UNIQUENESS THEOREM OF GENERALIZED MARKOV NUMBERS THAT ARE PRIME POWERS

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Abstract

In this paper, we study positive integer solutions to a generalized form of the Markov equation, given as $x^2 + y^2 + z^2 + k(yz + zx + xy) = (3 + 3k)xyz$, where $k \in \mathbb{Z}_{\geq 0}$. This equation extends the classical Markov equation $x^2 + y^2 + z^2 = 3xyz$. We generalize the concept of Cohn triples for the classical Markov equation to the generalized Markov equations. Using this, we provide a generalization of the uniqueness theorem of Markov numbers that are prime powers.

1. Introduction

1.1. Background

The *Markov equation*,

$$x^2 + y^2 + z^2 = 3xyz,$$

was studied by Markov in the late 1870s in the context of the Diophantine approximation [12, 13]. Every positive integer solution (up to order) of this equation, which is called a *Markov triple*, can be obtained by applying the *Vieta jumpings*

$$(a, b, c) \mapsto \left(a, \frac{a^2 + b^2}{c}, b\right) \quad \text{or} \quad (a, b, c) \mapsto \left(b, \frac{b^2 + c^2}{a}, c\right)$$

to $(1, 1, 1)$ repeatedly. A number appearing in Markov triples is called a *Markov number*. Due to the discoveries of various mathematicians, this combinatorial structure of Markov triples is known today to appear in a variety of theories, including

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combinatorics, continued fractions, hyperbolic geometry and cluster algebra theory (in detail, see [1]). For this equation, the following conjecture was proposed by Frobenius in 1913.

Conjecture 1.1 (The uniqueness conjecture for Markov numbers, [7])**.** *For any Markov number b, there is a unique Markov triple (a, b, c) up to order such that* $\max\{a, b, c\} = b$.

Conjecture 1.1 has been solved affirmatively in special cases, but it has not been fully solved yet. One known approach to Conjecture 1.1 is to use the following criterion.

Theorem 1.2 ([1, Corollary 3.17])**.** *For a Markov number $b \notin \{1, 2\}$, if the number of solutions to $x^2 + 1 \equiv 0 \pmod{b}$ is at most two, then Conjecture 1.1 is true for b.*

By using the criterion in Theorem 1.2, the following theorem is proved.

Theorem 1.3 ([1, Corollary 3.20])**.** *For a Markov number b, if there exists an odd prime p and $m \in \mathbb{Z}_{\geq 1}$ such that $b = p^m$ or $2p^m$, then Conjecture 1.1 is true for b.*

Theorem 1.3 has been (partially or fully) proved by some mathematicians using various methods other than the criterion, for example, Baragar [4], Button [5], Schmutz [15], Zhang [16], and Lang–Tan [10].

Recently, a generalization of the Markov equation endowed with the combinatorial structure of positive integer solutions was given in relation to cluster algebra theory by Banaian [2], the first author [8], and the first author and Matsushita [9]. In this sense of generalization, the most general class currently known is the following equation given by [9]:

$$x^2 + y^2 + z^2 + k_1yz + k_2zx + k_3xy = (3 + k_1 + k_2 + k_3)xyz, \tag{1.1}$$

where $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$. In the papers [3], [9], and [8], the authors attempted to extend the theory of the classical Markov equation to the theory of these generalized equations. In this context, Conjecture 1.1 gives the following question.

Question 1.4. For any number b appearing in a positive integer solution to (1.1), is there a unique positive integer solution (a, b, c) up to order such that $\max\{a, b, c\} = b$?

The answer to this question is *no* if k_1, k_2, k_3 are not the same number, and a counterexample is given by [9]. However, any counterexamples have not been found yet in the case where $k_1 = k_2 = k_3$.

1.2. Main Results

In this paper, we consider Question 1.4 for the case where $k_1 = k_2 = k_3$. The equation

$$x^2 + y^2 + z^2 + k(yz + zx + xy) = (3 + 3k)xyz$$

obtained by substituting $k = k_1 = k_2 = k_3$ to (1.1) is called the *k-generalized Markov equation*. We abbreviate this equation as GME(k). Throughout this paper, unless otherwise stated, we assume $k \in \mathbb{Z}_{\geq 0}$.

As is the classical case, a positive integer solution (up to order) of this equation is called the *k-generalized Markov triple*. According to [9], every k-generalized Markov number can be obtained by applying the *Vieta jumpings*

$$(a, b, c) \mapsto \left(a, \frac{a^2 + kab + b^2}{c}, b\right) \quad \text{or} \quad (a, b, c) \mapsto \left(b, \frac{b^2 + kbc + c^2}{a}, c\right)$$

to $(1, 1, 1)$ repeatedly. A number appearing in k-generalized Markov triples is called a *k-generalized Markov number*.

One of the main results in this paper is a generalization of Theorem 1.2.

Theorem 1.5. *Let $k \in \mathbb{Z}_{\geq 0}$. For a k-generalized Markov number $b \notin \{1, k+2\}$, if the number of solutions to $x^2 + kx + 1 \equiv 0 \pmod{b}$ is at most two, then there is a unique k-generalized Markov triple (a, b, c) up to order such that $\max\{a, b, c\} = b$.*

By using the criterion in Theorem 1.5, we will prove the following theorem.

Theorem 1.6. *Let $k \in \mathbb{Z}_{\geq 0}$. For a k-generalized Markov number b, if $b = p$ or $2p$, where p is a prime, then there is a unique k-generalized Markov triple (a, b, c) up to order such that $\max\{a, b, c\} = b$.*

Theorem 1.6 is the k-generalized version of Theorem 1.3 for the case where $m = 1$. For the case where $m > 1$, we can extend Theorem 1.3 to the following form.

Theorem 1.7. *Let $k \in \mathbb{Z}_{\geq 0}$. For a k-generalized Markov number b, suppose that there exists a prime p and $m \in \mathbb{Z}_{\geq 2}$ such that $b = p^m$ or $2p^m$. Assume one of the following three conditions about k is satisfied:*

(1) $k = 2$,

(2) $k \geq 4$ is even, and both $\dfrac{k}{2} + 1$ and $\dfrac{k}{2} - 1$ are not divided by p^2, and

(3) k is odd, and both $k + 2$ and $k - 2$ are not divided by p^2.

Then there is a unique k-generalized Markov triple (a, b, c) up to order such that $\max\{a, b, c\} = b$.

We prove Theorem 1.7 in Section 4.

Of the non-negative integers less than 100,

$$\begin{aligned} k = {} & 1, 2, 3, 4, 5, 8, 9, 12, 13, 15, 17, 19, 21, 24, 28, 31, 32, 33, \\ & 35, 36, 37, 39, 40, 41, 44, 45, 49, 53, 55, 57, 59, 60, 63, 64, \\ & 67, 68, 69, 71, 72, 75, 76, 80, 81, 84, 85, 87, 89, 91, 93, 95, 99 \end{aligned}$$

satisfy the assumption of Theorem 1.7 for every p. From the above theorems, it seems that Question 1.4 is a problem worth considering when $k = k_1 = k_2 = k_3$.

