ADDITIONAL CONSTRUCTIONS OF SEQUENCES OF ALTERNATING SUM AND DIFFERENCE DOMINATED SETS

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Abstract

A More Sums Than Differences (MSTD) set is a finite set of integers A where the cardinality of its sumset, $A+A$, is greater than the cardinality of its difference set, $A-A$. We address a problem posed by Samuel Allen Alexander that asks whether there exists an infinite sequence of sets alternating between being MSTD and More Differences Than Sums (MDTS), where each set properly contains the previous. While a companion paper resolved this using 'filling in' techniques, we solve the more challenging 'non-filling-in' version, where any missing integer between a set's minimum and maximum elements remains missing in all subsequent sets.

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1. Introduction

The *sumset* $A + A$ and the *difference set* $A - A$ of a finite set of integers A are defined to be the set of all possible sums or differences between elements of the set A. More formally, $A + A \coloneqq \{a + b : a, b \in A\}$ and $A - A \coloneqq \{a - b : a, b \in A\}$, with their respective cardinalities denoted $\left|A + A\right|$ and $\left|A - A\right|$. A *More Sums Than Differences* (MSTD) or *sum-dominated* set is a set A for which $\left|A + A\right| > \left|A - A\right|$. Similarly, a *More Differences Than Sums* (MDTS) or *difference-dominated* set is a set A for which $\left|A - A\right| > \left|A + A\right|$. We should generally expect a set A to be difference-dominated since addition is commutative while subtraction is not.

Early examples of sum-dominated sets were discovered by Conway [1], Marica [2] [4], and Freiman-Pigarev [3][1]. The set found by Conway was later shown by Hegarty [2] to be the smallest possible MSTD set with respect to cardinality and diameter. The problem was formalized by Nathanson, who noted that MSTD sets should be rare and proved methods of their construction along with several properties of such sets [9]. Despite these sets being rare, Martin and O'Bryant [5] proved that a positive percentage of subsets of $\{1, 2, \ldots, n\}$ are sum dominated as $n \to \infty$. Note this model (all subsets being equally likely to be chosen) is equivalent to each element of $\{1, 2, \ldots, n\}$ being in our set with probability 1/2; Hegarty and Miller [3] proved that if each element is chosen uniformly and independently with probability $p(n)$, then if $p(n) \to 0$, almost all sets are difference dominated. Work has also been done to find explicit constructions of generalized MSTD sets [6, 7, 10] and generalizations to groups [11]. Sumsets and difference sets are also present in many problems in number theory.

At the recent 2025 CANT (Combinatorial and Additive Number Theory) Conference, Samuel Allen Alexander posed the problem of finding an infinite sequence of sets with $A_{i-1} \subset A_i$ that alternate being MSTD and MDTS. In a companion paper [8], we addressed this problem at first by 'filling in' sets in order to generate subsequent sets in the sequence. Filling in a set A_i refers to the process of adding elements of $[a, b] \setminus A_i$ to A_i. However, filling in can distort the structure of earlier sets in the sequence, resulting in relatively trivial sequences with rapidly growing sets. Constructions with slower growth rates are more desirable, as they extend more naturally to algebraic settings where the construction of long alternating chains is constrained by the underlying structure, such as finite fields or abelian groups.

In [8], we also introduced a method that avoids filling in altogether and produced a sequence which retained the structural properties of the initial set A_1. In this paper, we build on this work by providing three additional methods which prohibit the use of filling in. Before introducing these methods, we first establish several known properties of both general and MSTD sets.

[1] $\{0, 2, 3, 4, 7, 11, 12, 14\}$
[2] $\{1, 2, 3, 5, 8, 9, 13, 15, 16\}$
[3] $\{0, 1, 2, 4, 5, 9, 12, 13, 14, 16, 17, 21, 24, 25, 26, 28, 29\}$

For any set of integers A, we define the *dilation* $x \cdot A \coloneqq \{xa : a \in A\}$. For all integers x, y such that $x \neq 0$, the set

$$B \;=\; x \cdot A + \{y\} \;=\; \{xa + y : a \in A\}$$

satisfies $\left|B+B\right| = \left|A+A\right|$ and $\left|B-B\right| = \left|A-A\right|$ [9]. This property ensures that any method developed on a restricted subset of the integers can be extended, via dilation and translation, to the full domain of integers.

Furthermore, through dilation and translation, we have that any non-trivial k-term arithmetic progression can be obtained from the interval

$$I \;=\; \{1, 2, \ldots, k\} \;:=\; [1, k].$$

It is also evident that

$$\begin{aligned} I + I \;&=\; [2, 2k], \text{ and} \\ I - I \;&=\; [1-k, k-1]. \end{aligned}$$

Therefore, such sets have the same number of sums and differences. For instance, if $A = \{0, 1, 2\}$, then

$$\begin{aligned} A + A \;&=\; \{0, 1, 2, 3, 4\}, \\ A - A \;&=\; \{-2, -1, 0, 1, 2\}, \text{ and} \\ \left|A + A\right| \;&=\; \left|A - A\right| \;=\; 5. \end{aligned}$$

For a set of integers A, if there exists an $a^* \in \mathbb{Z}$ such that $a^* - A = A$, then we say that A is *symmetric* with respect to a^*, and we have that $\left|A+A\right| = \left|A-A\right|$. If A is an arithmetic progression, then A is symmetric to $a^* = \min(A) + \max(A)$ [9].

Many methods that are used to construct MSTD sets make use of the symmetry property [9]. For example, the set $A = \{0, 2, 3, 7, 11, 12, 14\}$ is symmetric with respect to 14, so $\left|A+A\right| = \left|A-A\right|$. However, adding 4 to this set forms Conway's set, which is MSTD. We also use this symmetry property in one method presented in the paper.

Adding to the work of our previous paper [8], we provide additional non-trivial non-filling in methods to obtain the desired sequences. Each method achieves a more efficient growth rate than the filling in methods presented in [8]. Theorem 1 provides a method that generates the sequence by adjoining elements chosen to preserve the sum-difference imbalance at each step. Theorem 3 refines an approach of Nathanson by imposing additional constraints to achieve an even slower growth rate in the resulting sequence. Ideas from the proof of Theorem 3 are used to construct a single sequence which is our slowest-growing result.

2. Non-filling in Methods

We formalize the 'non-filling in' constraint on the alternating sequence we seek to generate. For any A_k, let $a = \min A_k$ and $b = \max A_k$. Then, if an $x \in [a, b]$ is not in A_k, we require that $x \notin A_n$ for all $n > k$. In other words, this restriction prohibits filling in when generating the sequence. We provide several non-filling in methods below.

2.1. Non-filling in Method 1

First, we construct a non-filling in method that essentially 'copies' the initial set A_1 around a suitably chosen modulus n, ensuring that the initial structure of A_1 is maintained in later sets in the sequence. Theorem 1 summarizes this method.

Theorem 1. *Let A_1 be a MSTD set with $0 \in A_1$, such that there is an integer $n > \max A_1$ satisfying the following conditions.*

(a) The number of unique elements in $A_1 + A_1$ modulo n is the same as the number of unique elements of $A_1 - A_1$ modulo n.

(b) Let x be the number of $a \in A_1$ such that $n + a \notin A_1 + A_1$, and let y be the number of $b \in A_1$ such that $n - b \notin A_1 - A_1$. We require that

$$2y - x - 1 > \left|A_1 + A_1\right| - \left|A_1 - A_1\right|.$$

For $l \geq 1$, we define

$$\begin{aligned} A_{2l} &:= A_{2l-1} \ \cup \ \{ln\}, \text{ and} \\ A_{2l+1} &:= A_{2l-1} \ \cup \ (A_1 + ln). \end{aligned}$$

Then A_{2l} is MDTS, A_{2l+1} is MSTD, and $A_1 \subset \cdots \subset A_{2l-1} \subset A_{2l} \subset A_{2l+1} \subset \cdots$ forms the desired alternating sequence.

Proof. First, set $A_2 = A_1 \cup \{n\}$. We show that A_2 is difference-dominated.

