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NEW FAMILIES OF WEIGHT SETS YIELDING PRESCRIBED WEIGHTED ZERO-SUM CONSTANTS

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Abstract

The problem of determining the values of the weighted generalizations of the classical zero-sum constants has received considerable attention over the past decade and a half. Although this topic has flourished for a wide range of weight sets and groups, there are still several challenging open problems. In a seminal paper, Adhikari, Chen, Friedlander, Konyagin, and Pappalardi posed the problem of determining weight sets A for which the weighted Gao constant satisfies $E_A(\mathbb{Z}_n) = n+j$, for specific small values of j. Due to a theorem of Grynkiewicz, Marchan, and Ordaz, this problem boils down to finding weight sets A for which the weighted Davenport constant satisfies $D_A(\mathbb{Z}_n) = j+1$. In this article, we construct large families of such weight sets satisfying these equalities for the finite cyclic group $\mathbb{Z}_n$ and we extend our methods to obtain analogous results for finite abelian groups of rank two and finite abelian p-groups, where p is a prime.

1. Introduction

Throughout this article, by G we shall mean a finite abelian group with an additive binary composition. By a sequence over G, we mean an unordered finite multi-set with elements from G, that is, a collection in which repetition of terms is allowed and the order of the elements is disregarded. A sequence $S = (g_1, \ldots, g_t)$ over G is said to be a *zero-sum sequence* if $g_1 + \cdots + g_t = 0$, where 0 denotes the identity element of G. Similarly, a *zero-sum subsequence* of a given sequence over G is simply a subsequence that is itself a zero-sum sequence.

The *Davenport constant* of G, which is usually denoted by $D(G)$, is defined as the least positive integer t such that every sequence over G of length t has at least one non-empty zero-sum subsequence.

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The study of the Davenport constant was first initiated by K. Rogers [17] and H. Davenport (see [15], for instance) in connection with factorization in an algebraic number field K. Subsequently, this constant was elegantly used to solve numerous other problems, notably in the proof of the infinitude of Carmichael numbers by Alford, Granville, and Pomerance (see [6]). For a more comprehensive historical overview on this topic, one may look into the expository articles [7] and [9].

The Davenport constant has a natural weighted analogue which was first introduced by Adhikari, Chen, Friedlander, Konyagin, and Pappalardi [3] and Adhikari and Rath [5] more than a decade and a half ago. Since then, this concept has received considerable attention from most of the mathematicians working in this area.

Throughout the paper, for integers $a \leq b$, we use the notation $[a, b]$ to denote the set $\{n \in \mathbb{Z} : a \leq n \leq b\}$.

Let G be a finite abelian group of exponent m. Then for any non-empty set $A \subset [1, m-1]$, the *Davenport constant of G with weight A,* denoted by $D_A(G)$, is defined to be the smallest positive integer t such that every sequence $S = (x_1, \ldots, x_t)$ over G of length t contains a non-empty *A-weighted zero-sum subsequence,* that is, there exist a subsequence $(x_{j_1}, \ldots, x_{j_k})$ of S and $a_1, \ldots, a_k \in A$ such that $\sum_{i=1}^{k} a_i x_{j_i} = 0$.

Clearly, when $A = \{1\}$, the weighted Davenport constant $D_A(G)$ coincides with the classical Davenport constant $D(G)$. The weighted Davenport constant has also found significant applications in various branches of mathematics. For example, one may look into [14] to see some interplays between coding theory and the weighted Davenport constant. In [12], Halter-Koch established a striking connection between the norms of principal ideals in a quadratic number field and the $\{\pm 1\}$-weighted Davenport constant of its class group.

In the above definition of the weighted Davenport constant, a particular restriction on the length of the A-weighted zero-sum subsequence gives rise to another zero-sum constant. Formally, for any finite abelian group G of exponent m and a non-empty set $A \subset [1, m-1]$, the constant $E_A(G)$ is defined to be the smallest positive integer t such that any sequence over G of length t has an A-weighted zero-sum subsequence of length exactly $|G|$.

Clearly, the classical *Gao constant* $E(G)$ can be retrieved by just setting $A = \{1\}$ in the above definition of $E_A(G)$.