Conjecture 1.8. *Let $k \in \mathbb{Z}_{\geq 0}$. For any k-generalized Markov number b, there is a unique k-generalized Markov triple (a, b, c) up to order such that $\max\{a, b, c\} = b$.*

1.3. Generalization of Cohn Triples

In 1950s, the *(classical) Cohn triple* was introduced by Cohn [6] in his study of the Markov equation. In the present paper, we will introduce a generalization of it. It plays an important role in the proof of Theorems 1.5–1.7. A classical Cohn triple is a triple

$$(P, Q, R) = \left(\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \right)$$

of elements of $SL(2, \mathbb{Z})$ which satisfies

- (p_{12}, q_{12}, r_{12}) is a Markov triple,
- $Q = PR$,
- $(\mathrm{tr}(P), \mathrm{tr}(Q), \mathrm{tr}(R)) = (3p_{12}, 3q_{12}, 3r_{12})$.

The Cohn triple is a "matrixization" of the Markov triple. Indeed, the following theorem holds.

Theorem 1.9 ([1, Theorem 4.7])**.** *Every Markov triple (a, b, c) with $b \geq \max\{a, c\}$ can be obtained by applying*

$$(P, Q, R) \mapsto (P, PQ, Q) \quad \text{or} \quad (P, Q, R) \mapsto (Q, QR, R) \tag{1.2}$$

successively to a Cohn triple with $(p_{12}, q_{12}, r_{12}) = (1, 1, 1)$. Moreover, the transformations of $(1, 2)$-entries in (1.2) coincide with the Vieta jumpings of the Markov triples.

We will explain how to generalize the Cohn triples to triples which are compatible with the k-generalized Markov equation. It is known that a triple (p_{12}, q_{12}, r_{12}) is a Markov triple if and only if $(3p_{12}, 3q_{12}, 3r_{12})$ is a positive integer solution to the *second Markov equation*,

$$x^2 + y^2 + z^2 = xyz$$

([1, Chapter 2]). We introduce the *k-generalized second Markov equation*,

$$x^2 + y^2 + z^2 + (k^2 + 2k)(x + y + z) + 2k^3 + 3k^2 = xyz.$$

We abbreviate the equation as GSME(k). If a triple (p_{12}, q_{12}, r_{12}) is a k-generalized Markov triple, then

$$((3 + 3k)p_{12} - k, (3 + 3k)q_{12} - k, (3 + 3k)r_{12} - k)$$

is a positive integer solution to GSME(k) by a straightforward calculation. Based on this, we define the *k-generalized Cohn triple* as a triple (P, Q, R) which satisfies

- (p_{12}, q_{12}, r_{12}) is a k-generalized Markov triple,
- $Q = PR - S$, where $S = \begin{bmatrix} k & 0 \\ 3k^2 + 3k & k \end{bmatrix}$,
- $(\mathrm{tr}(P), \mathrm{tr}(Q), \mathrm{tr}(R)) = ((3+3k)p_{12} - k, (3+3k)q_{12} - k, (3+3k)r_{12} - k)$,

and we prove the following theorem.

Theorem 1.10. *Every k-generalized Markov triple (a, b, c) with $b \geq \max\{a, c\}$ is obtained by applying*

$$(P, Q, R) \mapsto (P, PQ - S, Q) \quad \text{or} \quad (P, Q, R) \mapsto (Q, QR - S, R) \tag{1.3}$$

successively to a k-generalized Cohn triple with $(p_{12}, q_{12}, r_{12}) = (1, 1, 1)$. Moreover, the transformations of $(1, 2)$-entries in (1.3) coincide with the Vieta jumpings of the k-generalized Markov triples.

The generalized second Markov equation is derived from the following identity for any matrices A, B, C in $SL(2, \mathbb{C})$, which have been found by Luo [11] and Näätänen–Nakanishi [14]. For $a := -\mathrm{tr}(A)$, $b := -\mathrm{tr}(B)$, $c := -\mathrm{tr}(C)$, $d := -\mathrm{tr}(ABC)$, $x := -\mathrm{tr}(AB)$, $y := -\mathrm{tr}(BC)$, and $z := -\mathrm{tr}(CA)$, the equality

$$\begin{aligned} x^2 + y^2 + z^2 + (ad + bc)x + (bd + ca)y + (cd + ab)z \\ + a^2 + b^2 + c^2 + d^2 + abcd - 4 = xyz \end{aligned}$$

holds. For triples (A, B, C) and (P, Q, R) which satisfy

- $\mathrm{tr}(A) = \mathrm{tr}(B) = \mathrm{tr}(C) = -k$,
- $\mathrm{tr}(ABC) = -2$,
- $\mathrm{tr}(P) = -\mathrm{tr}(BC),\ \mathrm{tr}(Q) = -\mathrm{tr}(CA),\ \mathrm{tr}(R) = -\mathrm{tr}(AB)$,

$(\mathrm{tr}(P), \mathrm{tr}(Q), \mathrm{tr}(R))$ is a solution to the k-generalized second Markov equation. It turns out that a triple (A, B, C) of $SL(2, \mathbb{Z})$ satisfying the above conditions can be taken for each k-generalized Cohn triple (P, Q, R). Relations between this triple (A, B, C) and the k-generalized Cohn triple (P, Q, R) will be studied in a forthcoming paper.

In order to prove Theorem 1.2, we will make use of the following result from [1].

Proposition 1.11 ([1, Proposition 3.15])**.** *Let $m \in M$, $m \geq 5$, and suppose (m, m_2, m_3) and (m, n_2, n_3) are Markov triples with maximum m and corresponding characteristic numbers u_1 and u_2. If $u_1 = u_2$, then $\{m_2, m_3\} = \{n_2, n_3\}$; hence the triples are identical.*

Here, the *characteristic number* u_1 of the triple (m, m_2, m_3) is defined as the unique solution x to

$$\begin{cases} m_2 x \equiv \pm m_3 \pmod{m}, \\ 0 < x < m. \end{cases}$$

In [1], this proposition is proved using elementary arguments. When we prove Theorem 1.5, which is a generalization of Theorem 1.2, a k-generalized version of Proposition 1.11 is also used. However, in our case, the proof is carried out using the method based on generalized Cohn triples, instead of elementary arguments. See the last two sentences of the proof of Theorem 1.5 in Section 4. The reason why we use generalized Cohn triples instead of elementary arguments is that the proof of [1, Proposition 3.15] does not appear to extend easily to the case $k \neq 0$. We use the condition $k = 0$ essentially when we verify the equality in [1, (3.13)], that is,

$$m^2 \mid m_2 n_3 - m_3 n_2. \tag{1.4}$$

Indeed, assuming (m, m_2, m_3) and (m, n_2, n_3) are both k-generalized Markov triples, we have

$$\frac{m^2 + m_2^2 + m_3^2 + k(mm_2 + m_2m_3 + m_3m)}{m^2 + n_2^2 + n_3^2 + k(mn_2 + n_2n_3 + n_3m)} = \frac{m_2m_3}{n_2n_3},$$

and thus we have

$$\begin{aligned} &(m_2m_3 - n_2n_3)m + k(m_2m_3(n_2 + n_3) - n_1n_2(m_1 + m_2)) \\ &\qquad = \frac{m_2n_3 - m_3n_2}{m}(m_2n_2 - m_3n_3). \end{aligned}$$

Moreover, this implies

$$k(m_2m_3(n_2 + n_3) - n_1n_2(m_1 + m_2)) \equiv \frac{m_2n_3 - m_3n_2}{m}(m_2n_2 - m_3n_3) \pmod{m}.$$

Assuming $m \mid m_2n_3 - m_3n_2$, as in the proof of [1, Proposition 3.15], we get the fact that $\dfrac{m_2n_3 - m_3n_2}{m}$ is in $\mathbb{Z}$. Moreover, assuming that $m_2n_2 \neq m_3n_3$ as in the proof of [1, Proposition 3.15], and substituting $k = 0$, we have

$$0 \equiv \frac{m_2n_3 - m_3n_2}{m} \pmod{m},$$

and this is equivalent to (1.4). If we assume that k is not zero, we cannot proceed to (1.4); therefore, we need to find a form of generalization of (1.4), but so far we have not achieved it.