The additional sums created by n are $n + n$ and $\{n + a : a \in A_1\}$. In the latter case, a new sum is realized whenever $n + a \notin A_1 + A_1$. Thus,

$$\left|A_2 + A_2\right| \ = \ \left|A_1 + A_1\right| + x + 1.$$

The additional differences created by n are $\pm\{n - a, a \in A_1\}$; since a difference is realized if and only if its negative is realized, we only count differences of the form $(n - a)$ and double the result. This gives a new difference whenever $n - a \notin A_1 - A_1$. Thus,

$$\left|A_2 - A_2\right| \ = \ \left|A_1 + A_1\right| + 2y,$$

and we have

$$\begin{aligned}\left|A_2 - A_2\right| - \left|A_2 + A_2\right| &= \left|A_1 - A_1\right| - \left|A_1 + A_1\right| + 2y - x - 1 \\ &= 2y - x - 1 - \left(\left|A_1 + A_1\right| - \left|A_1 - A_1\right|\right).\end{aligned}$$

If assumption (b) holds, we get $\left|A_2 - A_2\right| - \left|A_2 + A_2\right| > 0$.

Let $A_3 = A_1 + \{0, n\}$ [5]. We show that A_3 is sum-dominated. For $A_3 + A_3$, there are two types of additional sums to consider. These are sums of the form $\{(a+n)+(b+n) : a, b \in A_1\}$, or $\{(a+n)+b : a, b \in A_1\}$. To count the first type, note that the largest element of $A_1 + A_1$ is

$$2 \cdot \max A_1 < 2n,$$

so all sums of this type are unaccounted for. Furthermore, every new sum corresponds to a unique value of $a+b$, so there are $\left|A + A\right|$ new sums. Sums of the form $\{(a+n)+b : a, b \in A_1\}$ give new sums unless there are $a', b' \in A_1$ with $a' + b' - n = a + b$ or $a' + b' + n = a + b$, as they occur in $A_1 + A_1$ or as the first type, respectively. As such, the new sums are the elements of $A_1 + A_1$ that occur exactly once modulo n.

Note that since $2n > \max(A_1 + A_1)$, for each element that appears in $A_1 + A_1$ modulo n, there are either one or two sums in $A_1 + A_1$ that are congruent to that element modulo n. If two, the element appears twice in $\left|A_1 + A_1\right|$ and is thus double-counted as the first type of sum. If one, the element appears as both types of sums, so is again double-counted. Since every element is double-counted, the amount of new sums is twice the number of unique elements of $A_1 + A_1$ modulo n.

Next, for $A_3 - A_3$, there are two types of additional differences to consider. The set of differences $\{(a+n)-(b+n) : a, b \in A_1\}$ reduces to exactly $A_1 - A_1$, so there are no new differences. To count the set of differences $\pm\{(a+n)-b : a, b \in A_1\}$, we again only count the new positive differences and then double the total. A positive difference is new unless there are $a', b' \in A_1$ with $a' - b' - n = a - b$, so the number of new positive differences is the number of unique differences modulo n.

Note that unlike the analogous case of the sum counting, we do not count only the differences which occur exactly once modulo n. If $a' - b' - n = a - b$, we reject $a - b$ but we already have $a' - b'$. Thus, the amount of new differences is twice the number of unique elements of $A_1 - A_1$ modulo n. If assumption (a) holds, we see that the number of new sums in $A_3 + A_3$ is the same as the number of new differences in $A_3 - A_3$. Thus,

$$\left|A_3 + A_3\right| - \left|A_3 - A_3\right| = \left|A_1 + A_1\right| - \left|A_1 - A_1\right|.$$

Since A_1 is sum-dominated, we have that A_3 is sum-dominated as well. Additionally, we have

$$A_1 \subset A_2 \subset A_3.$$

For $l \geq 2$, we define A_{2l+1} as in the statement of Theorem 1, i.e.,

$$A_{2l+1} \;=\; A_{2l-1} \cup (A_1 + ln).$$

The additional sums in $A_{2l+1} + A_{2l+1}$ that are not in $A_{2l-1} + A_{2l-1}$ have the form $a + ln + b + kn$, where $0 \leq k \leq l, k \in \mathbb{Z}$. For $k \neq l, l-1$, the sum

$$a + ln + b + kn \;=\; a + (l-1)n + b + (k+1)n$$

is not a new sum since $A_1 + \{(l-1)n\} \subset A_{2l-1}$ and $A_1 + \{(k+1)n\} \subset A_{2l-1}$. We then consider sums of the form $\{a + b + 2ln\}$ and $\{a + b + (2l-1)n\}$. The largest element of $A_{2l-1} + A_{2l-1}$ is

$$2 \cdot \max A_1 + 2(l-1)n < 2ln,$$

so all sums of the first type are unaccounted for. Therefore, there are $\left|A_1 + A_1\right|$ new sums. Sums of type $\{a + b + (2l-1)n\}$ give new sums unless there are $a', b' \in A$ with $a' + b' - n = a + b$ or $a' + b' + n = a + b$ (as they occur in $A_{2l-1} + A_{2l-1}$ or as the first type, respectively). The number of new sums is therefore equal to the number of elements of $A_1 + A_1$ that occur exactly once modulo n. Thus, we can show that as with $A_3 + A_3$, the number of new sums is twice the number of unique elements of $A_1 + A_1$ modulo n.

The additional differences in $A_{2l+1} - A_{2l+1}$ are $\pm\{a + ln - b - kn\}$, where $0 \leq k \leq l$ and $k \in \mathbb{Z}$. For $k \neq 0$,

$$a + ln - b - kn \;=\; a + (l-k)n - b$$

is not a new difference since $(A_1 - A_1) + \{(l-k)n\} \subset A_{2l-1} - A_{2l-1}$. The $\{a + ln - b\}$ give new differences unless there are $a', b' \in A_1$ with $a' - b' - n = a - b$. Thus, as with $A_3 - A_3$, the number of new differences is twice the number of unique elements of $A_1 - A_1$ modulo n. Therefore,

$$\begin{aligned} \left|A_{2l+1} + A_{2l+1}\right| - \left|A_{2l+1} - A_{2l+1}\right| \;&=\; \left|A_{2l-1} + A_{2l-1}\right| - \left|A_{2l-1} - A_{2l-1}\right| \\ &=\; \left|A_1 + A_1\right| - \left|A_1 - A_1\right|, \end{aligned} \tag{1}$$

and A_{2l+1} is sum-dominated.

We now define A_{2l} as in the statement of Theorem 1. Thus, we have

$$A_{2l} \;=\; A_{2l-1} \cup \{ln\}.$$

The additional sums in $A_{2l} + A_{2l}$ are $2ln$ and the set of $\{ln + a + kn\}$, where $a \in A_1$ and $0 \leq k \leq l-1$ and $k \in \mathbb{Z}$. For $k \neq l-1$,

$$a + ln + kn \;=\; a + (l-1)n + (k+1)n$$

is not a new sum since $0 \leq k+1 \leq l-1$. When $k = l-1$, we have that $a+(2l-1)n$ is a new sum whenever $a+(2l-1)n \notin A_1+A_1+\{(2l-2)n\}$, since $A_1+A_1+\{(2l-2)n\}$ is the only possible subset of $A_{2l-1}+A_{2l-1}$ that $a+(2l-1)n$ can be a part of. The total number of a that satisfy this is x since $a+(2l-1)n \notin A_1+A_1+\{(2l-2)n\}$ if and only if $n+a \notin A_1+A_1$. Thus,

$$\left|A_{2l}+A_{2l}\right| \;=\; \left|A_{2l-1}+A_{2l-1}\right|+x+1. \tag{2}$$

The additional differences in $A_{2l}-A_{2l}$ are $\pm\{ln-b-kn\}$, where $b \in A_1$, $0 \leq k \leq l-1$, and $k \in \mathbb{Z}$. For $k \neq 0$,

$$ln-b-kn \;=\; (l-k)n-b$$

is not a new difference since $0 \leq l-k \leq l-1$. When $k=0$, we see that $ln-b$ is a new (positive) difference whenever $ln-b \notin \{(l-1)n\}+A_1-A_1$, since $\{(l-1)n\}+A_1-A_1$ is the only possible subset of $A_{2l-1}-A_{2l-1}$ that $ln-b$ can be a part of. The total number of b that satisfy this is y, since $ln-b \notin \{(l-1)n\}+A_1-A_1$ if and only if $n-b \notin A_1-A_1$. Thus,

$$\left|A_{2l}-A_{2l}\right| \;=\; \left|A_{2l-1}-A_{2l-1}\right|+2y,$$

and combining this with Equations (1) and (2), we get

$$\begin{aligned}
\left|A_{2l}-A_{2l}\right|-\left|A_{2l}+A_{2l}\right| \;&=\; \left|A_{2l-1}-A_{2l-1}\right|-\left|A_{2l-1}+A_{2l-1}\right|+2y-x-1\\
&=\; \left|A_1-A_1\right|-\left|A_1+A_1\right|+2y-x-1\\
&=\; 2y-x-1-(\left|A_1+A_1\right|-\left|A_1-A_1\right|).
\end{aligned}$$