At this point, it is worth noting that the concepts of these weighted generalizations of the classical zero-sum constants were initially introduced for the finite cyclic group $\mathbb{Z}_n$ (see [3] and [5]), and were subsequently extended to arbitrary finite abelian groups (see [2] and [19]).

Determining the values of $D_A(G)$ and $E_A(G)$ for different groups and for different weight sets is a problem of broad interest and for a wide range of groups and weight sets, the values of these constants have been explicitly determined (see, for example, [1], [3], [5], [19], [18], [13], and [10]).

In a pioneering paper [11], Grynkiewicz, Marchan, and Ordaz showed that for a weight set A, the constants $D_A(G)$ and $E_A(G)$ do not take values randomly but follow a nice manner. Specifically, they proved that for any finite abelian group G and non-empty weight set A,

$$E_A(G) = D_A(G) + |G| - 1. \tag{1}$$

This result confirms a conjecture independently posed by Griffiths [10] and Thangadurai [19] and aligns with the expectations expressed earlier by Adhikari et al. [5]. The classical case, corresponding to the weight set $A = \{1\}$, of this result was proved long ago by Gao [8] in 1996.

In the seminal work of Adhikari et al. [3], where the weighted zero-sum constant $E_A(G)$ was first introduced, the authors proved, along with other important results, that $E_A(\mathbb{Z}_n) = n + 1$, for the weight set $A = \mathbb{Z}_n \setminus \{0\}$. Following this result, they posed the problem of identifying other weight sets $A \subset \mathbb{Z}_n \setminus \{0\}$ for which $E_A(\mathbb{Z}_n) = n+j$, for specific small values of j. By Equation (1), this problem becomes equivalent to determining weight sets $A \subset \mathbb{Z}_n \setminus \{0\}$ for which $D_A(\mathbb{Z}_n) = j + 1$, for certain small values of j. Continuing along this line of investigation, Xia [20] proved the following couple of results. In what follows, for a real number x, the notation $\lceil x \rceil$ will be used to denote the smallest integer greater than or equal to x.

Theorem 1 ([20])**.** *If the weight set $A \subset \mathbb{Z}_n \setminus \{0\}$ is an arithmetic progression with length $\lceil \frac{n}{2} \rceil$ and common difference 1, then*

$$D_A(\mathbb{Z}_n) = 2.$$

Theorem 2 ([20])**.** *Let n be a positive integer and p be a prime. Suppose that k is the largest integer such that p^k divides n. If $A = \{a : a \not\equiv 0 \pmod{p}\}$, then*

$$D_A(\mathbb{Z}_n) = k + 1.$$

In view of the above problem posed by Adhikari et al. [3] and the subsequent results obtained by Xia [20], it is natural to investigate additional families of weight sets $A \subset \mathbb{Z}_n \setminus \{0\}$ for which the weighted Davenport constant $D_A(\mathbb{Z}_n)$ assumes small values. In this article, we aim to further develop this line of research. In particular, we construct several new families of such weight sets and establish exact values for their associated Davenport constants. Moreover, we extend our investigation to finite abelian groups of rank two and to finite abelian p-groups. These results broaden the current scope of known results in this direction.

The remainder of the article is organized as follows. In Section 3, we present our results concerning finite cyclic groups. Section 4 is devoted to the case of finite abelian groups of rank two, while the final section addresses extensions of our results to finite abelian p-groups. Before proceeding to these, we dedicate the next section to discussing some necessary preliminaries.

2. Preliminaries

The precise determination of the Davenport constant $D(G)$ for an arbitrary finite abelian group G remains one of the most challenging open problems in zero-sum theory. Its exact value is known only for certain restricted classes of groups. In what follows, we collect a few known results that will serve as essential tools in the proofs of our main theorems.

For any positive integer n, a straightforward application of the pigeonhole principle yields the identity

$$D(\mathbb{Z}_n) = n. \tag{2}$$

This result was extended by J. E. Olson to broader classes of finite abelian groups. In particular, he determined the Davenport constant for groups of rank two (see [16]) and for finite abelian p-groups, where p is a prime (see [15]). These results are stated below.