2. Generalized Markov Equation

We consider the following binary tree, $\mathrm{WMT}(k)$:

(1) the root vertex is $(1,1,1)$,

(2) every vertex (a,b,c) has the following two children:

$$\begin{array}{ccc} & (a,b,c) & \\ \swarrow & & \searrow \\ \left(a, \dfrac{a^2+kab+b^2}{c}, b\right) & & \left(b, \dfrac{b^2+kbc+c^2}{a}, c\right). \end{array}$$

The tree $\mathrm{WMT}(k)$ is called the *wide k-generalized Markov tree*. The transformations

$$(a,b,c) \mapsto \left(a, \frac{a^2+kab+b^2}{c}, b\right) \quad \text{and} \quad (a,b,c) \mapsto \left(b, \frac{b^2+kbc+c^2}{a}, c\right)$$

are called the *left Vieta jumping* and the *right Vieta jumping*, respectively. Note that for every $(a,b,c) \in \mathrm{WMT}(k)$, b is the unique maximal number in a, b and c if (a,b,c) is not the root vertex. The following proposition is the special case of [9, Theorem 1].

Proposition 2.1. *All vertices in* $\mathrm{WMT}(k)$ *are k-generalized Markov triples, and each k-generalized Markov triple* (a,b,c) *with* $b > \max\{a,c\}$ *appears exactly twice in* $\mathrm{WMT}(k)$*. Moreover, one of them appears in the left descendants of the root, and the other does in the right descendants of the root.*

Since there are vertices in $\mathrm{WMT}(k)$ assigned to the same triple, we take the full subtree with the left child of $\mathrm{WMT}(k)$ as the root to resolve this. This tree is called the *k-generalized Markov tree* and is denoted by $\mathrm{MT}(k)$. By Proposition 2.1, we have the following corollary.

Corollary 2.2. *All vertices in* $\mathrm{MT}(k)$ *are k-generalized Markov triples, and each k-generalized Markov triple* (a,b,c) *with* $b > \max\{a,c\}$ *appears exactly once in* $\mathrm{MT}(k)$*.*

By using the k-generalized Markov equation, we have

$$\begin{aligned} \frac{a^2+kab+b^2}{c} &= (3+3k)ab - c - k(a+b), \\ \frac{b^2+kbc+c^2}{a} &= (3+3k)bc - a - k(b+c), \end{aligned}$$

and it is another presentation of the transformed elements.

The following is a basic property of the k-generalized Markov triple.

Proposition 2.3 ([9, Corollary 8])**.** *For every k-generalized Markov triple (a, b, c), all pairs in a, b and c are relatively prime.*

We discuss the relation between GME(k) and GSME(k). By a straightforward calculation, we have the following proposition.

Proposition 2.4. *A triple (a, b, c) is one of the rational solutions to* GME(k) *if and only if the triple*

$$((3+3k)a-k, (3+3k)b-k, (3+3k)c-k)$$

is one of the rational solutions to GSME(k)*.*

By Proposition 2.4, if (a, b, c) is a positive integer solution to GME(k), then $((3+3k)a-k, (3+3k)b-k, (3+3k)c-k)$ is a positive integer solution to GSME(k). In the case where $k = 0$, the converse holds ([1, Proposition 2.2]), but in general, this does not hold.

Example 2.5. We set $k = 4$. Then $(9, 9, 22)$ is a positive integer solution to GSME(4), but the corresponding solution $\left(\frac{13}{15}, \frac{13}{15}, \frac{26}{15}\right)$ to GME(4) is not an integer triple.

In parallel with GME(k), there is an algorithm to obtain a new positive integer solution to GSME(k) from a given solution.

Proposition 2.6. *Let (a, b, c) be an integer solution to* GSME(k)*. Then*

$$(a, ab-c-k^2-2k, b) \quad \textit{and} \quad (b, bc-a-k^2-2k, c)$$

are also integer solutions to GSME(k)*.*

Proof. We only prove that $(a, ab-c-k^2-2k, b)$ is an integer solution. By Proposition 2.4, $\left(\frac{a+k}{3+3k}, \frac{b+k}{3+3k}, \frac{c+k}{3+3k}\right)$ is a rational solution to GME(k). Applying the left Vieta jumping of GME(k), we have a rational solution to GME(k), $\left(\frac{a+k}{3+3k}, \frac{ab-c-k^2-k}{3+3k}, \frac{b+k}{3+3k}\right)$. By Proposition 2.4 again, $(a, ab-c-k^2-2k, b)$ is a rational solution to GSME(k). Clearly, it is an integer solution. □

A positive integer solution (a, b, c) to GSME(k) is called an *induced (positive) solution (from* GME(k)*)* if $\left(\frac{a+k}{3+3k}, \frac{b+k}{3+3k}, \frac{c+k}{3+3k}\right)$ is a positive integer triple.

Proposition 2.7. *Let (a, b, c) be an induced solution to* GSME(k)*. Then*

$$(a, ab-c-k^2-2k, b) \quad \textit{and} \quad (b, bc-a-k^2-2k, c)$$

are also induced solutions to GSME(k)*.*

Proof. We only prove the case where $(a, ab-c-k^2-2k, b)$. It suffices to show that $ab-c-k^2-2k$ is positive. Since $\dfrac{ab-c-k^2-k}{3+3k} \geq 1$, we have $ab-c-k^2-k \geq 3+3k$. Therefore, we have $ab-c-k^2-2k \geq 3+2k \geq 1$. □

In parallel with GME(k), the transformations

$$(a,b,c) \mapsto (a, ab-c-k^2-2k, b), \quad (a,b,c) \mapsto (b, bc-a-k^2-2k, c)$$

are called the *left Vieta jumping* and the *right Vieta jumping*, respectively.

We denote by $\mathrm{SM}\mathbb{T}(k)$ the tree obtained from $\mathrm{M}\mathbb{T}(k)$ by replacing (a,b,c) with $((3+3k)a-k, (3+3k)b-k, (3+3k)c-k)$. From the above discussion, we have the following corollary.

Corollary 2.8. *All vertices in* $\mathrm{SM}\mathbb{T}(k)$ *are induced solutions to* GSME(k)*, and each induced solution* (a,b,c) *to* GSME(k) *with* $b > \max\{a,c\}$ *appears exactly once in* $\mathrm{SM}\mathbb{T}(k)$*.*

3. Generalized Cohn Tree

3.1. k-Generalized Cohn Tree

We begin by defining the k-generalized Cohn matrix.

Definition 3.1. Let $k \in \mathbb{Z}_{\geq 0}$. An element $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ of $SL(2,\mathbb{Z})$ is called a *k-generalized Cohn matrix* if p_{12} is a k-generalized Markov number and $\mathrm{tr}(P) = (3+3k)p_{12} - k$.

