#A90 INTEGERS 26 (2026)

THINNED WALLIS-TYPE PRIME PRODUCTS IN RESIDUE CLASSES MODULO 2^m

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Received: 3/25/26, Revised: 5/22/26, Accepted: 6/20/26, Published: 7/29/26

Abstract

For odd primes p we consider the factors $A(p) = (p - \chi_4(p))/(p + \chi_4(p))$, where χ_4 is the quadratic Dirichlet character modulo 4, and study products of $A(p)$ restricted to unions of residue classes modulo 2^m. We give a simple criterion for the existence of a finite nonzero limit, prove a logarithmic asymptotic in the general case, express the limiting constant in terms of Mertens-type constants in arithmetic progressions and Dirichlet L-values, and give reproducible high-precision computations of the resulting constants.

1. Introduction

Define $A(p)$ as in the abstract. The full product over odd primes has a finite limit:

$$\prod_{p \text{ odd}} A(p) = 2.$$

The classical Wallis product is

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}.$$

Equivalently, after indexing by odd integers $r > 1$, it can be written as

$$\frac{\pi}{2} = \prod_{\substack{r>1 \\ r \text{ odd}}} \frac{r - \chi_4(r)}{r + \chi_4(r)}.$$

We call the factors in this odd-integer formulation the Wallis factors. Thus, $A(p)$ is obtained by replacing the odd integer r in a Wallis factor by an odd prime p. Thinning by congruence conditions modulo 2^m is natural here, since χ_4 is a character

DOI: 10.5281/zenodo.21678054

modulo 4 and thus constant on residue classes modulo 2^m. By *thinning* we simply mean restricting the product to primes in a prescribed set of residue classes. The aim is to understand what happens after restricting p to selected residue classes modulo $q = 2^m$: for which selections does the product still converge to a finite nonzero limit?

2. Notation and Preliminary Facts

Fix $q = 2^m$ with $m \geq 2$. Let $(\mathbb{Z}/q\mathbb{Z})^\times$ be the group of reduced residue classes modulo q. Throughout, p denotes an odd prime. For $S \subset (\mathbb{Z}/q\mathbb{Z})^\times$ define

$$P_S(x) = \prod_{\substack{p \leq x \\ p \bmod q \in S}} A(p).$$

Since 4 divides q, each $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ has a well-defined value $\chi_4(a) \in \{\pm 1\}$. We use Mertens' theorem in arithmetic progressions in the form

$$\sum_{\substack{p \leq x \\ p \equiv a \ (q)}} \frac{1}{p} = \frac{1}{\varphi(q)} \log\log x + B(q,a) + o(1), \qquad x \to \infty,$$

for each $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, with constants $B(q,a)$ depending on q and a; see [8].

3. A Structural Identity

The following elementary factorization is useful throughout the paper.

Lemma 1. *For every odd prime p one has*

$$A(p) = \frac{(1 - \chi_4(p)/p)^2}{1 - 1/p^2}.$$

Proof. We compute

$$\frac{p - \chi_4(p)}{p + \chi_4(p)} = \frac{(p - \chi_4(p))^2}{p^2 - \chi_4(p)^2} = \frac{(1 - \chi_4(p)/p)^2}{1 - 1/p^2},$$

since $\chi_4(p)^2 = 1$ for odd primes p. □

4. Asymptotic Behavior and the Balance Criterion

For $S \subset (\mathbb{Z}/q\mathbb{Z})^\times$ set

$$\mu(S) = \sum_{a \in S} \chi_4(a).$$

The main asymptotic statement is as follows.