Theorem 3 ([16]). *Let h, k be two positive integers such that $h \mid k$. Then*

$$D(\mathbb{Z}_h \oplus \mathbb{Z}_k) = h + k - 1.$$

Theorem 4 ([15]). *Let p be a prime and let $e_1, \ldots, e_r$ be positive integers. Then*

$$D(\mathbb{Z}_{p^{e_1}} \oplus \mathbb{Z}_{p^{e_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_r}}) = 1 + \sum_{i=1}^{r} (p^{e_i} - 1).$$

We shall also make frequent use of the following property: for any finite abelian group G of exponent n, and for non-empty subsets $B \subset A \subset [1, n-1]$, one has

$$D_B(G) \geq D_A(G). \tag{3}$$

We conclude this section with a number-theoretic observation that will be useful in proving our theorems. As we have not found this result explicitly stated in the literature, we include a complete proof for the sake of clarity and completeness.

Lemma 1. *Let $m, n \geq 2$ be two positive integers such that $m \mid n$. Then*

$$\left\lceil \frac{n}{\lceil \frac{n}{m-1} \rceil - 1} \right\rceil = m.$$

Proof. It suffices to show that

$$m - 1 < \frac{n}{\lceil \frac{n}{m-1} \rceil - 1} \leq m. \tag{4}$$

Since $\lceil \frac{n}{m-1} \rceil - 1 < \frac{n}{m-1}$, we immediately have

$$m - 1 < \frac{n}{\lceil \frac{n}{m-1} \rceil - 1}. \tag{5}$$

To prove the upper bound, let $n = mt$, for some integer $t \geq 1$. We consider two cases.

`Case 1:` Let $t \leq m - 1$. Then

$$\begin{aligned}
\frac{n}{\lceil \frac{n}{m-1} \rceil - 1} &= \frac{mt}{\lceil \frac{mt}{m-1} \rceil - 1} \\
&= \frac{mt}{\lceil t + \frac{t}{m-1} \rceil - 1} \\
&= \frac{mt}{t + 1 - 1} = m.
\end{aligned}$$

`Case 2:` Let $t \geq m$. Observe that

$$\begin{aligned}
\frac{n}{m-1} - \frac{n}{m} &= \frac{n}{m(m-1)} \\
&= \frac{mt}{m(m-1)} \\
&= \frac{t}{m-1} > 1, \text{as } t \geq m.
\end{aligned}$$

On the other hand, since $\frac{n}{m-1} - (\lceil \frac{n}{m-1} \rceil - 1) \leq 1$, it follows that

$$\frac{n}{m} < \lceil \frac{n}{m-1} \rceil - 1,$$

which implies

$$\frac{n}{\lceil \frac{n}{m-1} \rceil - 1} < m. \tag{6}$$

Thus, combining Inequalities (5) and (6), we obtain (4), as desired. $\square$

3. Results on Finite Cyclic Groups

In the pioneering paper [3] on weighted generalizations of the zero-sum constants, the key result obtained by the authors was the value of $E_A(\mathbb{Z}_n)$ corresponding to the weight set $A = \{\pm 1\}$, which is arguably the most basic choice after the classical case $A = \{1\}$. In the same article [3], they also considered the maximal weight set $A = \mathbb{Z}_n \setminus \{0\}$ and proved that $E_A(\mathbb{Z}_n) = n + 1$. Motivated by this, they proposed the problem of finding weight sets A for which $E_A(\mathbb{Z}_n) = n + j$, for specific small values of j. As mentioned in the introduction, in light of Identity (1), this problem boils down to characterizing weight sets A for which $D_A(\mathbb{Z}_n) = j + 1$, for some specific small values of j. In this direction, introducing a new family of weight sets, we prove the following theorem.

Theorem 5. *Let $n \geq 2$ be a positive integer and $m \geq 2$ be a positive divisor of n. Then, for any weight set A satisfying $\left\{\frac{n}{m}\right\} \subset A \subset \left\{1, 2, \ldots, \left\lceil \frac{n}{m-1} \right\rceil - 1\right\}$, we have*

$$D_A(\mathbb{Z}_n) = m.$$

Proof. First let $S = (x_1, \ldots, x_m)$ be an arbitrary sequence over $\mathbb{Z}_n$. Then, by (2), as $D(\mathbb{Z}_m) = m$, S has a non-empty subsequence $S' = (x_{j_1}, \ldots, x_{j_l})$ such that

