integers such that $p^x + (p+a)^y = z^2$. Since p and $p+a$ are odd primes, p^x and $(p+a)^y$ are also odd, and therefore their sum,

that is, z^2 must be even. Consequently, z must be even, and hence, $z^2 \equiv 0 \pmod 4$. If $p \equiv 1 \pmod 4$, then the congruence $a \equiv 0 \pmod 4$ leads to $(p+a) \equiv 1 \pmod 4$. Therefore, modulo 4, the given equation yields $1^x + 1^y \equiv 0 \pmod 4$; that is, $2 \equiv 0 \pmod 4$, which is a contradiction. Hence, we must have $p \equiv -1 \pmod 4$. We now consider the following cases separately.

`Case 1:` $x = 0$, $y = 0$. In this case, we obtain $z^2 = 2$, which is a contradiction.

`Case 2:` $y = 0$, $x \geq 1$. In this case, the equation simplifies to $p^x + 1 = z^2$. If $x = 1$, then $p = z^2 - 1 = (z-1)(z+1)$. This implies $z - 1 = 1$ and $z + 1 = p$, which in turn gives $z = 2$ and $p = 3$, contradicting our hypothesis. So, we assume $x > 1$ and rewrite the equation as $z^2 - p^x = 1$. The cases $z = 0$ and $z = 1$ lead to obvious contradictions. If $z > 1$, then $\min\{2,p,x,z\} > 1$, and so, by Theorem 1, $z^2 - p^x = 1$ has only the solution $(p,x,z) = (2,3,3)$, which is not possible, as p is an odd prime.

`Case 3:` $x = 0$, $y \geq 1$. In this case, the equation reduces to $1 + (p+a)^y = z^2$. If $y = 1$, then $p + a + 1 = z^2$. So, whenever $(p + a + 1)$ is a perfect square, $(x,y,z) = (0,1,\sqrt{p+a+1})$ is a solution. Next, consider $y > 1$. If $z = 0$, then $(p+a)^y = -1$, a contradiction. If $z = 1$, then $(p+a)^y = 0$, which is again not possible. When $z > 1$, we obtain $\min\{y,z,p+a\} > 1$, and so, by the Theorem 1, $z^2 - (p+a)^y = 1$ has no solution for the case $x = 0$, $y > 1$, since $p + a \neq 2$, being an odd prime.

`Case 4:` $x \geq 1$, $y \geq 1$. Since $p \equiv -1 \pmod 4$, and $a \equiv 0 \pmod 4$, it follows that $(p+a) \equiv -1 \pmod 4$. Also, $z^2 \equiv 0 \pmod 4$, since p and $p+a$ are both odd, and, therefore, $(-1)^x + (-1)^y \equiv 0 \pmod 4$. Hence, x and y must have different parities.

First, we consider the case where x is even, and y is odd. Let $x = 2t$, $t \in \mathbb{N}$. Then $(p+a)^y = z^2 - p^{2t} = (z - p^t)(z + p^t)$. Since $(p+a)$ is a prime, the only admissible factorization is $(p+a)^u = z - p^t$, and $(p+a)^{y-u} = z + p^t$, with $y > u$. This yields $2p^t = (p+a)^u((p+a)^{y-2u} - 1)$. Since p and $(p+a)$ are distinct primes, and $a \equiv -1 \pmod p$, this is possible only when $u = 0$. Therefore, $2p^t = (p+a)^y - 1$. Since $2p^t \equiv 0 \pmod p$, $(p+a)^y \equiv 1 \pmod p$, which further gives $a^y \equiv 1 \pmod p$. Since $a \equiv -1 \pmod p$, we must have $(-1)^y \equiv 1 \pmod p$. Since y is odd, we arrive at the contradiction $p = 2$.

Next, we consider the case when x is odd and y is even. Let $y = 2l$, $l \in \mathbb{N}$. Then $p^x = z^2 - (p+a)^{2l} = (z - (p+a)^l)(z + (p+a)^l)$. Therefore, $p^v = z - (p+a)^l$, and $p^{x-v} = z + (p+a)^l$, with $x > v$. This gives $2(p+a)^l = p^v(p^{x-2v} - 1)$, $x > v$. Since p and $p+a$ are primes, we must have $2(p+a)^l = p^x - 1$. Now, $p^x - 1 \equiv -1 \pmod p$, which implies $2(p+a)^l \equiv -1 \pmod p$, and hence $2a^l \equiv -1 \pmod p$. This is not possible for any odd prime p other than $p = 3$. Hence, the equation has no solution for any $p \equiv -1 \pmod 4$ other than $p = 3$. □

Corollary 1. *The exponential Diophantine equation* $19^x + 151^y = z^2$ *has no solution.*

Theorem 7. *If $p > 3$ and $p + a$ are odd primes, where $a \equiv 2 \pmod 4$, then the equation $p^x + (p + a)^y = z^2$, has no solution in the positive integers for $a \equiv -1 \pmod p$ and $\left(\frac{a}{p}\right) = -1$.*

Proof. Let x, y, and z be non-negative integers such that $p^x + (p + a)^y = z^2$. To prove the result, we break the argument into two separate cases: one for $p \equiv -1 \pmod 4$ and another for $p \equiv 1 \pmod 4$.