Theorem 1. *There exists* $K(q,S) \in (0,\infty)$ *such that*

$$P_S(x) = K(q,S)\,(\log x)^{-2\mu(S)/\varphi(q)}(1+o(1)), \qquad x \to \infty.$$

Proof. Write

$$A(p) = \frac{p-\chi_4(p)}{p+\chi_4(p)} = \frac{1-\chi_4(p)/p}{1+\chi_4(p)/p}.$$

With $u = \chi_4(p)/p$ we have $|u| \le 1/3$ for $p \ge 3$, hence

$$\log \frac{1-u}{1+u} = -2u + O(u^3), \qquad u \to 0,$$

and therefore

$$\log A(p) = -\frac{2\chi_4(p)}{p} + O\left(\frac{1}{p^3}\right).$$

Summing over primes $p \le x$ with $p \bmod q \in S$ yields

$$\log P_S(x) = -2 \sum_{\substack{p\le x\\ p \bmod q \in S}} \frac{\chi_4(p)}{p} + O\left(\sum_p \frac{1}{p^3}\right) = -2 \sum_{\substack{p\le x\\ p \bmod q \in S}} \frac{\chi_4(p)}{p} + O(1),$$

since $\sum_p p^{-3}$ converges. Decompose the main sum into residue classes:

$$\sum_{\substack{p\le x\\ p \bmod q \in S}} \frac{\chi_4(p)}{p} = \sum_{a\in S} \chi_4(a) \sum_{\substack{p\le x\\ p\equiv a\ (q)}} \frac{1}{p}.$$

By Mertens' theorem in arithmetic progressions,

$$\sum_{\substack{p\le x\\ p\equiv a\ (q)}} \frac{1}{p} = \frac{1}{\varphi(q)} \log\log x + B(q,a) + o(1), \qquad x \to \infty,$$

for each $a \in (\mathbb{Z}/q\mathbb{Z})^\times$. Hence,

$$\log P_S(x) = -\frac{2\mu(S)}{\varphi(q)} \log\log x + C(q,S) + o(1),$$

where $C(q,S) = -2\sum_{a\in S} \chi_4(a) B(q,a) + O(1)$ is finite. Exponentiating gives

$$P_S(x) = K(q,S)\,(\log x)^{-2\mu(S)/\varphi(q)}(1+o(1)),$$

with $K(q,S) = e^{C(q,S)} \in (0,\infty)$. □

The preceding theorem immediately gives the convergence criterion.

Corollary 1. *The limit* $\lim_{x\to\infty} P_S(x)$ *exists in* $(0,\infty)$ *if and only if* $\mu(S) = 0$.

5. The Limiting Constant

For each $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ define

$$C(q,a) = \lim_{x\to\infty} (\log x)^{1/\varphi(q)} \prod_{\substack{p\le x\\ p\equiv a\ (q)}} \left(1-\frac{1}{p}\right),$$

and

$$D(q,a) = \prod_{\substack{p\\ p\equiv a\ (q)}} \left(1-\frac{1}{p^2}\right).$$

Note that $D(q,a) = \zeta_{q,a}(2)^{-1}$, where $\zeta_{q,a}(s) = \prod_{p\equiv a\ (q)}(1-p^{-s})^{-1}$. This normalization matches the standard Mertens product asymptotic in arithmetic progressions; see [8]. We shall use the following formula of Williams.

Proposition 1 (Williams [8, Theorem 1])**.** *For each $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ one has*

$$C(q,a) = \left(e^{-\gamma} \frac{q}{\varphi(q)} \prod_{\substack{\chi \bmod q\\ \chi\ne\chi_0}} \left(\frac{K_{\mathrm{W}}(1,\chi)}{L(1,\chi)}\right)^{\overline{\chi}(a)} \right)^{1/\varphi(q)},$$

where χ_0 denotes the principal character, γ is Euler's constant, and $K_{\mathrm{W}}(1,\chi)$ is the nonzero constant appearing in Williams' theorem.

The next result identifies the constant $K(q,S)$ from Theorem 1. The balance condition is not needed for this identification; it is needed only for the disappearance of the logarithmic factor. The analytic framework for the Mertens product constants in arithmetic progressions is developed by Languasco and Zaccagnini [5, 6]. Efficient algorithms for high-precision evaluation of $C(q,a)$ are given in [6, (5)]; for $\zeta_{q,a}(2)$, and hence for $D(q,a)$, see Languasco and Moree [4, (5.16)].