$$x_{j_1} + \cdots + x_{j_l} \equiv 0 \pmod{m}.$$

Multiplying both sides by $\frac{n}{m}$, we obtain

$$\frac{n}{m}(x_{j_1} + \cdots + x_{j_l}) \equiv 0 \pmod{n}.$$

This shows that S' is an $\{\frac{n}{m}\}$-weighted zero-sum subsequence of S in $\mathbb{Z}_n$. Consequently, $D_{\{\frac{n}{m}\}}(\mathbb{Z}_n) \leq m$. So, for any weight set $A \supset \{\frac{n}{m}\}$, applying (3), we have

$$D_A(\mathbb{Z}_n) \leq D_{\{\frac{n}{m}\}}(\mathbb{Z}_n) \leq m. \tag{7}$$

For the reverse inequality, consider the sequence

$$S_1 = (\underbrace{1, 1, \ldots, 1}_{(m-1) \text{ times}})$$

over $\mathbb{Z}_n$ and suppose $B = \{1, 2, \ldots, \lceil \frac{n}{m-1} \rceil - 1\}$.
Then, for any non-empty subsequence $S_1' = (x_{j_1}, \ldots, x_{j_h})$ of S_1 and any choice of weights $a_1, \ldots, a_h \in B$, we have

$$1 \leq \sum_{i=1}^{h} a_i x_{j_i} \leq (m-1)\left(\left\lceil \frac{n}{m-1} \right\rceil - 1\right) < n.$$

Therefore, the sequence S_1 does not have any non-empty B-weighted zero-sum subsequence over $\mathbb{Z}_n$, which implies $D_B(\mathbb{Z}_n) \geq m$. So, for any weight set $A \subset B$, applying again inequality (3), we have

$$D_A(\mathbb{Z}_n) \geq D_B(\mathbb{Z}_n) \geq m. \tag{8}$$

Combining Inequalities (7) and (8), we get our desired result. $\square$

Remark 1. Note that the condition on the weight set A in Theorem 5 is always feasible. Specifically, since

$$\frac{n}{m} < \left\lceil \frac{n}{m-1} \right\rceil,$$

we have

$$\frac{n}{m} \leq \left\lceil \frac{n}{m-1} \right\rceil - 1,$$

which implies that

$$\frac{n}{m} \in \left\{1, 2, \ldots, \left\lceil \frac{n}{m-1} \right\rceil - 1\right\}.$$

Remark 2. Theorem 5 supplies us a rich family of weight sets for which the values of the weighted Davenport constants are the same. For fixed positive integers $n, m \geq 2$, we have $2^{(\lceil \frac{n}{m-1} \rceil - 2)}$ distinct weight sets having the same value of the weighted Davenport constant. Notably, *the lower the value of m, the higher the number of such distinct weight sets.*

Remark 3 (Quantitative Comparison with Known Results)**.** To appreciate the breadth of Theorem 5, it is helpful to compare it with existing results in the literature. In particular, let n be an even square-free integer. Then, using Theorems 1 and 2, one can construct a total of $\frac{n}{2} + \omega(n)$ distinct weight sets for which the weighted Davenport constant equals 2; where $\omega(n)$ denotes, for each integer n, the number of distinct prime divisors of n.
In contrast, our Theorem 5 yields 2^{n-2} different weight sets for which the weighted Davenport constant also equals 2 — *a quantity that grows exponentially in n and far exceeds the previous total.*
For instance, if we take $n = 2 \cdot 3 \cdot 5 = 30$, previously known constructions in the literature yield a total of 18 weight sets with weighted Davenport constant equal to 2, while our result produces $2^{28} \approx 2.68 \times 10^{8}$ distinct such sets, all sharing the same constant value.

Corollary 1. *Let n, r be positive integers with $1 < r < n$ and let $A = \{1, 2, \ldots, r\}$. Then, as established independently by Adhikari, David, and Urroz [4], and by Xia and Li [21], we have*

$$D_A(\mathbb{Z}_n) = \left\lceil \frac{n}{r} \right\rceil.$$

In particular, suppose $m|n$ and let $r = \lceil \frac{n}{m-1} \rceil - 1$. Then, applying Lemma 1, we obtain

$$D_A(\mathbb{Z}_n) = \left\lceil \frac{n}{r} \right\rceil = m.$$

This matches the value given by Theorem 5 for the same choice of weight set A, thereby showing that Theorem 5 recovers, as a special case, the aforementioned result of Adhikari, David, and Urroz [4], and Xia and Li [21].