The trace condition has the following presentation:

$$\mathrm{tr}(P) = \begin{bmatrix} 3+3k & 0 \end{bmatrix} P \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k.$$

We recall the definition of a k-generalized Cohn triple.

Definition 3.2. For $k \in \mathbb{Z}_{\geq 0}$, a triple (P,Q,R) is called a *k-generalized Cohn triple* if P, Q, and R are k-generalized Cohn matrices, $Q = PR - S$ where $S = \begin{bmatrix} k & 0 \\ 3k^2+3k & k \end{bmatrix}$, and (p_{12}, q_{12}, r_{12}) is a k-generalized Markov triple, where p_{12}, q_{12}, and r_{12} are the $(1,2)$-entries of P, Q, and R, respectively. The triple (P,Q,R) is said to be *associated with* (p_{12}, q_{12}, r_{12}).

Remark 3.3. Let (P,Q,R) be a k-generalized Cohn triple associated with (a,b,c). By the definition of k-generalized Cohn triple, $(\mathrm{tr}(P), \mathrm{tr}(Q), \mathrm{tr}(R))$ is an induced solution to GSME(k) corresponding to (a,b,c).

The definition of k-generalized Cohn matrix does not refer to its existence. We will prove that for a positive integer solution (a, b, c) to $\mathrm{GME}(k)$ with $b \geq \max\{a, c\}$, there exists a k-generalized Cohn triple (P, Q, R) such that $a = p_{12}, b = q_{12}$ and $c = r_{12}$.

First, we prove the case where $(a, b, c) = (1, 1, 1)$. For every $\ell \in \mathbb{Z}$, we set

$$P_{1;\ell} = \begin{bmatrix} \ell & 1 \\ -\ell^2 + 2k\ell + 3\ell - 1 & -\ell + 2k + 3 \end{bmatrix},$$
$$Q_{1;\ell} = \begin{bmatrix} k + \ell + 1 & 1 \\ k^2 - \ell^2 + 3k + \ell + 1 & k - \ell + 2 \end{bmatrix}, \text{ and}$$
$$R_{1;\ell} = \begin{bmatrix} 2k + \ell + 2 & 1 \\ -\ell^2 - 2k\ell + 2k - \ell + 1 & -\ell + 1 \end{bmatrix}.$$

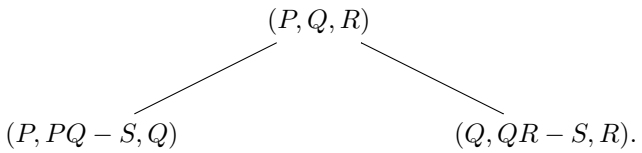
Proposition 3.4. *For every $\ell \in \mathbb{Z}$, the triple $(P_{1;\ell}, Q_{1;\ell}, R_{1;\ell})$ is a k-generalized Cohn triple. Conversely, for a k-generalized Cohn triple (P, Q, R) satisfying $(p_{12}, q_{12}, r_{12}) = (1, 1, 1)$, there exists $\ell \in \mathbb{Z}$ such that $(P, Q, R) = (P_{1;\ell}, Q_{1;\ell}, R_{1;\ell})$.*

Proof. The former statement can be checked directly. Since (P, Q, R) is determined by p_{11} and the defining conditions of the k-generalized Cohn triple, the latter statement follows (see also the proof of [1, Theorem 4.8], which is the special case $k = 0$ of this proposition). □

For a k-generalized Markov triple (a, b, c) with $b > \max\{a, c\}$, we prove the existence of a k-generalized Cohn triple corresponding to (a, b, c) by using induction on the distance from $(1, 1, 1)$ in $\mathrm{WMT}(k)$. To this end, we define another binary tree, the wide *k-generalized Cohn tree* $\mathrm{WGCT}(k, \ell)$ for $\ell \in \mathbb{Z}$ as follows:

(1) the root vertex is $(P_{1;\ell}, Q_{1;\ell}, R_{1;\ell})$, and

(2) every vertex (P, Q, R) has the following two children:



We prove the following theorem.

Theorem 3.5. *Let (a, b, c) be a k-generalized Markov triple. If (P, Q, R) is a k-generalized Cohn triple associated with (a, b, c), then $(P, PQ-S, Q)$ (resp. $(Q, QR-S, R)$) is a k-generalized Cohn triple associated with (a, c', b) (resp. (b, a', c)), where $c' = \dfrac{a^2 + kab + b^2}{c}$ and $a' = \dfrac{b^2 + kbc + c^2}{a}$.*

To prove Theorem 3.5, we introduce two lemmas.

Lemma 3.6 ([1, Lemma 4.2])**.** *Let A and $B \in SL(2,\mathbb{Z})$. The following equalities hold:*

(1) $\mathrm{tr}(A) = \mathrm{tr}(A^{-1})$,

(2) $\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB^{-1})$, *and*

(3) $A^2 = \mathrm{tr}(A)A - I$.

Lemma 3.7. *For a k-generalized Cohn matrix M, the following equalities hold:*

$$M \begin{bmatrix} 0 & 0 \\ 3+3k & 0 \end{bmatrix} M = (\mathrm{tr}(M)+k)M + \begin{bmatrix} 0 & 0 \\ 3+3k & 0 \end{bmatrix},$$
$$M^{-1} \begin{bmatrix} 0 & 0 \\ 3+3k & 0 \end{bmatrix} M^{-1} = -(\mathrm{tr}(M^{-1})+k)M^{-1} + \begin{bmatrix} 0 & 0 \\ 3+3k & 0 \end{bmatrix}.$$

Proof. We only prove the first equality. We set

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ 3+3k & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 3+3k & 0 \end{bmatrix}, \quad w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then we have

$$\begin{aligned} MEM = MwvM &= \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} \begin{bmatrix} (3+3k)m_{11} & (3+3k)m_{12} \end{bmatrix} \\ &= (3+3k) \begin{bmatrix} m_{11}m_{12} & m_{12}^2 \\ m_{11}m_{22} & m_{12}m_{22} \end{bmatrix} \\ &= (3+3k)m_{12}M + E \\ &= (\mathrm{tr}M + k)M + E. \end{aligned}$$

Note that $m_{11}m_{22} - m_{21}m_{12} = 1$ is used to show that the fourth equal sign is valid. □

Proof of Theorem 3.5. We only prove the statement for $(P, PQ-S, Q)$. First, we prove that the transformation $(\mathrm{tr}(P), \mathrm{tr}(Q), \mathrm{tr}(R)) \mapsto (\mathrm{tr}(P), \mathrm{tr}(PQ-S), \mathrm{tr}(Q))$ is the left Vieta jumping $(\alpha, \beta, \gamma) \mapsto (\alpha, \alpha\beta - \gamma - k^2 - 2k, \beta)$ of GSME(k). Since (P, Q, R) is a k-generalized Cohn triple, it suffices to show

$$\mathrm{tr}(PQ-S) = \mathrm{tr}(P)\mathrm{tr}(Q) - \mathrm{tr}(R) - k^2 - 2k.$$