Theorem 2. *For every subset $S \subset (\mathbb{Z}/q\mathbb{Z})^\times$ one has*

$$P_S(x) = \left(\prod_{a\in S} \left(\frac{C(q,a)^2}{D(q,a)}\right)^{\chi_4(a)}\right) (\log x)^{-2\mu(S)/\varphi(q)}(1+o(1)).$$

In particular,

$$K(q,S) = \prod_{a\in S} \left(\frac{C(q,a)^2}{D(q,a)}\right)^{\chi_4(a)}.$$

If $\mu(S)=0$, then

$$\lim_{x\to\infty} P_S(x) = \prod_{a\in S} \left(\frac{C(q,a)^2}{D(q,a)}\right)^{\chi_4(a)}.$$

Proof. By Lemma 1,

$$A(p) = \frac{(1-\chi_4(p)/p)^2}{1-1/p^2},$$

so

$$P_S(x) = \Bigg(\prod_{\substack{p \le x \\ p \bmod q \in S}} \Big(1 - \frac{\chi_4(p)}{p}\Big) \Bigg)^2 \cdot \prod_{\substack{p \le x \\ p \bmod q \in S}} \Big(1 - \frac{1}{p^2}\Big)^{-1}.$$

Fix a reduced residue class a modulo q. Since 4 divides q, the value $\chi_4(p)$ is constant on the progression $p \equiv a \pmod q$, namely $\chi_4(p) = \chi_4(a)$. Moreover,

$$1 - \frac{\chi_4(a)}{p} = \begin{cases} 1 - \dfrac{1}{p}, & \chi_4(a) = 1, \\ 1 + \dfrac{1}{p} = \dfrac{1-1/p^2}{1-1/p}, & \chi_4(a) = -1. \end{cases}$$

Consequently, for primes $p \equiv a \pmod q$,

$$\prod_{\substack{p \le x \\ p \equiv a\ (q)}} \Big(1 - \frac{\chi_4(a)}{p}\Big) = \Bigg(\prod_{\substack{p \le x \\ p \equiv a\ (q)}} \Big(1 - \frac{1}{p}\Big) \Bigg)^{\chi_4(a)} \cdot \Bigg(\prod_{\substack{p \le x \\ p \equiv a\ (q)}} \Big(1 - \frac{1}{p^2}\Big) \Bigg)^{(1-\chi_4(a))/2}.$$

In particular, the extra factors $(1-1/p^2)^{(1-\chi_4(a))/2}$ occur only when $\chi_4(a) = -1$ and are precisely compensated by the denominator $\prod(1-1/p^2)^{-1}$ coming from Lemma 1. Insert this identity into the previous display, multiply over all $a \in S$, and simplify. One obtains

$$P_S(x) = \prod_{a \in S} \Bigg(\prod_{\substack{p \le x \\ p \equiv a\ (q)}} \Big(1 - \frac{1}{p}\Big) \Bigg)^{2\chi_4(a)} \cdot \prod_{a \in S} \Bigg(\prod_{\substack{p \le x \\ p \equiv a\ (q)}} \Big(1 - \frac{1}{p^2}\Big) \Bigg)^{-\chi_4(a)} \cdot (1 + o(1)).$$

By the definitions of $C(q,a)$ and $D(q,a)$,

$$\prod_{\substack{p \le x \\ p \equiv a\ (q)}} \Big(1 - \frac{1}{p}\Big) = C(q,a)\,(\log x)^{-1/\varphi(q)}(1 + o(1))$$

and

$$\prod_{\substack{p \le x \\ p \equiv a\ (q)}} \Big(1 - \frac{1}{p^2}\Big) = D(q,a)(1 + o(1)).$$

Substituting these asymptotics yields

$$P_S(x) = \Bigg(\prod_{a \in S} \Big(\frac{C(q,a)^2}{D(q,a)} \Big)^{\chi_4(a)} \Bigg) (\log x)^{-2\mu(S)/\varphi(q)}(1 + o(1)).$$

This proves the asserted formula for $K(q,S)$. If $\mu(S)=0$, the logarithmic factor disappears and the finite nonzero limit equals the stated product. □

The following observation explains why the constants may be written in terms of Dirichlet L-values.