Hence, if assumption (b) holds, A_{2l} is difference-dominated. Finally, we note that

$$A_{2l-1} \subset A_{2l} \subset A_{2l+1}.$$

Therefore, we have a sequence $A_1 \subset A_2 \subset \cdots$ alternates between being MSTD and MDTS. $\square$

We briefly verify that the method does not fill in sets in the sequence. For $l \geq 1$,

$$\begin{aligned}
\max A_{2l-1} \;&<\; ln, \text{ and}\\
A_{2l} \setminus A_{2l-1} \;&=\; \{ln\},
\end{aligned}$$

and

$$\begin{aligned}
\max A_{2l} \;&=\; ln, \text{ and}\\
A_{2l+1} \setminus A_{2l} &\subset [ln,(l+1)n-1].
\end{aligned}$$

As such, this sequence satisfies the non-filling in constraint of this paper.

We give an example of a sequence generated by non-filling in method 1. For our base set, we use the Conway set, $A_1 = \{0, 2, 3, 4, 7, 11, 12, 14\}$. For $n = 17$, the sets $A_1 + A_1$ and $A_1 - A_1$ both have 17 unique elements modulo n, so assumption (a) of Theorem 1 is met. Furthermore, it can be computationally verified that for $n = 17$, we have $x = 3$, $y = 4$, $\left|A_1 + A_1\right| = 26$, and $\left|A_1 - A_1\right| = 25$. Therefore, $2y - x - 1 > \left|A_1 + A_1\right| - \left|A_1 - A_1\right|$, and assumption (b) of 1 is met.

Applying the method, we get

$$\begin{aligned} A_2 &= A_1 \cup \{17\}, \\ A_3 &= A_1 \cup \{a + 17 \mid a \in A_1\}, \\ A_4 &= A_3 \cup \{34\}, \\ A_5 &= A_4 \cup \{a + 34 \mid a \in A_1\}, \end{aligned}$$

and so on. The cardinalities and diameters of the sets in this sequence are given in Table 1. The density of a set refers to the cardinality of that set divided by its diameter. Note that all entries in the density column are rounded to 3 decimal places, and this convention will be followed throughout the rest of the paper.

Set	$\left\|A_i + A_i\right\|$	$\left\|A_i - A_i\right\|$	Cardinality	Diameter	Density
A_1	26	25	8	14	0.571
A_2	30	33	9	17	0.529
A_3	60	59	16	31	0.516
A_4	64	67	17	34	0.500
A_5	94	93	24	48	0.500
A_6	98	101	25	51	0.490
A_7	128	127	32	65	0.492
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: Non-filling in method 1 example sequence.

As seen in Table 1, the diameter of each MSTD set in the chain is exactly $n = 17$ larger than the most recent MSTD set in the sequence. Furthermore, the cardinality of each MSTD set in the chain is exactly $\left|A_1\right| = 8$ larger than the most recent MSTD set in the sequence. As such, we conclude that the growth rate of the cardinality and diameter between consecutive MSTD sets in a sequence generated by this method is linear. This also implies that the limiting density of MSTD sets in the sequence is $8/17$, or approximately 0.471.

Remark 1. We have not proven that there exists an n for every MSTD set that satisfies the initial two conditions. However, empirical observations suggest that nearly every MSTD set does have such a valid n, and many have more than one. For the Conway set, we could use $n = 17, 18$ or 20 to generate the sequence.

As to why this is, we note that both x and y increase at roughly the same rate as n increases for most MSTD sets, so assumption (b) is likely to be met for larger n. On the other hand, we reason that assumption (a) is more likely to be met for smaller n. This is because values greater than n in A_1+A_1 and values less than 0 in A_1-A_1 are more likely to fill in the gaps. In other words, for smaller n, there is a higher probability that the number of unique elements modulo n in both the sumset and difference set equals n itself. We also note that assumption (a) is never satisfied if $n > 2 \cdot \max A_1$ because the number of unique sums and differences modulo n are $\left|A_1+A_1\right|$ and $\left|A_1-A_1\right|$, respectively.

2.2. Non-filling in Method 2

We now construct a second method that generates the desired sequence and has $\left|A_{i+1}\right| = \left|A_i\right| + 1$ for all A_i in the sequence. Note that this is the smallest possible cardinality growth rate for any sequence. We use the following results taken from Nathanson.

Theorem 2 ([9]). *Let $m, d,$ and k be integers such that $m \geq 4$, $1 \leq d \leq m-1$, $d \neq m/2$, and*

- *$k \geq 3$ if $d < m/2$, or*
- *$k \geq 4$ if $d > m/2$.*

Define

$$
\begin{aligned}
B &= [0, m-1] \setminus \{d\}, \\
L &= \{m-d, 2m-d, \ldots, km-d\}, \\
a^* &= (k+1)m - 2d, \\
A^* &= B \cup L \cup (a^* - B), \text{ and} \\
A &= A^* \cup \{m\}.
\end{aligned}
$$

Then A is a MSTD set of integers such that both $2m \in (A+A) \setminus (A^*+A^*)$ and $A^* - \{m\} \subset A^* - A^*$.

Lemma 1 ([9]). *Let $m \geq 4$ and $2 \leq r \leq m-3$, with $m, r \in \mathbb{N}$. Additionally, let $B = [0, m-1] \setminus \{r\}$. Then $B+B = [0, 2m-2]$, and $B-B = [-(m-1), m-1]$.*

We now build upon these results to construct our method, which is summarized in the theorem below.

Theorem 3. *Let A be a MSTD set that is constructed in the way described in Theorem 2, with the additional constraints that*

$$
\begin{aligned}
m &\equiv 0 \bmod 4, \text{ and} \\
d &\in \left\{\frac{m}{4}, \frac{3m}{4}\right\}.
\end{aligned}
$$

Then $A_1 = A \cup \{-d, (k+1)m - d\}$ *is MSTD. For* $r \geq 1$, *define*

$$\begin{aligned} A_{2r} &\coloneqq A_{2r-1} \cup \{(k+r+1)m - d\}, \text{ and} \\ A_{2r+1} &\coloneqq A_{2r-1} \cup \{-rm - d, (k+r+1)m - d\}. \end{aligned}$$

Then A_{2r} *is MDTS,* A_{2r+1} *is MSTD, and* $A_1 \subset \cdots \subset A_{2r-1} \subset A_{2r} \subset A_{2r+1} \subset \cdots$ *forms the desired alternating sequence. Additionally, if* $A_i^* = A_i \setminus \{m\}$, *we have*

1. $m + A_{2r-1}^* \subset A_{2r-1}^* + A_{2r-1}^*$,
2. $m - A_{2r-1}^* \subset A_{2r-1}^* - A_{2r-1}^*$,
3. $\left|A_{2r-1} + A_{2r-1}\right| = \left|A_{2r-1} - A_{2r-1}\right| + 1$, *and*
4. $\left|A_{2r} + A_{2r}\right| \leq \left|A_{2r} - A_{2r}\right| - m + 3$.