4. Results on Finite Abelian Groups of Rank 2

In this section, we consider general finite abelian groups of rank 2 and establish lower bounds for their A-weighted Davenport constant for weight sets analogous to those studied in Section 3. Under additional restrictions, we also determine the exact values of these constants.

Let $m, n \geq 2$ be two positive integers such that $m \mid n$. For the remainder of this section, we define $\alpha_{m,n} := \lceil \frac{n}{m-1} \rceil - 1$ and to avoid notational clutter, we shall write α in place of $\alpha_{m,n}$.

Lemma 2. *Let $m, n \geq 2$ be positive integers with $m \mid n$ and let $k \geq 1$. Then there exists a sequence over $\mathbb{Z}_n \oplus \mathbb{Z}_{kn}$ of length $m + \lceil \frac{kn}{\alpha} \rceil - 2$ that contains no non-empty $B = \{1, 2, \ldots, \alpha\}$-weighted zero-sum subsequence, that is,*

$$D_B(\mathbb{Z}_n \oplus \mathbb{Z}_{kn}) \geq m + \left\lceil \frac{kn}{\alpha} \right\rceil - 1.$$

Proof. Let us consider the following sequence over $\mathbb{Z}_n \oplus \mathbb{Z}_{kn}$:

$$S = (\underbrace{(1,0),(1,0),\ldots,(1,0)}_{m-1 \text{ terms}}, \underbrace{(0,1),(0,1),\ldots,(0,1)}_{(\lceil \frac{kn}{\alpha} \rceil - 1) \text{ terms}}).$$

Now, we claim that for any non-empty subsequence S' of S, its B-weighted sum cannot be zero. In fact, if S' includes at least one $(1,0)$, then following the argument used in Theorem 5 to prove the lower bound, one can verify that any B-weighted sum of S' yields an element whose value in the first co-ordinate lies in the interval $[1, n-1]$, so it is non-zero in $\mathbb{Z}_n$. Similarly, if S' includes at least one $(0,1)$, any B-weighted sum of S' yields an element whose value in the second co-ordinate lies in the interval $[1, kn-1]$, which is also non-zero in $\mathbb{Z}_{kn}$. Thus, no non-empty B-weighted subsequence of S sums to zero in $\mathbb{Z}_n \oplus \mathbb{Z}_{kn}$. Since the sequence S has length $m + \lceil \frac{kn}{\alpha} \rceil - 2$, it follows that

$$D_B(\mathbb{Z}_n \oplus \mathbb{Z}_{kn}) \geq m + \left\lceil \frac{kn}{\alpha} \right\rceil - 1.$$

□

We now establish that under certain additional assumptions, the lower bound established in Lemma 2 is in fact sharp. This is formalized in the following theorem.

Theorem 6. *Let $n, m \geq 2$ be two positive integers such that $m \mid n$. Further, suppose that A is any non-empty subset of $B = \{1, 2, \ldots, \alpha\}$ with $\frac{n}{m} \in A$, where $\alpha = \lceil \frac{n}{m-1} \rceil - 1$.*

(i) If $\alpha \mid n$, then for every positive integer $k \geq 1$, we have

$$D_A(\mathbb{Z}_n \oplus \mathbb{Z}_{kn}) = m + \left\lceil \frac{kn}{\alpha} \right\rceil - 1.$$

(ii) If $\alpha \nmid n$, and $n \equiv r \pmod{\alpha}$, for some $1 \leq r < \alpha$. Then, for every positive integer $k \geq 1$, if $(k-1)\alpha < kr$, we have

$$D_A(\mathbb{Z}_n \oplus \mathbb{Z}_{kn}) = m + \left\lceil \frac{kn}{\alpha} \right\rceil - 1.$$

Proof. (i) : Suppose that $\alpha \mid n$. Then, by Lemma 1, it follows that $\lceil \frac{kn}{\alpha} \rceil = km$, for every integer $k \geq 1$. Let $t = m + \lceil \frac{kn}{\alpha} \rceil - 1 = m + km - 1$ and let $S = (\mathbf{x_1}, \dots, \mathbf{x_t})$ be an arbitrary sequence over $G = \mathbb{Z}_n \oplus \mathbb{Z}_{kn}$, where each $\mathbf{x_i} = (a_i, b_i)$, for $1 \leq i \leq t$.