Remark 1. Let $G=(\mathbb{Z}/q\mathbb{Z})^\times$. By character orthogonality on G, the indicator of a subset $S\subset G$ admits a Fourier expansion

$$1_S(b)=\frac{1}{|G|}\sum_{\chi \bmod q}\widehat{1_S}(\chi)\chi(b), \qquad \widehat{1_S}(\chi)=\sum_{a\in S}\overline{\chi(a)}.$$

Inserting this expansion into the relevant Euler products yields expressions for $\log C(q,a)$ and $\log D(q,a)$ as finite rational linear combinations of $\log L(1,\chi)$ and $\log L(2,\chi)$.

6. Examples

In the first example we evaluate the full odd-prime product explicitly; in the second we compute a nontrivial thinned product in closed form; the third example illustrates how the accelerated character formula in Appendix A resolves the first case not covered by an elementary closed form.

Example 1. Let $q=4$, and take $S=(\mathbb{Z}/4\mathbb{Z})^\times=\{1,3\}$. Then $\mu(S)=0$, and

$$\prod_{p\ \mathrm{odd}}A(p)=2.$$

Indeed, by Lemma 1,

$$\prod_{p\ \mathrm{odd}}A(p)=\frac{\left(\prod_{p\ \mathrm{odd}}\left(1-\frac{\chi_4(p)}{p}\right)\right)^2}{\prod_{p\ \mathrm{odd}}\left(1-\frac{1}{p^2}\right)}.$$

The Euler product for $L(1,\chi_4)$ gives

$$\prod_{p\ \mathrm{odd}}\left(1-\frac{\chi_4(p)}{p}\right)=\frac{1}{L(1,\chi_4)}=\frac{4}{\pi}.$$

Moreover,

$$\prod_{p\ \mathrm{odd}}\left(1-\frac{1}{p^2}\right)=\frac{1}{\zeta(2)}\cdot\frac{1}{1-2^{-2}}=\frac{8}{\pi^2}.$$

Hence, $\prod_{p\ \mathrm{odd}}A(p)=(4/\pi)^2/(8/\pi^2)=2$.

Example 2. Let $q = 8$ and $S = \{\pm 1\}$. Let χ_8 be the real character modulo 8 defined by $\chi_8(p) = 1$ for $p \equiv \pm 1 \pmod 8$ and $\chi_8(p) = -1$ for $p \equiv \pm 3 \pmod 8$ for odd primes p. Write

$$P_+(x) = \prod_{\substack{p \le x \\ p \text{ odd} \\ p \equiv \pm 1\ (8)}} A(p), \qquad P_-(x) = \prod_{\substack{p \le x \\ p \text{ odd} \\ p \equiv \pm 3\ (8)}} A(p),$$

and denote $P_+ = \lim_{x\to\infty} P_+(x)$ and $P_- = \lim_{x\to\infty} P_-(x)$. Set $S_+ = \{\pm 1\}$ and $S_- = \{\pm 3\}$ in $(\mathbb{Z}/8\mathbb{Z})^\times$. By Corollary 1 both limits P_+ and P_- exist in $(0, \infty)$, since $\mu(S_+) = \mu(S_-) = 0$. Then $P_+P_- = \prod_{p \text{ odd}} A(p) = 2$ by Example 1. Moreover,

$$\frac{P_+}{P_-} = \prod_{p \text{ odd}} A(p)^{\chi_8(p)}.$$

Set $\psi = \chi_4\chi_8$, which is the real character modulo 8 with $\psi(p) = 1$ for $p \equiv 1, 3 \pmod 8$ and $\psi(p) = -1$ for $p \equiv 5, 7 \pmod 8$. A check on residue classes shows

$$A(p)^{\chi_8(p)} = \frac{p - \psi(p)}{p + \psi(p)} \qquad (p \text{ odd}).$$

Using Lemma 1 with ψ in place of χ_4 we obtain

$$\prod_{p \text{ odd}} A(p)^{\chi_8(p)} = \frac{\left(\prod_{p \text{ odd}}(1 - \psi(p)/p)\right)^2}{\prod_{p \text{ odd}}(1 - 1/p^2)} = \left(\frac{1}{L(1,\psi)}\right)^2 \cdot \frac{\pi^2}{8}.$$