Proof. Consider a MSTD set A constructed as in Theorem 3, and let $A^* = A \setminus \{m\}$. Now, let

$$\begin{aligned} A_1 &= A \cup \{-d, (k+1)m - d\}, \\ A_1^* &= A^* \cup \{-d, (k+1)m - d\}, \text{ and} \\ L_1 &= L \cup \{-d, (k+1)m - d\}. \end{aligned}$$

We show that A_1 is MSTD. Since $-d$ and $(k+1)m-d$ are symmetric with respect to a^*, we have $\left|A_1^* + A_1^*\right| = \left|A_1^* - A_1^*\right|$.

We know $2m \notin A^* + A^*$, so showing $2m \notin A_1^* + A_1^*$ amounts to showing there exists no $a_1, a_2 \in A_1^*$ such that $a_1 - d = 2m$ or $a_2 + (k+1)m - d = 2m$. If such an a_1 existed, then $a_1 = 2m + d$. To prove that $2m + d \notin A_1^*$, we show individually that $2m + d$ is not in B, L_1, or $a^* - B$.

Since $\max(B) < 2m + d$, we have that $2m + d \notin B$. Next, if we suppose $2m + d \in L_1$, then since $d < m$, it follows that $2m + d = 3m - d$, which implies $d = m/2$. This is a contradiction since we assumed $d \neq m/2$.

If we assume $2m + d \in a^* - B$, then $2m + d = (k+1)m - 2d - b$ for some $b \in B$. Rearranging, we get $b = (k-1)m - 3d$. If $d = m/4$, then $b = km - 7m/4$. Since $k \geq 3$, this implies $b > m$, which is a contradiction. If $d = 3m/4$, then $b = km - 13m/4$. If $k > 4$, then $b > m$. If $k = 4$, then $b = d$. In either case, $b \notin B$. Therefore, $2m + d \notin a^* - B$, so no possible $a_1 \in A_1^*$ exists. Next, we solve for a_2 to get $a_2 = (1-k)m + d$. However, $a_2 \notin A_1^*$ because $a_2 < \min(A_1^*) = -d$. Thus, $2m \notin A_1^*$.

Next, we prove that $(A_1 + A_1) \setminus (A_1^* + A_1^*) = \{2m\}$ by showing $m + A_1^* \subseteq A_1^* + A_1^*$. First, we show $m + B \subset A_1^* + A_1^*$. We have $m + B = \left[m, 2m-1\right] \setminus \left\{d + m\right\}$. Since $2m - d \in L_1$, and $d - 1 \in B$, it follows that $(2m-d) + (d-1) = 2m - 1 \in A_1^* + A_1^*$. We have four cases to consider: (i) $1 < d < m - 2$, (ii) $d = 1$, (iii) $d = m - 2$, (iv) $d = m - 1$.

If (i) is true, then by Lemma 1, $B+B=[0,2m-2]$, so $m+B \subset A_1^*+A_1^*$. If (ii) is true, then $B+B=[0,2m-2]\setminus\{2\}$, so $m+B\subset A_1^*+A_1^*$. If (iii) is true, then $d=6$, $m=8$, and $B+B=[0,14]\setminus\{13\}$. We know $2m-d=10\in L$, and $3\in B$, hence $13\in A_1^*+A_1^*$ as well, so $m+B\subset A_1^*+A_1^*$. If (iv) is true, then $d=3$, $m=4$, and $B+B=[0,4]$. Next, $2m-d=5\in L_1$, and $0,1\in B$, so $5,6\in A_1^*+A_1^*$, and $m+B\subset A_1^*+A_1^*$. Combining all cases, we conclude

$$m+B \ \subset\ A_1^*+A_1^*. \tag{3}$$

Next, we show $m+L_1\subset A_1^*+A_1^*$. Take $l\in L_1$ such that $l\neq \max(L_1)$. Then, $l+m\in L_1$, and since $0\in B$, we have $l+m\in A_1^*+A_1^*$. If $l=\max(L_1)=(k+1)m-d$, then $l+m=(k+2)m-d$. If $d=m/4$, then $2m-d=7m/4\in L_1$ and $a^*-m/2=km\in a^*-B$. Therefore, $km+7m/4=l+m\in A_1^*+A_1^*$. If $d=3m/4$, then $3m-d=9m/4\in L_1$ and $a^*-m/2=(k-1)m\in a^*-B$. Therefore, $9m/4+(k-1)m=l+m\in A_1^*+A_1^*$. Thus,

$$m+L_1 \ \subset\ A_1^*+A_1^*. \tag{4}$$

Lastly, we show $m+a^*-B\subset A_1^*+A_1^*$. First, $x\in m+a^*-B$ implies that $x=a^*+(m-b)$, where $b\in B$. Clearly, $a^*\in a^*-B$. We show $m-b\in A_1^*$. First, let $b\neq 0$. For all $b\in B\setminus\{0\}$, we have $m-B\setminus\{0\} \ =\ [1,m-1]\setminus\{m-d\}$. If $m-b\neq d$, then $m-b\in B$ and $x\in A_1^*+A_1^*$. If $m-b=d$ and $d=m/4$, then $b=3m/4$ and $x=km+3m/4=(k+1)m-d$, so $x\in L_1$, and $x\in A_1^*+A_1^*$. If $m-b=d$ and $d=3m/4$, then $b=m/4$ and $x=km+m/4=(k+1)m-d$, so $x\in L_1$ and $x\in A_1^*+A_1^*$. If $b=0$, then $x \ =\ (k+2)m-2d$. Taking $(k+1)m-d,\ m-d\in L_1$, we get $(k+1)m-d+(m-d) \ =\ (k+2)m-2d$. Thus, $x\in A_1^*+A_1^*$, and

$$m+a^*-B \ \subset\ A_1^*+A_1^*. \tag{5}$$

From the subset relations (3), (4), and (5), we conclude that $m+A_1^*\subset A_1^*+A_1^*$. Therefore, $(A_1+A_1)\setminus(A_1^*+A_1^*)=\{2m\}$, and

$$\left|A_1+A_1\right| \ =\ \left|A_1^*+A_1^*\right|+1. \tag{6}$$

Next, we show $\left|A_1-A_1\right|=\left|A_1^*-A_1^*\right|$ by showing $\{m\}-A_1^*\subseteq A_1^*-A_1^*$. In the proof of Theorem 2, Nathanson showed that $\{m\}-A^*\subseteq A^*-A^*$ [9]. It remains to be shown that $\{m\}-\{-d,(k+1)m-d\}\subseteq A_1^*-A_1^*$.

First, $((k+1)m-d)-m=km-d\in A_1^*-A_1^*$, since $km-d\in L_1$. Now, consider $m-(-d)=m+d$. If $d=m/4$, then $m-2d\in B$ and $2m-d\in L_1$, so we have that $(2m-d)-(m-2d)=m+d\in A_1^*-A_1^*$. If $d=3m/4$. Then $m-d<m/2$, so $2m-2d<m$. This gives $2m-2d\in B$ since $2m-2d\neq d$. Taking $3m-d\in L_1$, we have $(3m-d)-(2m-2d)=m+d$, so $m+d\in A_1^*-A_1^*$. Thus, $m-A_1^*\subseteq A_1^*-A_1^*$, and since $A_1^*\subset A_1$,

$$\left|A_1-A_1\right| \ =\ \left|A_1^*-A_1^*\right|. \tag{7}$$

From Equations (6) and (7), we have that A_1 satisfies $\left|A_1 + A_1\right| = \left|A_1 - A_1\right| + 1$.