Now consider the group $G' = \mathbb{Z}_m \oplus \mathbb{Z}_{km}$. Clearly, G' is a finite abelian group of rank 2. By Theorem 3, we have $D(G') = m+km-1 = t$. Consequently, the sequence S, viewed as a sequence over G', contains a non-empty zero-sum subsequence. That is, there exists a non-empty subsequence $S' = (\mathbf{x_{j_1}}, \dots, \mathbf{x_{j_r}})$ of S such that

$$\mathbf{x_{j_1}} + \cdots + \mathbf{x_{j_r}} = \mathbf{0} \in G'.$$

Equivalently,

$$\sum_{i=1}^{r} a_{j_i} \equiv 0 \pmod{m}, \quad \sum_{i=1}^{r} b_{j_i} \equiv 0 \pmod{km}.$$

Multiplying both congruences by $\frac{n}{m}$ (which is an integer since $m \mid n$), we obtain

$$\sum_{i=1}^{r} \frac{n}{m} a_{j_i} \equiv 0 \pmod{n}, \quad \sum_{i=1}^{r} \frac{n}{m} b_{j_i} \equiv 0 \pmod{kn}.$$

Therefore, we have

$$\sum_{i=1}^{r} \frac{n}{m} \mathbf{x_{j_i}} = \mathbf{0} \in G.$$

This shows that S' is a non-empty $\{\frac{n}{m}\}$-weighted zero-sum subsequence of S over G. Hence $D_{\{\frac{n}{m}\}}(G) \leq t$. Now, for any weight set $A \subset B$ with $\frac{n}{m} \in A$, applying the monotonicity property (see Inequality (3)), we have

$$D_A(G) \leq D_{\{\frac{n}{m}\}}(G) \leq t = m + \left\lceil \frac{kn}{\alpha} \right\rceil - 1. \tag{9}$$

On the other hand, by Lemma 2, we have

$$D_A(G) \geq D_B(G) \geq m + \left\lceil \frac{kn}{\alpha} \right\rceil - 1. \tag{10}$$

Combining Inequalities (9) and (10), we conclude that

$$D_A(\mathbb{Z}_n \oplus \mathbb{Z}_{kn}) = m + \left\lceil \frac{kn}{\alpha} \right\rceil - 1,$$

as desired.

(ii) : Suppose now that $\alpha \nmid n$ and $n \equiv r \pmod{\alpha}$, for some $1 \le r < \alpha$. Let k be an integer chosen as specified in the statement of the theorem. Then, by the choice of k, it follows that $\lceil \frac{kn}{\alpha} \rceil = km$. The remainder of the proof proceeds in exactly the same manner as in part (i). □

Corollary 2. *Let $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$, where $n \ge 2$ be a positive integer and $m \ge 2$ be a positive divisor of n. Then, for any weight set $A \subset \{1, 2, \ldots, \lceil \frac{n}{m-1} \rceil - 1\}$ with $\frac{n}{m} \in A$, we have $D_A(G) = 2m - 1$.*

Determining the values of the weighted Davenport constant in part (ii) of Theorem 6 without imposing the additional inequality condition presents a natural and interesting direction for future research.

5. Results on Finite Abelian p-Groups

We now focus our attention on the finite abelian p-groups, where p is a prime. As in the previous section, we first prove a lower bound of the weighted Davenport constant for weight sets analogous to those studied in previous sections. Subsequently, we demonstrate that this lower bound is attainable under specified conditions.

Let $p \ge 2$ be a prime and let e, e_1 be positive integers with $e \le e_1$. For the remainder of this section, we define $\beta_{e,e_1} := \lceil \frac{p^{e_1}}{p^e - 1} \rceil - 1$ and we write simply β in place of β_{e,e_1}, to avoid notational complexity. Note that by Lemma 1 in Section 2, we have $\lceil \frac{p^{e_1}}{\beta} \rceil = p^e$.