The classical value $L(1,\psi) = \pi/(2\sqrt{2})$ gives $P_+/P_- = 1$. Therefore, $P_+ = P_- = \sqrt{2}$, hence

$$\prod_{\substack{p \text{ odd} \\ p \equiv \pm 1\ (8)}} A(p) = \sqrt{2}.$$

We use the classical evaluation $L(1,\psi) = \pi/(2\sqrt{2})$ for the real primitive character ψ modulo 8; see, for instance, Davenport [1, Chapter 6].

Example 3. Let $q = 16$ and $S = \{\pm 1\} = \{1, 15\}$. Here $\mu(S) = 0$. Theorem 2 gives

$$K(16, S) = \frac{C(16,1)^2 D(16,15)}{C(16,15)^2 D(16,1)}.$$

A direct computation of the partial products

$$P_S(x) = \prod_{\substack{p \le x \\ p \text{ odd} \\ p \equiv \pm 1\ (16)}} A(p)$$

is numerically misleading, because the convergence is slow and not monotone:

$$P_S(10^6) = 1.0000256406\ldots, \qquad P_S(10^7) = 0.9999838172\ldots.$$

Using instead the character-sum evaluation described in Appendix A, with working precision 100 decimal digits and truncation parameter $n_{\max} = 60$, one obtains

$$\log K(16, \{\pm 1\}) = -2.5619785140821339862366439219079601352364461 \cdot 10^{-56}.$$

Thus, $\log K(16, \{\pm 1\}) = 0$ to at least 40 decimal places, and hence

$$K(16, \{\pm 1\}) = 1.0000000000000000000000000000000000000000\ldots.$$

This explains why the direct partial products oscillate very close to 1, while still being inadequate as a reliable method for determining the constant.

7. Concluding Remarks

The arguments above apply verbatim to any modulus q with 4 dividing q. Indeed, for such q the function χ_4 is constant on each reduced residue class modulo q, and the proofs of Theorems 1 and 2 rely only on Mertens-type asymptotics in arithmetic progressions and on the identity in Lemma 1. In particular, for any $S \subset (\mathbb{Z}/q\mathbb{Z})^\times$ the product $P_S(x)$ admits the logarithmic asymptotic of Theorem 1, and it converges to a finite nonzero limit precisely when the balance condition $\mu(S) = 0$ holds. For $q = 2^m$ there are infinitely many balanced selections S, hence infinitely many associated limits $K(q, S)$. A simple family is given by $S = \{\pm 1 \pmod{2^m}\}$, and more generally by any union of residue classes containing equally many classes congruent to 1 and to 3 modulo 4. By character orthogonality one may express the constants $K(q, S)$ in terms of finitely many Dirichlet values $L(1, \chi)$ and $L(2, \chi)$, but we do not pursue explicit evaluations beyond the examples given above.

References

[1] H. Davenport, *Multiplicative Number Theory*, third edition, revised by H. L. Montgomery, Grad. Texts in Math. 74, Springer-Verlag, New York, 2000.

[2] F. Johansson and others, *mpmath: a Python library for arbitrary-precision floating-point arithmetic*, `https://mpmath.org`.

[3] A. Languasco, *Programs*, `https://www.dei.unipd.it/~languasco/Programs.html`.

[4] A. Languasco and P. Moree, Euler constants from primes in arithmetic progression, *Math. Comp.* **95** (2026), 363–387.

[5] A. Languasco and A. Zaccagnini, On the constant in the Mertens product for arithmetic progressions. I: identities, *Funct. Approx. Comment. Math.* **42** (2010), 17–27.

[6] A. Languasco and A. Zaccagnini, On the constant in the Mertens product for arithmetic progressions. II: numerical values, *Math. Comp.* **78** (2009), 315–326.

[7] Python Software Foundation, *The Python Language Reference*, Python Software Foundation, 2026, `https://docs.python.org/3/reference/`.

[8] K. S. Williams, Mertens' theorem for arithmetic progressions, *J. Number Theory* **6** (1974), 353–359.