Now, for $r \geq 1$, assume that A_{2r-1} is a MSTD set constructed as

$$\begin{aligned} L_{2r-1} &= \{-(r-1)m - d, \ldots, (k+r-1)m - d,\ (k+r)m - d\}, \\ A^*_{2r-1} &= B \,\cup\, L_{2r-1} \,\cup\, a^* - B, \text{ and} \\ A_{2r-1} &= A^*_{2r-1} \,\cup\, \{m\}. \end{aligned}$$

Assume further that A_{2r-1} satisfies

$$\begin{aligned} m - A^*_{2r-1} &\subseteq A^*_{2r-1} - A^*_{2r-1}, \\ 2m &\notin A^*_{2r-1} + A^*_{2r-1}, \text{ and} \\ \left|A_{2r-1} + A_{2r-1}\right| &= \left|A_{2r-1} - A_{2r-1}\right| + 1. \end{aligned} \tag{8}$$

We have already proven that A_1 satisfies Equation (8). Let

$$A_{2r+1} \;=\; A_{2r-1} \,\cup\, \{-rm - d, (k+r+1)m - d\}.$$

We show A_{2r+1} is MSTD and satisfies Equation (8). First, we define

$$\begin{aligned} L_{2r+1} &= L_{2r-1} \,\cup\, \{-rm - d, (k+r+1)m - d\}, \\ A^*_{2r+1} &= B \,\cup\, L_{2r+1} \,\cup\, a^* - B, \text{ and} \\ A_{2r+1} &= A^*_{2r+1} \cup \{m\}. \end{aligned}$$

Since A^*_{2r+1} is symmetric with respect to a^*, we have that

$$\left|A^*_{2r+1} + A^*_{2r+1}\right| \;=\; \left|A^*_{2r+1} - A^*_{2r+1}\right|. \tag{9}$$

We now show that $2m \notin A^*_{2r+1} + A^*_{2r+1}$. From our inductive assumption we know $2m \notin A^*_{2r-1} + A^*_{2r-1}$, so we need to show that there exists no $a_1, a_2 \in A^*_{2r+1}$ such that $a_1 - rm - d = 2m$ or $a_2 + (k+r+1)m - d = 2m$. We immediately know that $a_2 \notin A^*_{2r+1}$ since $a_2 < -rm - d = \min(A^*_{2r+1})$. For a_1, we rearrange and get $a_1 = (r+2)m + d$. Clearly, $a_1 \notin B$ since $m - 1 < a_1$. Additionally, $d < m$ and $a_1 \in L_{2r+1}$ implies $(r+2)m + d = (r+3)m - d$, which gives $m = 2d$, a contradiction. Thus, $a_1 \notin L_{2r+1}$.

If $a_1 \in a^* - B$, then there is $b \in B$ such that $(r+2)m + d = (k+1)m - 2d - b$, which gives $b = (k-r-1)m - 3d$. If $d = m/4$, then

$$\begin{aligned} b &= \left(k - r - \frac{7}{4}\right) m \Rightarrow \\ 0 &\leq \left(k - r - \frac{7}{4}\right) m \;<\; m \Rightarrow \\ 0 &\leq k - r - \frac{7}{4} \;<\; 1. \end{aligned} \tag{10}$$

Since $k, r \in \mathbb{Z}$, the only instance where Equation (10) occurs is when $k-r=2$. Then $k-r-7/4 = 1/4 = d$. Since $d \notin B$, there exists no $b \in B$ such that $a_1 = a^* - b$ for this case. Next, if $d = 3m/4$, then

$$
\begin{aligned}
b &= \left(k - r - \frac{13}{4}\right) m \Rightarrow \\
0 &\leq \left(k - r - \frac{13}{4}\right) m \ < \ m \Rightarrow \\
0 &\leq k - r - \frac{13}{4} \ < \ 1. \qquad (11)
\end{aligned}
$$

Since $k, r \in \mathbb{Z}$, the only instance where Equation (11) occurs is when $k-r=4$. Then $k-r-13/4 = 3/4 = d$. Since $d \notin B$, there exists no $b \in B$ such that $a_1 = a^* - b$ for this case. Therefore, $a_1 \notin A^*_{2r+1}$ for all possible d, so we have that $2m \notin A^*_{2r+1} + A^*_{2r+1}$.

We now prove Statement 1 of Theorem 3. In the construction of A_1 we proved Equations (3) and (5). Since $A^*_1 \subset A^*_{2r+1}$, we have

$$
\begin{aligned}
m + B &\subset A^*_{2r+1} + A^*_{2r+1}, \text{ and} \\
m + a^* - B &\subset A^*_{2r+1} + A^*_{2r+1}. \qquad (12)
\end{aligned}
$$

We prove that $m + L_{2r+1} \subset A^*_{2r+1} + A^*_{2r+1}$. If $l \in L_{2r+1}$ and $l \neq \max(L_{2r+1})$, then we have $l + m \in L_{2r+1}$, so $l + m \in A^*_{2r+1} + A^*_{2r+1}$.

If $l = \max(L_{2r+1})$, then $l + m = (k+r+2)m - d$. We consider the two possible values of d. If $d = m/4$, then since $k \geq 3$, it follows that $(r+2)m - d = rm + 7m/4$, which is in L_{2r+1}. Next, in the construction of A_1, we showed that $a^* - m/2 = km$, which is in $a^* - B$. Then

$$
l + m = \left(k + r + \frac{7}{4}\right) m = km + \left(rm + \frac{7m}{4}\right) \in A^*_{2r+1} + A^*_{2r+1}. \qquad (13)
$$

If $d = 3m/4$, then since $k \geq 4$, it follows that $(r+3)m - d = rm + 9m/4 \in L_{2r+1}$. Next, in the construction of A_1, we showed that $a^* - m/2 = (k-1)m \in a^* - B$. Then

$$
l + m = \left(k + r + \frac{5}{4}\right) m = (k-1)m + \left(rm + \frac{9m}{4}\right) \in A^*_{2r+1} + A_{2r+1}. \qquad (14)
$$

Thus,

$$
m + L_{2r+1} \subset A^*_{2r+1} + A^*_{2r+1}. \qquad (15)
$$

Taking Equations (12) and (15), we get $m + A^*_{2r+1} \subset A^*_{2r+1} + A^*_{2r+1}$, which yields

$$
|A_{2r+1} + A_{2r+1}| = |A^*_{2r+1} + A^*_{2r+1}| + 1. \qquad (16)
$$

Now we prove Statement 2 of Theorem 3. In the proof of Theorem 2, Nathanson [9] showed that

$$\begin{aligned} m - B \ &\subset \ A^*_{2r+1} - A^*_{2r+1}, \text{ and} \\ a^* - B - m \ &\subset \ A^*_{2r+1} - A^*_{2r+1}. \end{aligned} \tag{17}$$

We show that $L_{2r+1} - m \subset A^*_{2r+1} - A^*_{2r+1}$. If $l \in L_{2r+1}$ and $l \neq \min(L_{2r+1})$, then $l - m \in L_{2r+1}$, so $l - m \in A^*_{2r+1} - A^*_{2r+1}$.

If $l = \min(L_{2r+1}) = -rm - d$, then $m - l \ = \ (r+1)m + d$. We consider the two possible values of d. If $d = m/4$, then $m - 2d \in B$. Consider $(r+2)m - d \in L_{2r+1}$. Then $(r+2)m-d-(m-2d) = m-l \in A^*_{2r+1}-A^*_{2r+1}$. If $d = 3m/4$, then $2m-2d \in B$. Consider $(r+3)m-d \in L_{2r+1}$. Then $(r+3)m-d-(2m-2d) = m-l \in A^*_{2r+1}-A^*_{2r+1}$. Thus,

$$m - L_{2r+1} \ \subset \ A^*_{2r+1} - A^*_{2r+1}. \tag{18}$$

Taking Equations (17) and (18) gives $m - A^*_{2r+1} \ \subset \ A^*_{2r+1} - A^*_{2r+1}$, which yields

$$\left|A_{2r+1} - A_{2r+1}\right| \ = \ \left|A^*_{2r+1} - A^*_{2r+1}\right|. \tag{19}$$

Combining Equations (9), (16), and (19) shows that A_{2r+1} is a MSTD set which satisfies Equation (8), and equivalently, Statement 3 of Theorem 3.