`Case 1:` $p \equiv -1 \pmod 4$. We further divide the case into the following subcases.

`Subase 1.1:` $x = 0$, $y = 0$. In this case, we obtain $z^2 = 2$, which is a contradiction.

`Subcase 1.2:` $y = 0$, $x \geq 1$. In this case, the equation simplifies to $p^x + 1 = z^2$. If $x = 1$, then $p = z^2 - 1 = (z - 1)(z + 1)$. This implies $z - 1 = 1$ and $z + 1 = p$; that is, $z = 2$ and $p = 3$, contradicting our hypothesis. So, we assume $x > 1$, and rewrite the equation as $z^2 - p^x = 1$. The cases $z = 0$ and $z = 1$ clearly lead to impossibilities. If $z > 1$, then $\min\{2,p,x,z\} > 1$, and so, by Theorem 1, $z^2 - p^x = 1$ has only the solution $(p,x,z) = (2,3,3)$, which is not possible, as p is an odd prime.

`Subase 1.3:` $x = 0$, $y \geq 1$. In this case, the equation reduces to $1 + (p + a)^y = z^2$. If $y = 1$, then $p + a + 1 = z^2$, which gives $z^2 \equiv 2 \pmod 4$, which is not possible because $z^2 \equiv 0,1 \pmod 4$. Next, consider $y > 1$. If $z = 0$, then $(p + a)^y = -1$, a contradiction. If $z = 1$, then $(p + a)^y = 0$, which is not possible. Therefore, $z > 1$, and so, $\min\{2,y,z,p + a\} > 1$. Hence, by Theorem 1, $z^2 - (p + a)^y = 1$ has no solution for $p \equiv -1 \pmod 4$, $a \equiv 2 \pmod 4$, and $a \equiv -1 \pmod p$.

`Subcase 1.4:` $x \geq 1$, $y \geq 1$. Since $a \equiv 2 \pmod 4$, and, by our assumption, $p \equiv -1 \pmod 4$, we must have $(p + a) \equiv 1 \pmod 4$. As p and $p + a$ are both odd, z^2 must be even, and therefore, z must also be even. Consequently, $z^2 \equiv 0 \pmod 4$, and hence, $(-1)^x + 1^y \equiv 0 \pmod 4$. This is possible only when x is odd. We consider the following two cases separately.

`Subsubcase 1.4.1:` y is even. Let $y = 2l$, $l \in \mathbb{N}$. Then the given equation becomes

$$p^x = z^2 - (p + a)^{2l} = \left(z - (p + a)^l\right)\left(z + (p + a)^l\right).$$

Since p is a prime, we must have $p^v = z - (p + a)^l$, $p^{x-v} = z + (p + a)^l$ for some $v \in \mathbb{N}$. Therefore, $2(p + a)^l = p^v(p^{x-2v} - 1)$. Since p and $p + a$ are both primes, we obtain $v = 0$. Therefore, $2(p + a)^l = p^x - 1$. This leads to the contradiction $2(p + a)^l \equiv -1 \pmod p$, as $a \equiv -1 \pmod p$, and $p \neq 3$ is a prime.

`Subsubcase 1.4.2:` y is odd. In this case, we reduce the equation modulo p to obtain

$$(p + a)^y \equiv z^2 \pmod p.$$

By The Division Algorithm, we can write $y = (p - 1)m + l$, where $m,l \in \mathbb{Z}, 0 \leq l < p - 1$. By Fermat's Little Theorem (Theorem 2), $(p + a)^{p-1} \equiv 1 \pmod p$, which

implies $(p+a)^{(p-1)m+l} \equiv (p+a)^l \pmod{p}$; that is, $(p+a)^y \equiv (p+a)^l \pmod{p}$, and therefore, $z^2 \equiv (p+a)^l \pmod{p}$. Hence, $\left(\frac{(p+a)^l}{p}\right) = (\frac{z^2}{p}) = 1$, thereby giving $\left(\frac{p+a}{p}\right) = 1$, as l is odd. This, in turn, yields $\left(\frac{a}{p}\right) = 1$, which is a contradiction.

`Case 2:` $p \equiv 1 \pmod{4}$. We again break the argument into the following subcases.