[9] M. Winkler, *Python script*, `https://www.mikematics.de/prime-numbers/`.

Appendix A. High-Precision Computation of the Constants

The direct truncation of the prime product defining $P_S(x)$ is useful as a consistency check, but it is not an efficient method for determining the limiting constants. We therefore evaluate these constants by means of a character-sum formula obtained by combining the formula of Languasco and Zaccagnini for the Mertens product constants $C(q,a)$ with the corresponding formula of Languasco and Moree for $\zeta_{q,a}(2)$; see [6, (5)] and [4, (5.16)]. The background identities for these constants are given in [5]. Throughout this appendix let $q = 2^m$ and let χ_0 denote the principal character modulo q. To avoid a conflict with the notation $\mu(S)$ used in the main text, the Möbius function is denoted by μ_{M}. For $S \subset (\mathbb{Z}/q\mathbb{Z})^\times$ set

$$c_S(\chi) = \sum_{a \in S} \chi_4(a)\chi(a).$$

Then the constant in Theorem 2 may be evaluated from

$$\begin{aligned} \log K(q,S) = 2^{1-m}\mu(S) &\left(2(\log 2 - \gamma) + \sum_{k \geq 1} \frac{1}{k} \sum_{j \geq 1} \frac{\mu_{\mathrm{M}}(j)}{j} \log L(2kj, \chi_0) \right) \\ &+ 2^{1-m} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} c_S(\chi) \sum_{k \geq 1} \frac{1}{k} \sum_{j \geq 1} \frac{\mu_{\mathrm{M}}(j)}{j} \\ &\quad \times \left(\log L(2kj, \chi^j) - 2 \log L(kj, \chi^j) \right). \end{aligned} \tag{1}$$

Here χ^j denotes the pointwise j-th power of χ. In the balanced case $\mu(S) = 0$, the first line of Equation (1) vanishes, and the computation involves only nonprincipal characters in the outer character sum. In practice the sums over j may be restricted to squarefree j, since $\mu_{\mathrm{M}}(j) = 0$ otherwise. For the numerical implementation it is convenient to reorganize the double sum by putting $n = kj$:

$$\sum_{k,j \geq 1} \frac{\mu_{\mathrm{M}}(j)}{kj} F(kj, j) = \sum_{n \geq 1} \frac{1}{n} \sum_{j \mid n} \mu_{\mathrm{M}}(j) F(n, j).$$

In the principal-character part this immediately collapses to the term $n = 1$, because $\sum_{j|n} \mu_{\mathrm{M}}(j) = 0$ for $n > 1$. The computations below were carried out in Python using arbitrary-precision arithmetic from `mpmath` [2, 7]. Dirichlet L-values were evaluated via Hurwitz zeta functions, with the usual digamma formula for $L(1, \chi)$ when χ is nonprincipal. For speed, all characters modulo 2^m were evaluated by using the decomposition

$$(\mathbb{Z}/2^m\mathbb{Z})^\times = \{\pm 1\} \times \langle 5 \rangle.$$

The scripts used for the tables are available from the author's webpage [9]. Languasco's publicly available PARI/GP programs provide an independent reference implementation for the same underlying constants [3]. Table 1 gives the computed values for the family $S_m = \{1, 2^m - 1\}$. Table 2 gives the corresponding stability check.

m	$q = 2^m$	$\log K(q, S_m)$	$K(q, S_m)$
4	16	$\lvert \log K \rvert < 3 \cdot 10^{-56}$	1.0000000000000000000000000000000000000000
5	32	0.055785887828552438941573772577458625844	1.057371263440564119535037000028605726981
6	64	0.000372834967240502990072479904582710305	1.000372904478835417597736161858272147763
7	128	0.008748938687609497210384882738233790907	1.008787322509263329452892922676343582795
8	256	−0.008048465895392386947216571944447322214	0.991983836287144624613877998261566861441
9	512	0.0008587933600776863752417291695811209997	1.000859162228681755059086980611386764540
10	1024	0.0005667492584726388994062662029803796723	1.000566909891178350882758632030919205021

Table 1: Computed values of $\log K(q, S_m)$ and $K(q, S_m)$ for $S_m = \{1, 2^m - 1\} \subset (\mathbb{Z}/2^m\mathbb{Z})^\times$. The computations used 100 decimal digits and $n_{\max} = 60$ in the truncated character sum. All values are rounded to the last displayed decimal place.