Lemma 3. *Let $G = \mathbb{Z}_{p^{e_1}} \oplus \mathbb{Z}_{p^{e_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_r}}$ be a finite abelian p-group, where p is a prime and $e_1, e_2, \ldots, e_r$ are positive integers with $e_1 \le e_2 \le \cdots \le e_r$. Let e be a positive integer such that $e \le e_1$. Then there exists a sequence over G of length $\sum_{i=1}^{r} (\lceil \frac{p^{e_i}}{\beta} \rceil - 1)$ that contains no non-empty $B = \{1, 2, \ldots, \beta\}$-weighted zero-sum subsequence, that is,*

$$D_B(G) \ge 1 + \sum_{i=1}^{r} \left(\left\lceil \frac{p^{e_i}}{\beta} \right\rceil - 1 \right).$$

Proof. Let us consider the sequence S over G, consisting of

$$\left(\left\lceil \frac{p^{e_1}}{\beta} \right\rceil - 1\right) \text{ copies of } (1,0,\ldots,0),$$
$$\left(\left\lceil \frac{p^{e_2}}{\beta} \right\rceil - 1\right) \text{ copies of } (0,1,\ldots,0),$$
$$\vdots$$
$$\left(\left\lceil \frac{p^{e_r}}{\beta} \right\rceil - 1\right) \text{ copies of } (0,0,\ldots,1).$$

Clearly, the total length of S is $\sum_{i=1}^{r}\left(\left\lceil \frac{p^{e_i}}{\beta} \right\rceil - 1\right)$. Let S' be any non-empty subsequence of S. Consider any arbitrary element from S' and suppose that its i-th coordinate is non-zero (it is, in fact, 1) for some $1 \le i \le r$. We now observe that any B-weighted sum of S' produces an element whose value in the i-th coordinate lies in the interval $[1, p^{e_i} - 1]$, and is therefore non-zero in $\mathbb{Z}_{p^{e_i}}$. It follows that no non-empty B-weighted subsequence of S sums to the zero element of G. Therefore, we conclude that

$$D_B(G) \ge 1 + \sum_{i=1}^{r}\left(\left\lceil \frac{p^{e_i}}{\beta} \right\rceil - 1\right).$$

□

In the following theorem, we consider two special cases in which the lower bound established in Lemma 3 is tight, that is, it matches the precise value of the corresponding weighted Davenport constant.

Theorem 7. *Let $p \ge 2$ be a prime and let $G = \mathbb{Z}_{p^{e_1}} \oplus \mathbb{Z}_{p^{e_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_r}}$ be a finite abelian p-group, where $e_1, e_2, \ldots, e_r$ are positive integers with $e_1 \le e_2 \le \cdots \le e_r$. Let e be a positive integer such that $e \le e_1$ and let A be any non-empty subset of $B = \{1, 2, \ldots, \beta\}$ with $p^{e_1 - e} \in A$, where $\beta = \lceil \frac{p^{e_1}}{p^e - 1} \rceil - 1$.*

(i) If $\beta \mid p^{e_1}$, we have

$$D_A(G) = 1 + \sum_{i=1}^{r}\left(\left\lceil \frac{p^{e_i}}{\beta} \right\rceil - 1\right).$$

(ii) If $\beta \nmid p^{e_1}$, writing $p^{e_1} \equiv s \pmod{\beta}$, for some $1 \le s < \beta$, suppose that for every $j \in \{2, \ldots, r\}$, the inequality $(p^{e_j - e_1} - 1)\beta < sp^{e_j - e_1}$ holds. Then we have

$$D_A(G) = 1 + \sum_{i=1}^{r}\left(\left\lceil \frac{p^{e_i}}{\beta} \right\rceil - 1\right).$$

Proof. (i) : By Lemma 1 of Section 2, we have $\lceil \frac{p^{e_1}}{\beta} \rceil = p^e$. Since $\beta \mid p^{e_1}$, for any positive integer e_j with $e_j \geq e_1$, we have $\lceil \frac{p^{e_j}}{\beta} \rceil = p^{e_j - e_1 + e}$.
Let $t = 1 + \sum_{i=1}^{r} (\lceil \frac{p^{e_i}}{\beta} \rceil - 1) = 1 + \sum_{i=1}^{r} (p^{e_i - e_1 + e} - 1)$ and let $S = (\mathbf{x_1}, \ldots, \mathbf{x_t})$ be an arbitrary sequence over G of length t, where each $\mathbf{x_i} = (a_{i_1}, \ldots, a_{i_r})$, for $1 \leq i \leq t$.