Now, we prove Statement 4 of Theorem 3. Define $A_{2r} = A_{2r-1} \cup \{(k+r+1)m-d\}$. We show that

$$|A_{2r} + A_{2r}| \ \leq \ |A_{2r-1} + A_{2r-1}| + m. \tag{20}$$

First, denote $p = (k + r + 1)m - d$. Since $\max(A_{2r-1}) < p$, we have that $2p$ is a new sum. Now, consider $p + L_{2r-1}$. Take $l \in L_{2r-1}$. If $l < \max(L_{2r-1})$, then $l + m \in L_{2r-1}$ and

$$l + p \ = \ (l + m) + (p - m). \tag{21}$$

Clearly, $p-m \in L_{2r-1}$, so $l+p \in A_{2r-1}+A_{2r-1}$. If $l = \max(L_{2r-1}) \ = \ (k+r)m-d$, then $l + p$ is a new sum.

Now, consider $m + p = (k + r + 2)m - d$. If $d = m/4$, then

$$m + p \ = \ \left(k + r + \frac{7}{4}\right) m.$$

Since $(r + 2)m - d \in L_{2r-1}$, we apply the same sum used in Equation (13) to show $m + p \in A_{2r-1} + A_{2r-1}$. If $d = 3m/4$, then

$$m + p \ = \ \left(k + r + \frac{5}{4}\right) m.$$

Since $(r+3)m-d \in L_{2r-1}$ we apply the same sum used in Equations (14) to show $m+p \in A_{2r-1}+A_{2r-1}$. Therefore, $m+p$ is not a new sum.

Consider $p+B=[p,p+m-1]\setminus\{p+d\}$. We have two possibilities for the value of d: (i) $d=m/4$, (ii) $d=3m/4$. If (i) is true, then for $b\in B$, we have

$$p+b \;=\; \left(k+r+\frac{3}{4}\right)m+b.$$

Let $x\in\mathbb{Z}$ such that $-r+1\le x\le k+r$. Then $xm-d=(x-1/4)m\in L_{2r-1}$. We now show that there exists some $b_1\in B$ such that

$$\begin{aligned}\left(x-\frac{1}{4}\right)m+a^*-b_1 \;&=\; p+b,\\ \left(x-\frac{1}{4}\right)m+\left(k+\frac{1}{2}\right)m-b_1 \;&=\; \left(k+r+\frac{3}{4}\right)m+b, \text{ and}\\ \left(x-r-\frac{1}{2}\right)m-b \;&=\; b_1.\end{aligned} \tag{22}$$

Recall that $0\le b,\ b_1<m$. If we pick $x=r+1$, then $b_1=m/2-b$, so $b\le m/2$. Furthermore, we know $m/2-b\ne d$ since $b\ne d$. Thus, for all $b\le m/2$, choosing $x=r+1$ and $b_1=m/2-b$ satisfies Equation (22). If we pick $x=r+2$, then $b_1=3m/2-b$, so $b>m/2$. Furthermore, we know $3m/2-b\ne d$ since $b<m$. Thus, for all $b>m/2$, choosing $x=r+2$ and $b_1=3m/2-b$ satisfies Equation (22). Therefore, for every $b\in B$ there exists an x and b_1 such that $(xm-d)+a^*-b_1=p+b$, so we have $p+B\subset A_{2r-1}+A_{2r-1}$ if (i) is true. If (ii) is true, then

$$p+b \;=\; \left(k+r+\frac{1}{4}\right)m+b.$$

Let $x\in\mathbb{Z}$ with $-r+1\le x\le k+r$. Then $xm-d=(x-3/4)m\in L_{2r-1}$. We now show that there exists some $b_1\in B$ such that

$$\begin{aligned}\left(x-\frac{3}{4}\right)m+a^*-b_1 \;&=\; \left(k+r+\frac{1}{4}\right)m+b,\\ \left(x-\frac{1}{4}\right)m+\left(k-\frac{1}{2}\right)m-b_1 \;&=\; \left(k+r+\frac{3}{4}\right)m+b, \text{ and}\\ \left(x-r-\frac{3}{2}\right)m-b \;&=\; b_1.\end{aligned} \tag{23}$$

Again, recall that $0\le b,\ b_1<m$. If we pick $x=r+2$, then $b_1=m/2-b$, so $b\le m/2$. Furthermore, we know $m/2-b\ne d$ since $b\ge 0$. Thus, for all $b\le m/2$, choosing $x=r+2$ and $b_1=m/2-b$ satisfies Equation (23). If we pick $x=r+3$. Then $b_1=3m/2-b$, so $b>m/2$. Furthermore, we know $3m/2-b\ne d$ since $b\ne d$. Thus, for all $b>m/2$, choosing $x=r+3$ and $b_1=3m/2-b$ satisfies Equation (23).

Therefore, for every $b \in B$ there exists an x and b_1 such that $(xm-d)+a^*-b_1 = p+b$, so we have $p+B \subset A_{2r-1}+A_{2r-1}$ if (ii) is true. Therefore, since it is true for both (i) and (ii), we have $p+B \ \subset \ A_{2r-1}+A_{2r-1}$.

Next, we observe that $p+a^*-B$ has exactly $m-1$ elements. We show that there exists $l \in L_{2r-1}$ such that $l \neq \max(L_{2r-1})$ and $l \in a^*-B$. Since elements of L_{2r-1} are separated by exactly m and since

$$\begin{aligned} a^*-B \ &= \ \big[km-2d+1, km-2d+m\big] \setminus \big\{(k+1)m-3d\big\}, \text{ and} \\ \min(L_{2r-1}) \ < \ \min(a^*-B) \ &< \ \max(a^*-B) \ < \ \max(L_{2r-1}), \end{aligned}$$

there must exist some $l \in L_{2r-1}$ such that $l \in [km-2d+1, km-2d+m]$ and $l \neq (k+r)m-d$. We take this l and let $l = xm-d$ for some $x \in \mathbb{Z}$ with $-r+1 < x < k+r$. If $l = (k+1)m-3d$, then

$$\begin{aligned} xm-d \ &= \ (k+1)m-3d \\ 2d \ &= \ (k-x+1)m. \end{aligned} \tag{24}$$

However, since $d = m/4$ or $3m/4$, we have $2d \not\equiv 0 \bmod m$. Thus, Equation (24) is impossible, and so $l \neq (k+1)m-3d$, which implies $l \in a^*-B$. Then we have that $l+p \in A_{2r-1}+A_{2r-1}$ from Equation (21), so although there are $m-1$ elements in $p+a^*-B$, there are only up to $m-2$ total possible new sums in this set. Recall that the only other new sums we found in $A_{2r}+A_{2r}$ were $2p$ and $p+(k+r)m-d$. Thus, the maximum number of new sums in $A_{2r}+A_{2r}$ that are not in $A_{2r-1}+A_{2r-1}$ is $(m-2)+2 = m$. Hence, Equation (20) is true.

Now, consider $A_{2r}-A_{2r}$. First, we note that

$$A_{2r-1}-A_{2r-1} \ \subset \ \big[-\big((k+r)m-d\big), (k+r)m-d\big].$$

Considering $p-B$, we have

$$p-B \ = \ [(k+r)m-d+1, (k+r+1)m-d] \ \setminus \{(k+r+1)m-2d\}.$$

There are exactly $m-1$ elements in $p-B$, and since none of them lie in the range of $A_{2r-1}-A_{2r-1}$, we have that $p-B$ gives a set of $m-1$ differences in $A_{2r}-A_{2r}$ that are not in $A_{2r-1}-A_{2r-1}$. Similarly, $B-p$ contributes another $m-1$ new differences. Adding these up, we have that A_{2r} has at least $2m-2$ new differences, that is,

$$\big|A_{2r}-A_{2r}\big| \ \geq \ \big|A_{2r-1}-A_{2r-1}\big| + 2m-2. \tag{25}$$

If we take the results from Equations (8), (20), and (25), we have

$$\begin{aligned} \big|A_{2r}+A_{2r}\big| \ \leq \ \big|A_{2r-1}+A_{2r-1}\big| + m \ &= \ \big|A_{2r-1}-A_{2r-1}\big| + m + 1, \\ \big|A_{2r-1}-A_{2r-1}\big| + m + 1 \ &\leq \ \big|A_{2r}-A_{2r}\big| - m + 3, \text{ and} \\ \big|A_{2r}+A_{2r}\big| \ &\leq \ \big|A_{2r}-A_{2r}\big| - m + 3, \end{aligned}$$

which proves Statement 4 of Theorem 3. Next, note that since $m \geq 4$, we have that $\left|A_{2r} + A_{2r}\right| < \left|A_{2r} - A_{2r}\right|$. Thus, A_{2r} is a difference-dominated set. Furthermore, from our constructions of A_{2r} and A_{2r+1}, we observe that $A_{2r-1} \subset A_{2r} \subset A_{2r+1}$.