Since $Q = PR - S$, making use of Lemma 3.6 (2), we have

$$\begin{aligned} \mathrm{tr}(PQ-S) &= \mathrm{tr}(P(PR-S)) - 2k = \mathrm{tr}(P^2R) - \mathrm{tr}(PS) - 2k \\ &= \mathrm{tr}(P)\mathrm{tr}(PR) - \mathrm{tr}(PR^{-1}P^{-1}) - \mathrm{tr}(PS) - 2k \\ &= \mathrm{tr}(P)\mathrm{tr}(PR) - \mathrm{tr}(R) - \mathrm{tr}(PS) - 2k. \end{aligned}$$

On the other hand, since

$$\begin{aligned}\mathrm{tr}(SP^{-1}) &= \mathrm{tr}\left(\begin{bmatrix} p_{22}k & * \\ * & p_{11}k - p_{12}(3k^2+3k)\end{bmatrix}\right)\\ &= k\cdot\mathrm{tr}(P) - (3k^2+3k)p_{12} = -k^2,\end{aligned}$$

making use of Lemma 3.6 (2), we have

$$\begin{aligned}\mathrm{tr}(P)\mathrm{tr}(Q) - \mathrm{tr}(R) - k^2 - 2k &= \mathrm{tr}(P)\mathrm{tr}(PR) - \mathrm{tr}(P)\mathrm{tr}(S)\\ &\quad - \mathrm{tr}(R) - k^2 - 2k\\ &= \mathrm{tr}(P)\mathrm{tr}(PR) - \mathrm{tr}(SP)\\ &\quad - \mathrm{tr}(SP^{-1}) - \mathrm{tr}(R) - k^2 - 2k\\ &= \mathrm{tr}(P)\mathrm{tr}(PR) - \mathrm{tr}(PS) - \mathrm{tr}(R) - 2k.\end{aligned}$$

Therefore, we have the desired equality. Next, we prove that $\mathrm{tr}(PQ-S) = (3+3k)m_{12} - k$, where m_{12} is the $(1,2)$-entry of $PQ-S$. Set $v = \begin{bmatrix}3+3k & 0\end{bmatrix}$, $w = \begin{bmatrix}0\\1\end{bmatrix}$, and $E = \begin{bmatrix}0 & 0\\ 3+3k & 0\end{bmatrix}$. By the above, and making use of Lemmas 3.7 and 3.6 (3), we have

$$\begin{aligned}\mathrm{tr}(PQ-S) &= \mathrm{tr}(P)\mathrm{tr}(Q) - \mathrm{tr}(R) - k^2 - 2k\\ &= (vPw-k)(vQw-k) - (vRw-k) - k^2 - 2k\\ &= vPEPRw - kvPRw - kvPw - vRw - k\\ &= v\big((\mathrm{tr}P + k)P + E\big)Rw - kvPRw - kvPw - vRw - k\\ &= vP^2Rw - kvPw - k\\ &= v(P^2R - PS - S)w - k\\ &= v(PQ-S)w - k.\end{aligned}$$

Finally, we prove $PQ - S \in SL(2,\mathbb{Z})$. We set $PQ = \begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}$. By the above discussion, we have $\mathrm{tr}(PQ-S) = (3+3k)x_{12} - k$, and thus $\mathrm{tr}(PQ) = (3+3k)x_{12} + k$. Therefore, we have

$$\begin{aligned}\det(PQ-S) &= (x_{11}-k)(x_{22}-k) - x_{12}(x_{21} - 3k^2 - 3k)\\ &= \det(PQ) - k\cdot\mathrm{tr}(PQ) + k^2 + (3k^2+3k)x_{12}\\ &= \det(PQ) = 1.\end{aligned}$$

This finishes the proof. □

As a corollary, for a k-generalized Markov triple (a,b,c) with $b > \max\{a,c\}$, we obtain the existence of a k-generalized Cohn triple associated with (a,b,c). Before describing the statement, we will introduce the *canonical graph isomorphism* between two trees.

Definition 3.8. Let $\mathbb{T}$ and $\mathbb{T}'$ be rooted binary trees such that every vertex has either exactly two children or no child. Such a tree is called a *full binary tree*. If a graph isomorphism $f\colon \mathbb{T} \to \mathbb{T}'$ preserves the left child and the right child, then f is called *the canonical graph isomorphism*.

Corollary 3.9. *Let $\ell \in \mathbb{Z}$. The correspondence between (P, Q, R) in $\mathrm{WGC}\mathbb{T}(k, \ell)$ and (p_{12}, q_{12}, r_{12}) induces the canonical graph isomorphism between $\mathrm{WGC}\mathbb{T}(k, \ell)$ and $\mathrm{WM}\mathbb{T}(k)$. In particular, for every k-generalized Markov triple with $b \geq \max\{a, c\}$, there is a k-generalized Cohn matrix associated with (a, b, c).*

Moreover, a stronger result can be obtained.

Proposition 3.10. *Let (P, Q, R) be a k-generalized Cohn triple associated with (a, b, c). We assume that $b \geq \max\{a, c\}$. Then, there exists $\ell \in \mathbb{Z}$ such that (P, Q, R) is contained in $\mathrm{WGC}\mathbb{T}(k, \ell)$.*

To prove Proposition 3.10, we introduce the transformations from one vertex in $\mathrm{WGC}\mathbb{T}(k, \ell)$, except for the root, to its parent.

Lemma 3.11. *If (P, Q, R) is a k-generalized Cohn triple associated with (a, b, c), then $(P, R, P^{-1}(R + S))$ (resp. $((P + S)R^{-1}, P, R)$) is a k-generalized Cohn triple associated with (a, c, b') (resp. (b', a, c)), where $b' = \dfrac{a^2 + kac + c^2}{b}$.*

Proof. We only prove the statement for $(P, R, P^{-1}(R+S))$. First, we prove that the transformation $(\mathrm{tr}(P), \mathrm{tr}(Q), \mathrm{tr}(R)) \mapsto (\mathrm{tr}(P), \mathrm{tr}(R), \mathrm{tr}(P^{-1}(R + S)))$ is the inverse of the left Vieta jumping $(\alpha, \beta, \gamma) \mapsto (\alpha, \gamma, \alpha\gamma - \beta - k^2 - 2k)$ of $\mathrm{GSME}(k)$. Since (P, Q, R) is a k-generalized Cohn matrix, it suffices to show

$$\mathrm{tr}(P^{-1}(R + S)) = \mathrm{tr}(P)\mathrm{tr}(R) - \mathrm{tr}(Q) - k^2 - 2k.$$

Since $Q = PR - S$, making use of Lemma 3.6 (2), we have

$$\mathrm{tr}(P)\mathrm{tr}(R) - \mathrm{tr}(Q) - k^2 - 2k = \mathrm{tr}(P)\mathrm{tr}(R) - \mathrm{tr}(PR) - k^2.$$

On the other hand, since $\mathrm{tr}(P^{-1}S) = -k^2$ (see the proof of Theorem 3.5), we have

$$\mathrm{tr}(P^{-1}(R + S)) = \mathrm{tr}(P^{-1}R) + \mathrm{tr}(P^{-1}S) = \mathrm{tr}(P)\mathrm{tr}(R) - \mathrm{tr}(PR) - k^2.$$