`Subcase 2.1:` $x = 0$, $y = 0$. In this case, we get $z^2 = 2$, which is a contradiction.

`Subcase 2.2:` $y = 0$, $x \geq 1$. In this case, the equation reduces to $1 + p^x = z^2$. If $x = 1$, then $p = z^2 - 1 = (z-1)(z+1)$. This implies $z - 1 = 1$ and $z + 1 = p$; that is, $z = 2$ and $p = 3$, contradicting our hypothesis. So, we assume $x > 1$, and rearrange the equation as $z^2 - p^x = 1$. If $z = 0$, then $p^x = -1$, a contradiction. If $z = 1$, then $p^x = 0$, again a contradiction. If $z > 1$, then $\min\{2,p,x,z\} > 1$, and so, by Theorem 1, the equation $z^2 - p^x = 1$ has only the solution $(p,x,z) = (2,3,3)$, which is not possible, as p is an odd prime.

`Subcase 2.3:` $x = 0$, $y \geq 1$. If $y = 1$, then $p + a + 1 = z^2$ with $z^2 \equiv 0 \pmod{4}$. In this case, $(x,y,z) = (0,1,\sqrt{p+a+1})$ is a solution whenever $p + a + 1$ is a perfect square. Next, consider $y > 1$. If $z = 0$, then $(p+a)^y = -1$, a contradiction. If $z = 1$, then $(p+a)^y = 0$, which is not possible. Therefore, $z > 1$, and so $\min\{2,y,z,p+a\} > 1$. Hence, by Theorem 1, $z^2 - (p+a)^y = 1$ has no solution in this case, since $p + a \neq 2$, being an odd prime.

`Subcase 2.4:` $x \geq 1$, $y \geq 1$. In this case, $(p+a) \equiv -1 \pmod{4}$, and since $z^2 \equiv 0 \pmod{4}$, we must have $1^x + (-1)^y \equiv 0 \pmod{4}$. This shows that y must be odd. We proceed by considering two cases as follows.

`Subsubcase 2.4.1:` x is even. Let $x = 2m$, $m \in \mathbb{N}$. Then $(p+a)^y = z^2 - p^{2m}$, so that

$$(p+a)^y = (z - p^m)(z + p^m).$$

Since $p + a$ is a prime, we must have $(p+a)^u = z - p^m$, and $(p+a)^{y-u} = z + p^m$ for some $u \in \mathbb{N}$. Therefore,

$$2p^m = (p+a)^{y-u} - (p+a)^u = (p+a)^u \left((p+a)^{y-2u} - 1\right).$$

Since p and $(p+a)$ are both primes, the above equation gives $u = 0$. Therefore, $2p^{\frac{x}{2}} = (p+a)^y - 1$. Again, $2p^{\frac{x}{2}} \equiv 0 \pmod{p}$, and so $(p+a)^y \equiv 1 \pmod{p}$, thereby leading to $a^y \equiv 1 \pmod{p}$. Since y is odd, we must have $a \equiv 1 \pmod{p}$, which is not true, as per our hypothesis.

`Subsubcase 2.4.2:` x is odd. In this case, from the equation, we have $p^x \equiv z^2 \pmod{p+a}$. By The Division Algorithm, we can write $x = (p+a-1)m + l$, where $m,l \in \mathbb{Z}$, and $0 \leq l < p + a - 1$. Since x is odd, l must be odd. By Fermat's Little Theorem (Theorem 2), we have $p^{p+a-1} \equiv 1 \pmod{p+a}$, which implies $p^{(p+a-1)m+l} \equiv p^l \pmod{p+a}$. Thus, $p^x \equiv p^l \pmod{p+a}$, and, consequently

$z^2 \equiv p^l \pmod{p+a}$. So, $\left(\frac{p^l}{p+a}\right) = 1$, and, since l is odd, we obtain $\left(\frac{p}{p+a}\right) = 1$. Again, by The Law of Quadratic Reciprocity (Theorem 4), we get

$$\left(\frac{p}{p+a}\right)\left(\frac{p+a}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{p+a-1}{2}}.$$

Since $p + a \equiv 1 \pmod 4$, we have $\left(\frac{p}{p+a}\right)\left(\frac{p+a}{p}\right) = 1$, and therefore, we obtain $\left(\frac{p+a}{p}\right) = 1$, which, in turn, leads to the contradiction $\left(\frac{a}{p}\right) = 1$. □

Remark 1. When $p = 3$, the Diophantine equation $3^x+(3+a)^y = z^2$ always has at least one solution, namely (1,0,2), for any choice of a. For instance, the Diophantine equation $3^x + 11^y = z^2$, which aligns with the assumptions made in Theorem 6, has solutions like (1,0,2) and (5,4,122), while the Diophantine equation $3^x + 17^y = z^2$, conforming to the conditions of Theorem 7, has the solution (1,0,2).