Since the construction of A_{2r} and A_{2r+1} works for all $r \geq 1$, and since A_1 satisfies the necessary conditions in Equation (8) when $r = 1$, we are thus able to generate an infinite chain of sets

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

such that A_{2r-1} is MSTD and A_{2r} is MDTS for all $r \in \mathbb{N}$.

□

For an example, we apply non-filling in method 2 to $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$, the Conway set, and we get

$$\begin{aligned} A_1 &= \{-1, 0, 2, 3, 4, 7, 11, 12, 14, 15\}, \\ A_2 &= A_1 \cup \{19\}, \\ A_3 &= A_2 \cup \{-5\}, \end{aligned}$$

and so on. Table 2 summarizes the growth characteristics of this sequence.

Set	$\|A_i + A_i\|$	$\|A_i - A_i\|$	Cardinality	Diameter	Density
A_1	32	31	10	16	0.625
A_2	36	37	11	20	0.550
A_3	40	39	12	24	0.500
A_4	44	45	13	28	0.464
A_5	48	47	14	32	0.437
A_6	52	53	15	36	0.416
A_7	56	55	16	40	0.400
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2: Non-filling in method 2 example sequence.

As can be seen, non-filling in method 2 gives a linear cardinality growth rate of 1 and a linear diameter growth rate of $m = 4$ between consecutive sets in the chain. Thus, the limiting density of MSTD sets is 0.25.

Remark 2. The reason we start the chain with A_1 instead of A is that if if $d = m/4$, then $A \cup \{(k+1)m - d\}$ is not guaranteed to be MDTS. For instance, if A is the Conway set, then $A \cup \{15\}$ is actually sum-difference balanced. Additionally, since $(k+1)m - d > \max(A)$ and $-d < \min(A)$, the 'no filling in' constraint is met.

2.3. Non-filling in Method 3

Building off of non-filling in method 2, we now construct a more specific method that has a smaller diameter growth rate between consecutive MSTD sets. For $k \geq 0$, let

$$\begin{aligned} A_{4k+1} &= \{-5k-24, -5k-23, -5k-19, -5k-18, \ldots, -14, -13\} \\ &\quad \cup \{21, 22, 26, 27, \ldots, 5k+31, 5k+32\} \\ &\quad \cup \{-8, -7, -4, -3, 0, 5, 8, 11, 12, 15, 16\} \\ &= \{-7, 0, 5, 8, 15\} \cup (5 \cdot [-k-5, k+6] + \{1, 2\}) \setminus \{-9, 1, 2, 6, 7, 17\}. \end{aligned}$$

We show A_{4k+1} is MSTD. Using the shorthand

$$B_n = A_n \setminus \{5\},$$

we see that B_{4k+1} is symmetric with respect to 8, so

$$\left|B_{4k+1} + B_{4k+1}\right| = \left|B_{4k+1} - B_{4k+1}\right|.$$

Note that $B_{4k+1} + B_{4k+1}$ does not include 10, as B_{4k+1} does not contain any of $\{-5, 2, 10, 17\}$, and the remaining elements of B_{4k+1} are congruent to 1 or 2 modulo 5. Since $5 \in A_{4k+1}$, we know $A_{4k+1} + A_{4k+1}$ does include 10. On the other hand, if we let $l \in [-k-5, k+5]$ and $\delta \in \{1, 2\}$ we form

$$\begin{aligned} (5l + \delta) + 5 &= (5(l+1) + \delta) + 0, \\ (5(k+6) + \delta) + 5 &= (5(k+4) + \delta) + 15, \\ -7 + 5 &= 11 + (-13), \\ 0 + 5 &= 8 + (-3), \\ 8 + 5 &= 16 + (-3), \text{ and} \\ 15 + 5 &= 8 + 12, \end{aligned} \tag{26}$$

so $\left|A_{4k+1} + A_{4k+1}\right| = \left|B_{4k+1} + B_{4k+1}\right| + 1$.

To see that $\left|A_{4k+1} - A_{4k+1}\right| = \left|B_{4k+1} - B_{4k+1}\right|$, we let $l \in [-k-4, k+6]$ and $\delta \in \{1, 2\}$. We form

$$\begin{aligned} (5l + \delta) - 5 &= (5(l-1) + \delta) + 0, \\ (5(-k-5) + \delta) - 5 &= (5(-k-3) + \delta) - 15, \\ -7 - 5 &= 0 - 12, \\ 0 - 5 &= 11 - 16, \\ 8 - 5 &= 11 - 8, \text{ and} \\ 15 - 5 &= 22 - 12. \end{aligned} \tag{27}$$

Thus,

$$\left|A_{4k+1} + A_{4k+1}\right| \;=\; \left|A_{4k+1} - A_{4k+1}\right| + 1. \tag{28}$$

Now, let

$$\begin{aligned} A_{4k+3} \;&=\; \{-5k-28, -5k-24, -5k-19, -5k-18, \ldots, -14, -13\} \\ &\quad \cup\; \{21, 22, 26, 27, \ldots, 5k+32, 5k+36\} \\ &\quad \cup\; \{-8, -7, -4, -3, 0, 5, 8, 11, 12, 15, 16\} \\ &=\; \{-5k-28, -7, 0, 5, 8, 15, 5k+36\} \\ &\quad \cup\; (5 * [-k-5, k+6] + \{1, 2\}) \setminus \{-9, 6, 7, 17\} \\ &=\; A_{4k+1} \cup \{-5k-28, 5k+36\}. \end{aligned}$$

As B_{4k+3} is symmetric with respect to 8, we have

$$\left|B_{4k+3} + B_{4k+3}\right| \;=\; \left|B_{4k+3} - B_{4k+3}\right|.$$

Note $B_{4k+3} + B_{4k+3}$ does not contain 10 as $B_{4k+3} = B_{4k+1} \cup \{-5k-28, 5k+36\}$ and $5k+38, -5k-26 \notin B_{4k+3}$. However, $A_{4k+3} + A_{4k+3}$ contains 10. Using the sums in Equations (26) and

$$\begin{aligned} (-5k-28) + 5 \;&=\; (-5k-23) + 0, \text{ and} \\ (5k+36) + 5 \;&=\; (5k+26) + 15, \end{aligned}$$

we see that $\left|A_{4k+3} + A_{4k+3}\right| = \left|B_{4k+3} + B_{4k+3}\right| + 1$.

To show that $\left|A_{4k+3} - A_{4k+3}\right| = \left|B_{4k+3} - B_{4k+3}\right|$, we use the differences in Equations (27) and

$$\begin{aligned} (-5k-28) - 5 \;&=\; (-5k-18) - 15, \text{ and} \\ (5k+36) - 5 \;&=\; (5k+31) - 0. \end{aligned}$$

Thus,

$$\left|A_{4k+3} + A_{4k+3}\right| \;=\; \left|A_{4k+3} - A_{4k+3}\right| + 1. \tag{29}$$

Let

$$A_{4k+2} \;=\; A_{4k+1} \cup \{5k+36\}.$$

For $l \in [-k-5, k+5]$ and $\delta \in \{1, 2\}$,

$$\begin{aligned} (5l+\delta) + (5k+36) \;&=\; (5(l+1)+\delta) + (5k+31), \\ -7 + (5k+36) \;&=\; (5k+32) + (-3), \\ 0 + (5k+36) \;&=\; (5k+31) + 5, \\ 5 + (5k+36) \;&=\; (5k+26) + 15, \text{ and} \\ 8 + (5k+36) \;&=\; (5k+32) + 12. \end{aligned}$$

Since $\max(A_{4k+1}+A_{4k+1}) = 10k+64$ and 15 is the largest element of A_{4k+1} congruent to 0 mod 5, the new sums are

$$\begin{aligned}(5k+31)+(5k+36) &= 10k+67,\\ (5k+32)+(5k+36) &= 10k+68,\\ 2(5k+36) &= 10k+72, \text{ and}\\ 15+(5k+36) &= 5k+51.\end{aligned}$$

Thus, there are 4 new sums.