Therefore, we have the desired equality. Next, we prove that $\mathrm{tr}(P^{-1}(R + S)) = (3 + 3k)m_{12} - k$, where m_{12} is the $(1, 2)$-entry of $P^{-1}(R + S)$. By the above, we

have

$$\begin{aligned}
\mathrm{tr}(P^{-1}(R+S)) =& \mathrm{tr}(P)\mathrm{tr}(R) - \mathrm{tr}(Q) - k^2 - 2k \\
=& \left(- \begin{bmatrix} 3+3k & 0 \end{bmatrix} P^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k \right) \left(\begin{bmatrix} 3+3k & 0 \end{bmatrix} R \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k \right) \\
& - \left(\begin{bmatrix} 3+3k & 0 \end{bmatrix} Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k \right) - k^2 - 2k \\
=& - \begin{bmatrix} 3+3k & 0 \end{bmatrix} P^{-1} \begin{bmatrix} 0 & 0 \\ 3+3k & 0 \end{bmatrix} P^{-1} Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
& - \begin{bmatrix} 3+3k & 0 \end{bmatrix} P^{-1} \begin{bmatrix} 0 & 0 \\ 3+3k & 0 \end{bmatrix} P^{-1} S \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
& - k \begin{bmatrix} 3+3k & 0 \end{bmatrix} P^{-1} Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k \begin{bmatrix} 3+3k & 0 \end{bmatrix} P^{-1} S \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
& + k \begin{bmatrix} 3+3k & 0 \end{bmatrix} P^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 3+3k & 0 \end{bmatrix} Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k \\
=& \begin{bmatrix} 3+3k & 0 \end{bmatrix} \left((\mathrm{tr}P^{-1} + k)P^{-1} - \begin{bmatrix} 0 & 0 \\ 3+3k & 0 \end{bmatrix} \right) Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
& + \begin{bmatrix} 3+3k & 0 \end{bmatrix} \left((\mathrm{tr}P^{-1} + k)P^{-1} - \begin{bmatrix} 0 & 0 \\ 3+3k & 0 \end{bmatrix} \right) S \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
& - k \begin{bmatrix} 3+3k & 0 \end{bmatrix} P^{-1} Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k \begin{bmatrix} 3+3k & 0 \end{bmatrix} P^{-1} S \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
& + k \begin{bmatrix} 3+3k & 0 \end{bmatrix} P^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 3+3k & 0 \end{bmatrix} Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k \\
=& \begin{bmatrix} 3+3k & 0 \end{bmatrix} (P^{-2}Q + P^{-2}S + P^{-1}S) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k \\
=& \begin{bmatrix} 3+3k & 0 \end{bmatrix} (P^{-1}(R+S)) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k.
\end{aligned}$$

Note that in the second equality from the last, we use the fact that $S \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix}$.

Finally, we prove $P^{-1}(R+S) \in SL(2,\mathbb{Z})$. By setting $R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$, we have

$$\begin{aligned}
\det(R+S) &= (r_{11}+k)(r_{22}+k) - r_{12}(r_{21}+3k^2+3k) \\
&= \det(R) + k \cdot \mathrm{tr}(R) + k^2 - (3k^2+3k)r_{12} \\
&= \det(R) = 1.
\end{aligned}$$

This finishes the proof. □

Proof of Proposition 3.10. We consider the transformations

$$(P,Q,R) \mapsto (P,R,P^{-1}(R+S)) \quad \text{and} \quad (P,Q,R) \mapsto ((P+S)R^{-1},P,R).$$

We assume that (P, Q, R) is associated with (a, b, c). If $a \leq c$ (respectively, $a \geq c$), then by using the first transformation (respectively, the second transformation), we obtain another k-generalized Cohn triple (P', Q', R') by Lemma 3.11. Let a', b', and c' be the $(1,2)$-entries of P', Q', and R', respectively. Then, we have $b' < b$. By repeating this operation, we obtain a k-generalized Cohn triple (P'', Q'', R'') associated with $(1,1,1)$. According to Proposition 3.4, there exists $\ell \in \mathbb{Z}$ such that $P'' = P_{1;\ell}$, $Q'' = Q_{1;\ell}$, and $R'' = R_{1;\ell}$. This implies that (P, Q, R) is contained in $\mathrm{WGC}\mathbb{T}(k, \ell)$. □

Remark 3.12. The integer ℓ in Proposition 3.10 is not unique if $(a, b, c) \neq (1, 1, 1)$. Since there are two operations to get the k-generalized Cohn triple associated with $(1, 1, 1)$ from one associated with $(1, k + 2, 1)$, there are just two ℓ's. Therefore, (P, Q, R) appears in just two wide k-generalized Cohn trees, one in the left half of one tree and the other in the right half of the other tree.

The discussion thus far proves Theorem 1.10.

Proof of Theorem 1.10. The statement follows from Theorem 3.5 and Proposition 3.10. □

Next, we prove the following theorem.

Theorem 3.13. *Let $k \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}$. The second entries of k-generalized Cohn triples in $\mathrm{WGC}\mathbb{T}(k, \ell)$ are distinct.*

To prove Theorem 3.13, we introduce the index of a k-generalized Cohn matrix.

Definition 3.14. For a k-generalized Cohn matrix $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$, the *index* of M is the value $I_M := \dfrac{m_{11}}{m_{12}}$.

Proof of Theorem 3.13. For each (P, Q, R), it is enough to show that $I_P < I_Q < I_R$. First, we prove $I_Q < I_R$. Since $Q = PR - S$ holds, we have $P = (Q + S)R^{-1}$. By comparing the $(1,2)$-entries of this equality, we have

$$p_{12} = r_{11}q_{12} - (q_{11} + k)r_{12} \leq r_{11}q_{12} - q_{11}r_{12}.$$

Therefore, we have

$$0 < \frac{p_{12}}{q_{12}r_{12}} \leq \frac{r_{11}}{r_{12}} - \frac{q_{11}}{q_{12}} = I_R - I_Q,$$

and $I_Q < I_R$ holds as desired. Next, we prove $I_P < I_Q$. We obtain $R = P^{-1}(Q+S)$ from $Q = PR - S$. By comparing the $(1,2)$-entries of this equality, we have

$$\begin{aligned} r_{12} &= p_{22}q_{12} - p_{12}(q_{22} + k) \\ &= ((3 + 3k)p_{12} - k - p_{11})q_{12} - p_{12}((3 + 3k)q_{12} - q_{11}) \\ &= -kq_{12} + q_{11}p_{12} - q_{12}p_{11} \leq q_{11}p_{12} - q_{12}p_{11}. \end{aligned}$$

Therefore, we have

$$0 < \frac{r_{12}}{p_{12}q_{12}} \leq \frac{q_{11}}{q_{12}} - \frac{p_{11}}{p_{12}} = I_Q - I_P,$$

and $I_P < I_Q$ holds as desired. □

We fix $k \in \mathbb{Z}_{\geq 0}$. We denote by $\mathrm{GC}\mathbb{T}(k,\ell)$ the full subtree of $\mathrm{WGC}\mathbb{T}(k,\ell)$ whose root is the left child of $(P_{1;\ell}, Q_{1;\ell}, R_{1;\ell})$, and we call this subtree the *k-generalized Cohn tree*. By the results of $\mathrm{WGC}\mathbb{T}(k,\ell)$, we have the following corollaries.

Corollary 3.15. *Let $\ell \in \mathbb{Z}$. The correspondence between (P,Q,R) in $\mathrm{GC}\mathbb{T}(k,\ell)$ and (p_{12}, q_{12}, r_{12}) induces the canonical graph isomorphism between $\mathrm{GC}\mathbb{T}(k,\ell)$ and $\mathrm{M}\mathbb{T}(k)$.*

Corollary 3.16. *Let $k \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}$. The second entries of k-generalized Cohn triples in $\mathrm{GC}\mathbb{T}(k,\ell)$ are distinct.*

Moreover, by Proposition 3.10 and Remark 3.12, the following proposition holds.