Theorem 8. *Let p and $p+a$ be primes such that $p \equiv -1 \pmod 4$, $p \neq 3$, and let a be a positive integer satisfying $a \equiv 2 \pmod 4$. The Diophantine equation $(p+a)^x - p^y = z^2$ has no non-trivial integer solution other than $(x,y,z) = (1,0,\sqrt{p+a-1})$, whenever $\sqrt{p+a-1}$ is an integer, provided that $a \equiv \pm 1 \pmod p$ and $(\frac{a}{p}) = -1$.*

Proof. Let x, y, and z be non-negative integers such that $(p + a)^x - p^y = z^2$. We consider the following cases separately.

`Case 1:` $x = 0$, $y = 0$. In this case, we get $z^2 = 0$, which gives $(x,y,z) = (0,0,0)$.

`Case 2:` $x = 0$, $y \geq 1$. In this case, the equation reduces to $1 - p^y = z^2$, and so $z^2 < 0$, which is a contradiction.

`Case 3:` $y = 0$, $x \geq 1$. In this case, the equation reduces to $(p + a)^x - z^2 = 1$. If $x = 1$, then $p+a-1 = z^2$ with $z^2 \equiv 0 \pmod 4$. So, $(x,y,z) = (1,0,\sqrt{p+a-1})$ is a solution, whenever $p+a-1$ is a perfect square. Next, consider $x > 1$. If $z = 0$, then $(p + a)^x = 1$, which gives $x = 0$, a contradiction. If $z = 1$, we obtain $(p + a)^x = 2$, which is not possible. Therefore, $z > 1$, and so $\min\{2,y,z,p + a\} > 1$. Hence, by Theorem 1, the equation $(p + a)^y - z^2 = 1$ has no solution, as $p + a \equiv 1 \pmod 4$, by our hypothesis, and so $p + a \neq 3$.

`Case 4:` $x \geq 1, y \geq 1$. Since p and $p + a$ are both odd, so $z^2 \equiv 0 \pmod 4$, and therefore, $1^x - (-1)^y \equiv 0 \pmod 4$. Hence, y is even.

We consider the parity of x separately. If x is even, say $x = 2m$, then $p^y = (p + a)^{2m} - z^2$, and so $p^y = ((p + a)^m - z)((p + a)^m + z)$. Since p is prime, so

$$p^{y-u} = (p + a)^m - z,\ p^u = (p + a)^m + z, \quad u \in \mathbb{Z},\ y \geq u.$$

Adding these, we get $2(p+a)^m = p^u(p^{y-2u} + 1)$. Since p and $p+a$ are odd primes, we have $2(p+a)^{\frac{x}{2}} = p^y+1$. Now, we have $p^y+1 \equiv 1 \pmod p$, and so $2(p+a)^{\frac{x}{2}} \equiv 1$

$(\text{mod } p)$, and hence, $2a^{\frac{x}{2}} \equiv 1 \pmod{p}$, which is a contradiction, as $a \equiv \pm 1 \pmod{p}$ and $p \neq 3$.

If x is odd, from the equation, we have $-p^y \equiv z^2 \pmod{p+a}$. By The Division Algorithm, we have $y = (p+a-1)m + l$, where m,$l \in \mathbb{Z}$, and $0 \leq l < p+a-1$. Since y is odd, l also must be odd. Applying Fermat's Little Theorem (Theorem 2), we have $p^{p+a-1} \equiv 1 \pmod{p+a}$, which implies $p^{(p+a-1)m+l} \equiv p^l \pmod{p+a}$. Thus, $p^y \equiv p^l \pmod{p+a}$; that is, $-p^y \equiv -p^l \pmod{p+a}$, and, consequently, $z^2 \equiv -p^l \pmod{p+a}$. Therefore, $\left(\frac{-p^l}{p+a}\right) = 1$, thereby giving $\left(\frac{-1}{p+a}\right)\left(\frac{p^l}{p+a}\right) = 1$. Hence, $\left(\frac{p^l}{p+a}\right) = 1$, as $p+a \equiv 1 \pmod 4$. Since l is odd, we obtain $\left(\frac{p}{p+a}\right) = 1$. Again, by The Law of Quadratic Reciprocity (Theorem 4), we get

$$\left(\frac{p}{p+a}\right)\left(\frac{p+a}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{p+a-1}{2}}.$$

Since $p+a \equiv 1 \pmod 4$, we have $\left(\frac{p}{p+a}\right)\left(\frac{p+a}{p}\right) = 1$, and therefore, $\left(\frac{p+a}{p}\right) = 1$, which, in turn, leads to the contradiction $\left(\frac{a}{p}\right) = 1$. □

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