For $l \in [-k-4, k+6]$ and $\delta \in \{1,2\}$,

$$\begin{aligned}(5k+36)-(5l+\delta) &= (5k+31)-(5(l-1)+\delta),\\ (5k+36)-0 &= (5k+32)-(-4),\\ (5k+36)-5 &= (5k+31)-0,\\ (5k+36)-8 &= (5k+21)-(-7), \text{ and}\\ (5k+36)-15 &= (5k+21)-0.\end{aligned}$$

Since $\max(A_{4k+1}-A_{4k+1}) = 10k+56$ and -7 is the smallest element of A_{4k+1} congruent to 3 mod 5, the new positive differences are

$$\begin{aligned}(5k+36)-(-5k-23) &= 10k+59,\\ (5k+36)-(-5k-24) &= 10k+60, \text{ and}\\ (5k+36)-(-7) &= 5k+43.\end{aligned}$$

Thus, there are 6 new differences. Combining the amount of new sums and differences with Equation (28), we get

$$\left|A_{4k+2}-A_{4k+2}\right| = \left|A_{4k+2}+A_{4k+2}\right|+1.$$

Let

$$A_{4k+4} = A_{4k+3}\cup\{5k+37\}.$$

For $l \in [-k-5, k+5]$ and $\delta \in \{1,2\}$,

$$\begin{aligned}(5l+\delta)+(5k+37) &= (5(l+1)+\delta)+(5k+32),\\ -7+(5k+37) &= (5k+22)+8,\\ 0+(5k+37) &= (5k+32)+5,\\ 5+(5k+37) &= (5k+27)+15, \text{ and}\\ 15+(5k+37) &= (5k+36)+16.\end{aligned}$$

Since $\max(A_{4k+3}+A_{4k+3}) = 10k+72$, we see that $(5k+33) \notin A_{4k+3}$, and since 8 is the largest element of A_{4k+1} congruent to 3 mod 5, the new sums are

$$\begin{aligned}(5k+32)+(5k+37) &= 10k+69,\\ (5k+36)+(5k+37) &= 10k+73,\\ 2(5k+37) &= 10k+74, \text{ and}\\ 8+(5k+37) &= 5k+45.\end{aligned}$$

Thus, there are 4 new sums. Next, for $l \in [-k-4, k+6]$ and $\delta \in \{1,2\}$,

$$\begin{aligned}(5k+37)-(5l+\delta) &= (5k+32)-(5(l-1)+\delta),\\ (5k+37)-(-7) &= (5k+36)-(-8),\\ (5k+37)-5 &= (5k+32)-0,\\ (5k+37)-8 &= (5k+22)-(-7), \text{ and}\\ (5k+37)-15 &= (5k+22)-0.\end{aligned}$$

Since $\max(A_{4k+3}-A_{4k+3}) = 10k+64$, we see that $(-5k-25) \notin A_{4k+3}$, and since 0 is the smallest element of A_{4k+3} congruent to 0 mod 5, the new positive differences are

$$\begin{aligned}(5k+37)-(-5k-24) &= 10k+61,\\ (5k+37)-(-5k-28) &= 10k+65, \text{ and}\\ (5k+37)-0 &= 5k+37.\end{aligned}$$

Thus, there are 6 new differences. Combining the amount of new sums and differences with Equation (29), we get

$$\left|A_{4k+4}-A_{4k+4}\right| = \left|A_{4k+4}+A_{4k+4}\right|+1.$$

Furthermore, we have

$$A_{4k+1} \subset A_{4k+2} \subset A_{4k+3} \subset A_{4k+4} \subset A_{4k+5},$$

so the sequence $A_1 \subset A_2 \subset \cdots$ alternates between being MSTD and MDTS. The growth characteristics of this sequence are given in Table 3.

As seen in the table, each set in the sequence has one more element than the previous. The diameter alternates between growing by 2 and 8 between consecutive MSTD sets in this sequence, which gives an 'average' diameter growth rate of 5. This is the best diameter growth rate yet, at the expense of a rather large initial set. The limiting density of MSTD sets is 0.4.

Remark 3. Other sequences were found that had even smaller 'average' diameter growth rates than 5. For instance, the set $\{0, 1, 2, 4, 5, 9, 12, 13, 14\}$ can be

Set	$\lvert A_i + A_i \rvert$	$\lvert A_i - A_i \rvert$	Cardinality	Diameter	Density
A_1	98	97	23	56	0.410
A_2	102	103	24	60	0.400
A_3	106	105	25	64	0.391
A_4	110	111	26	65	0.400
A_5	114	113	27	66	0.409
A_6	118	119	28	70	0.400
A_7	122	121	29	74	0.392
A_8	126	127	30	75	0.400
A_9	130	129	31	76	0.408
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3: Non-filling in method 3 sequence

extended to form a sequence with an 'average' diameter growth rate of 4.8. However, the period of the diameter growth rates for this sequence was 5 as opposed to 2, so the more concise proof in non-filling in method 3 was given instead. A similar method using a symmetric set with every integer odd modulo 7, excluding $\{-25, -23, -6, 5, 22, 24\}$ and including $\{-10, 7, 9\}$, gives an 'average' diameter growth rate of 14/3, cardinality growth rate of 2, and period of 6.

3. Growth Rates

We give a table (Table 4 below) that compares the growth characteristics of the various methods we constructed. The first three rows of the table cover the methods in [8], while the last three rows cover the methods presented in this paper. Note that the $\lvert A_1 \rvert$ and A_1 Diam. columns give the smallest possible cardinality and diameter for a set A_1 which a method can be applied to. Additionally, the Card. Rate and Diam. Rate columns measure the growth rate between consecutive MSTD sets in the sequence generated by the minimal A_1 for that method.

Method	$\lvert \mathbf{A_1} \rvert$	$\mathbf{A_1}$ **Diam.**	**Card. Rate**	**Diam. Rate**
Filling in 1	8	14	$> 3 \cdot \lvert A_{2i-1} \rvert$	$> 3 \cdot$ Diam(A_{2i-1})
Filling in 2	11	19	20	20
Non-filling in	11	18	4	8
Method 1	8	14	8	17
Method 2	10	16	2	8
Method 3	23	56	2	5 (avg)

Table 4: Minimal method growth characteristics

4. Future Directions

We conclude by briefly discussing some avenues for future research that have arisen in the course of this paper. Firstly, one might investigate the properties of the modulus n used in Theorem 1. For instance, we have not been able to prove when such an n exists for a given MSTD set. Also, the ideas used in Section 2.3 could be modified, possibly in a similar manner to the remark following 2.3, to give a slowly growing set for all odd integers. If a strategy for even integers is found, such a set would have a diameter growth rate of 4 between consecutive MSTD sets.

One could also investigate how these methods behave in other algebraic settings. One natural direction is to study fixed endpoint constructions, where one generates the desired alternating chains inside a subset with fixed endpoints. For example, given a prime p, one may ask how long of an alternating chain of sum- and difference-dominated supersets can be obtained within the finite field $\mathbb{F}_p$. More generally, one might explore how these constructions behave in fields of prime power order, or more broadly, in groups that are not necessarily abelian. In this context especially, methods with slower, controlled growth rates are far more applicable, hence the emphasis on developing efficient constructions.

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