Proposition 3.17. *Let (P,Q,R) be a k-generalized Cohn triple associated with (a,b,c). We assume that $b > \max\{a,c\}$. Then, there exist a unique $\ell \in \mathbb{Z}$ and a unique vertex v in $\mathrm{GC}\mathbb{T}(k,\ell)$ such that $v = (P,Q,R)$.*

Remark 3.18. We denote by $\mathrm{GC}\mathbb{T}^*(k,\ell)$ the full subtree of $\mathrm{WGC}\mathbb{T}(k,\ell)$ whose root is the right child of $(P_{1;\ell}, Q_{1;\ell}, R_{1;\ell})$. In fact, $\mathrm{GC}\mathbb{T}^*(k,\ell) = \mathrm{GC}\mathbb{T}(k, k+\ell+1)$ holds. This fact is proved in the following way. Let (P,Q,R) be a k-generalized triple of the root in $\mathrm{GC}\mathbb{T}^*(k,\ell)$, and p_{11} be the $(1,1)$-entry of P. We note that (P,Q,R) is a k-generalized Cohn triple associated with $(1,k+2,1)$. By the generation rule of $\mathrm{WGC}\mathbb{T}(k,\ell)$, we have $p_{11} = k+\ell+1$. By Lemma 3.11, $(P,R,P^{-1}(R+S))$ is a k-generalized Cohn triple associated with $(1,1,1)$. By Proposition 3.4, $(P,R,P^{-1}(R+S))$ is uniquely determined by p_{11} and we have $(P,R,P^{-1}(R+S)) = (P_{1;k+\ell+1}, Q_{k+\ell+1}, R_{k+\ell+1})$. Therefore, (P,Q,R) is the left child of $(P_{1;k+\ell+1}, Q_{k+\ell+1}, R_{k+\ell+1})$, and it is the root of $\mathrm{GC}\mathbb{T}(k, k+\ell+1)$. Hence we have $\mathrm{GC}\mathbb{T}^*(k,\ell) = \mathrm{GC}\mathbb{T}(k, k+\ell+1)$.

3.2. Farey Tree and Fraction Labeling

In this subsection, we introduce the Farey tree, and we label k-generalized Cohn matrices in $\mathrm{GC}\mathbb{T}(k,\ell)$ with irreducible fractions. As a preparation, we extend the concept of relatively prime.

Definition 3.19. Non-negative integers a and b with $(a,b) \neq (0,0)$ are said to be *relatively prime* if there are no a' and $b' \in \mathbb{Z}_{\geq 0}$ such that $ca' = a$ and $cb' = b$ for any $c \in \mathbb{Z}_{>1}$.

If a and b are both positive integers, then the above definition is the same as the usual sense. We consider the case $a = 0$. If $b = 1$, then a and b are relatively prime, and otherwise, a and b are not relatively prime.

Definition 3.20. Let $q \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$ and n and $d \in \mathbb{Z}_{\geq 0}$. The symbol $\frac{n}{d}$ is called the *reduced expression* of q if n and d are relatively prime and $q = \frac{n}{d}$, where $\frac{n}{d}$ is regarded as ∞ when $d = 0$ and $n > 0$. Moreover, a fraction $\frac{n}{d}$ is said to be *irreducible* if there exists $q \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$ such that $\frac{n}{d}$ is the reduced expression of q.

Definition 3.21. For $\frac{a}{b}$ and $\frac{c}{d}$, we denote $ad-bc$ by $\det\left(\frac{a}{b}, \frac{c}{d}\right)$. A triple $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)$ is called a *Farey triple* if $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are irreducible fractions and

$$\left|\det\left(\frac{a}{b}, \frac{c}{d}\right)\right| = \left|\det\left(\frac{c}{d}, \frac{e}{f}\right)\right| = \left|\det\left(\frac{e}{f}, \frac{a}{b}\right)\right| = 1.$$

We define the *Farey tree* $\mathrm{F}\mathbb{T}$ as follows. Its root vertex is $\left(\frac{0}{1}, \frac{1}{1}, \frac{1}{0}\right)$, and every vertex $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)$ has the two children shown in Figure 1:

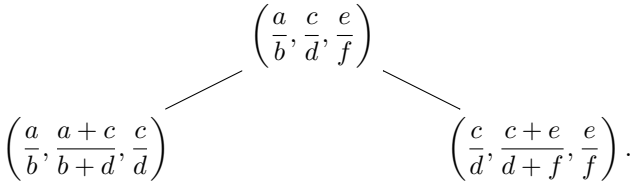


Figure 1: The two children of a vertex $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)$ in the Farey tree.

Proposition 3.22 ([1, Section 3.2])**.**

(1) If $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)$ *is a Farey triple, then so are* $\left(\frac{a}{b}, \frac{a+c}{b+d}, \frac{c}{d}\right)$ *and* $\left(\frac{c}{d}, \frac{c+e}{d+f}, \frac{e}{f}\right)$. *In particular, each vertex in* $\mathrm{F}\mathbb{T}$ *is a Farey triple.*

(2) For every irreducible fraction $\frac{a}{b} \in \mathbb{Q}_{>0}$*, there exists a unique Farey triple* F *in* $\mathrm{F}\mathbb{T}$ *such that* $\frac{a}{b}$ *is the second entry of* F*.*

(3) For $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)$ *in* $\mathrm{F}\mathbb{T}$*,* $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$ *holds.*

By using the canonical graph isomorphism from the Farey tree to the k-generalized Cohn tree, we provide the correspondence from Farey triples in $\mathrm{F}\mathbb{T}$

to k-generalized Cohn triples in $\mathrm{GC}\mathbb{T}(k,\ell)$. This correspondence induces the map from irreducible fractions in $(0,\infty)$ which are the second components of k-generalized Cohn triples in $\mathrm{GC}\mathbb{T}(k,\ell)$ to k-generalized Cohn matrices. This map is called the *fraction labeling to k-generalized Cohn matrices*. We denote by $C_q(k,\ell)$ the k-generalized Cohn matrix labeled with a fraction q. By Theorem 3.13, its proof and Proposition 3.22 (3), we have the following corollary.

Corollary 3.23. *The fraction labeling to k-generalized Cohn matrices is injective. Moreover, if $p<q$ for $p,q\in\mathbb{Q}_{\geq 0}\cup\{\infty\}$, then $I_{C_p(k,\ell)}<I_{C_q(k,\ell)}$ holds.*

4. Proof of Main Theorems

In this section, we prove the criterion theorem (Theorem 1.5) and find k-generalized Markov numbers to which this theorem can be applied.