The first two exact cases, $K(4, \{1, 3\}) = 2$ and $K(8, \{1, 7\}) = \sqrt{2}$, are treated in Examples 1 and 2. Table 1 begins with the first case not covered by an elementary closed form. The row $m = 4$ resolves the case $q = 16$, $S = \{\pm 1\}$ from Example 3: the raw partial products alone do not determine the constant reliably, whereas the character formula gives $K(16, S) = 1$ to at least 40 decimal places.

m	q	$\log K$ at $n_{\max} = 30$	$\log K$ at $n_{\max} = 60$
4	16	$-5.6599497521 \cdot 10^{-29}$	$-2.5619785141 \cdot 10^{-56}$
5	32	0.055785887828552438941573772577458625844	0.055785887828552438941573772577458625844
6	64	0.000372834967240502990072479904582710305	0.000372834967240502990072479904582710305
7	128	0.008748938687609497210384882738233790907	0.008748938687609497210384882738233790907
8	256	−0.008048465895392386947216571944447322214	−0.008048465895392386947216571944447322214
9	512	0.0008587933600776863752417291695811209997	0.0008587933600776863752417291695811209997
10	1024	0.0005667492584726388994062662029803796723	0.0005667492584726388994062662029803796723

Table 2: Stability check for the character-sum computation of Table 1. The working precision was 100 decimal digits. All displayed values are real; the imaginary parts arising from the complex-valued character computation were below 10^{-102} in absolute value and are omitted.

Appendix B. Comparison with Direct Partial Products

For comparison, Table 3 records direct truncations for the balanced thinnings $p \equiv \pm 1 \pmod{2^m}$. These values should not be used as the primary numerical determination of the constants; their role is only to check qualitative consistency with Table 1. The agreement is visible, but the convergence is much slower than the computation based on Equation (1). For example, the direct value for $q = 32$ at $x = 10^7$ is still only close to the accelerated value $1.05737126344056\ldots$.

x	$q = 8$	$q = 16$	$q = 32$	$q = 64$
10^4	1.4113275632	0.9999436166	1.0579775029	1.0006591649
10^5	1.4130836158	0.9995823760	1.0572678232	1.0001694785
10^6	1.4141098422	1.0000256406	1.0573224354	1.0003905341
10^7	1.4141721971	0.9999838172	1.0573538661	1.0003650135

Table 3: Direct partial products for balanced thinnings $p \equiv \pm 1 \pmod{q}$. The columns report $\prod_{p \le x,\ p \equiv \pm 1\ (q)} A(p)$ for the indicated values of q.

Now consider $q = 16$ and $S = \{3, 11\}$. Both residue classes are congruent to 3 modulo 4, hence

$$\mu(S) = \chi_4(3) + \chi_4(11) = -2,$$

and the predicted exponent is

$$-\frac{2\mu(S)}{\varphi(16)} = \frac{1}{2}.$$

The character formula in Equation (1), evaluated with 100 decimal digits and $n_{\max} = 160$, gives

$$K(16, \{3, 11\}) = 1.4215455276657927384479770851205632758 09\ldots.$$

Thus,

$$P_{\{3,11\}}(x) \sim 1.421545527665792738447977085120563275809\ldots (\log x)^{1/2}.$$

The direct normalized values in Table 4 are consistent with this value.

x	$P_{\{3,11\}}(x)$	$(\log x)^{-1/2} P_{\{3,11\}}(x)$
10^4	4.3149324585	1.4217923138
10^5	4.8235475228	1.4215878898
10^6	5.2838015898	1.4215529197
10^7	5.7070707662	1.4215311910
10^8	6.1011757220	1.4215452732

Table 4: Direct values for the unbalanced example $q = 16$ and $S = \{3, 11\}$.