Now consider the group $G' = \mathbb{Z}_{p^e} \oplus \mathbb{Z}_{p^{e_2 - e_1 + e}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_r - e_1 + e}}$. Observe that G' is a finite abelian p-group, and by Theorem 4, we have

$$D(G') = 1 + \sum_{i=1}^{r} (p^{e_i - e_1 + e} - 1) = t.$$

Consequently, the sequence S, viewing its elements modulo G', has a non-empty zero-sum subsequence. That is, there exists a non-empty subsequence $S' = (\mathbf{x_{j_1}}, \ldots, \mathbf{x_{j_l}})$ of S such that

$$\mathbf{x_{j_1}} + \cdots + \mathbf{x_{j_l}} = \mathbf{0} \in G'.$$

Equivalently,

$$\sum_{i=1}^{l} a_{j_{i_k}} \equiv 0 \pmod{p^{e_k - e_1 + e}}, \quad \text{for } 1 \leq k \leq r.$$

Now multiplying the congruences by $p^{e_1 - e}$ (which is an integer since $e \leq e_1$), we obtain

$$\sum_{i=1}^{l} p^{e_1 - e} \cdot a_{j_{i_k}} \equiv 0 \pmod{p^{e_k}}, \quad \text{for } 1 \leq k \leq r.$$

Therefore, it follows that

$$\sum_{i=1}^{l} p^{e_1 - e} \mathbf{x_{j_i}} = \mathbf{0} \in G.$$

This shows that S' is a non-empty $\{p^{e_1 - e}\}$-weighted zero-sum subsequence of S over G. The arbitrariness of S implies that $D_{\{p^{e_1 - e}\}}(G) \leq t$. Now, for any weight set $A \subset B$ with $p^{e_1 - e} \in A$, applying the monotonicity property (see Inequality (3)), we have

$$D_A(G) \leq D_{\{p^{e_1 - e}\}}(G) \leq t = 1 + \sum_{i=1}^{r} \left(\left\lceil \frac{p^{e_i}}{\beta} \right\rceil - 1 \right). \tag{11}$$

On the other hand, by Lemma 3, we have

$$D_A(G) \geq D_B(G) \geq 1 + \sum_{i=1}^{r} \left(\left\lceil \frac{p^{e_i}}{\beta} \right\rceil - 1 \right). \tag{12}$$

Combining Inequalities (11) and (12), we conclude that

$$D_A(G) = 1 + \sum_{i=1}^{r} \left(\left\lceil \frac{p^{e_i}}{\beta} \right\rceil - 1 \right),$$

as claimed.

(ii) : Now suppose that $\beta \nmid p^{e_1}$ and $p^{e_1} \equiv s \pmod{\beta}$, for some $1 \leq s < \beta$. Let $e_2, \ldots, e_r$ be positive integers chosen as specified in the statement of the theorem. Under these assumptions, one can verify that $\lceil \frac{p^{e_j}}{\beta} \rceil = p^{e_j - e_1 + e}$, for all $j \in \{1, \ldots, r\}$. The remaining part of the proof follows in exactly the same manner as in part (i). $\square$

Corollary 3. *Let p be a prime and let a, e be two positive integers such that $a \leq e$. If $G = \underbrace{\mathbb{Z}_{p^e} \oplus \mathbb{Z}_{p^e} \oplus \cdots \oplus \mathbb{Z}_{p^e}}_{r \text{ copies}}$, then for any weight set $A \subset \{1, 2, \ldots, \lceil \frac{p^e}{p^a - 1} \rceil - 1\}$ with $p^{e-a} \in A$, we have*

$$D_A(G) = 1 + r(p^a - 1).$$

It would be a natural and interesting direction for future research to determine the values of the weighted Davenport constant in part (ii) of Theorem 7 without assuming the additional inequality conditions.

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