Let $\mathrm{LGC}\mathbb{T}(k,\ell)$ be the full subtree of $\mathrm{GC}\mathbb{T}(k,\ell)$ whose root is the left child of the root of $\mathrm{GC}\mathbb{T}(k,\ell)$. In this section, we consider the case $\ell=-k$. In this case, the root $\left(C_{\frac{0}{1}}(k,-k),C_{\frac{1}{2}}(k,-k),C_{\frac{1}{1}}(k,-k)\right)$ of $\mathrm{LGC}\mathbb{T}(k,-k)$ is given by

$$C_{\frac{0}{1}}(k,-k)=\begin{bmatrix}-k & 1\\ -3k^2-3k-1 & 3k+3\end{bmatrix},$$
$$C_{\frac{1}{2}}(k,-k)=\begin{bmatrix}k+2 & 2k^2+6k+5\\ 3k^2+9k+5 & 6k^3+24k^2+31k+13\end{bmatrix},\ \text{and}$$
$$C_{\frac{1}{1}}(k,-k)=\begin{bmatrix}1 & k+2\\ 3k+2 & 3k^2+8k+5\end{bmatrix}.$$

All k-generalized Cohn matrices labeled with irreducible fractions between 0 and 1 are included in $\mathrm{LGC}\mathbb{T}(k,\ell)$. Moreover, if we restrict $\mathrm{M}\mathbb{T}(k)$ to the same region (we denote by $\mathrm{LM}\mathbb{T}(k)$ this tree), then all positive integer solutions to $\mathrm{GME}(k)$ but $(1,1,1)$ and $(1,k+2,1)$ appear exactly once without overlap (here, triples that differ only in order are regarded as the same solution). Therefore, the correspondence between the Farey triple (r,t,s) and the $(1,2)$-entries of the k-generalized Cohn triple $(C_r(k,\ell),C_t(k,\ell),C_s(k,\ell))$ induces a bijection from the set of Farey triples in $[0,1]^3$ to the set of all k-generalized Markov triples but $(1,1,1)$ and $(1,k+2,1)$. If we take the second entries of (r,t,s), then this bijection gives each k-generalized Markov number except for 1 and $k+2$ a fraction labeling. We call it the *fraction labeling to k-generalized Markov numbers*, and for every irreducible fraction $t\in(0,1)$, we denote by m_t the corresponding k-generalized Markov number. Also, we set $m_{\frac{0}{1}}=1$ and $m_{\frac{1}{1}}=k+2$.

We will use the following proposition to prove Theorem 1.5.

Proposition 4.1. *Let $k \in \mathbb{Z}_{\geq 0}$. For a k-generalized Markov number b, there is a unique k-generalized Markov triple (a, b, c) up to order such that $\max\{a, b, c\} = b$ if and only if there is a unique irreducible fraction $t \in [0, 1]$ such that $b = m_t$.*

Proof. For $b = 1$ and $b = k + 2$, there exists a unique k-generalized Markov triple up to order such that $\max\{a, b, c\} = b$, and a unique irreducible fraction $t \in [0, 1]$ such that $b = m_t$. Let $b \notin \{1, k + 2\}$. We assume that there is a unique irreducible fraction $t \in [0, 1]$ such that $b = m_t$. If m_t is the maximal number in two k-generalized Markov triples (m_r, m_t, m_s) and $(m_{r'}, m_t, m_{s'})$ in $\mathrm{LM}\mathbb{T}(k)$, then the corresponding Farey triples are (r, t, s) and (r', t, s'). By Proposition 3.22, we have $r = r'$ and $s = s'$, and thus $m_r = m_{r'}$ and $m_s = m_{s'}$. Conversely, we assume that there is a unique k-generalized Markov triple (a, b, c) up to order such that $\max\{a, b, c\} = b$. If $b = m_t = m_{t'}$, then by the assumption, there exist unique fractions r and s such that (m_r, b, m_s) is a k-generalized Markov triple in $\mathrm{LM}\mathbb{T}(k)$. Since (r, t, s) and (r, t', s) are both Farey triples, by the definition of the Farey tree, we have $t = t'$. □

This proposition yields the following.

Corollary 4.2. *Conjecture 1.8 is true if and only if the fraction labeling to the k-generalized Markov numbers $t \mapsto m_t$ is an injective map.*

4.1. Proof of Theorem 1.5

We fix $k \in \mathbb{Z}_{\geq 0}$ and a k-generalized Markov triple (m_r, m_t, m_s) in $\mathrm{LM}\mathbb{T}(k)$. Note that $m_t > \max\{m_r, m_s\}$ and $m_r \neq m_s$. We consider solutions x to the congruences

$$m_r x \equiv \pm m_s \pmod{m_t} \tag{4.1}$$

(it means that $m_r x$ is congruent to m_s or $-m_s$). Since m_r and m_t are relatively prime from Proposition 2.3, there is a unique solution to (4.1) in the range $\left(0, \dfrac{m_t}{2}\right)$.

Definition 4.3. The *characteristic number* u_t of (m_r, m_t, m_s) is the unique solution to (4.1) in the range $\left(0, \dfrac{m_t}{2}\right)$.

Lemma 4.4. *For an irreducible fraction $t \in (0, 1)$, the $(1, 1)$-entry of $C_t(k, -k)$ is positive.*

Proof. By the monotonically increasing property of indices (Corollary 3.23), it suffices to show the statement for the case $t = \dfrac{1}{n}$. First, the $(1, 1)$-entry of $C_{\frac{1}{1}}(k, -k)$ is 1. Since the $(1, 1)$-entry of $C_{\frac{1}{2}}(k, -k)$ is $k + 2$, it is larger than that of $C_{\frac{1}{1}}(k, -k)$. We assume that the $(1, 1)$-entry of $C_{\frac{1}{i}}(k, -k)$ is positive, and that of $C_{\frac{1}{i+1}}(k, -k)$

is larger than that of $C_{\frac{1}{i}}(k,-k)$. We will prove that the $(1,1)$-entry of $C_{\frac{1}{i+2}}(k,-k)$ is larger than that of $C_{\frac{1}{i+1}}(k,-k)$, in particular, it is positive. We set

$$C_{\frac{1}{i}}(k,-k)=\begin{bmatrix}a & b\\ c & d\end{bmatrix},\quad C_{\frac{1}{i+1}}(k,-k)=\begin{bmatrix}a' & b'\\ c' & d'\end{bmatrix},\quad C_{\frac{1}{i+2}}(k,-k)=\begin{bmatrix}a'' & b''\\ c'' & d''\end{bmatrix}.$$

Then, we have

$$\begin{aligned}C_{\frac{1}{i+1}}(k,-k)&=C_{\frac{1}{1}}(k,-k)C_{\frac{1}{i}}(k,-k)-S\\&=\begin{bmatrix}-k & 1\\ -3k^2-3k-1 & 3k+3\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}-\begin{bmatrix}k & 0\\ 3k^2+3k & k\end{bmatrix}\\&=\begin{bmatrix}-ka+c-k & *\\ -(3k^2+3k+1)a+(3k+3)c-3k^2-3k & *\end{bmatrix}.\end{aligned}$$

Therefore, we have

$$a'=-ka+c-k,\quad c'=-(3k^2+3k+1)a+(3k+3)c-3k^2-3k.$$

By assumption, we have $-ka+c-k\geq a>0$. Since

$$C_{\frac{1}{i+2}}(k,-k)=C_{\frac{1}{1}}(k,-k)C_{\frac{1}{i+1}}(k,-k)-S,$$

we have

$$\begin{aligned}a''-a'&=-(k+1)a'+c'-k\\&=-(k+1)(-ka+c-k)-(3k^2+3k+1)a+(3k+3)c-3k^2-4k\\&=(-2k^2-2k-1)a+(2k+2)c-2k^2-3k\\&\geq(-2k^2-2k-1)a+(2k+2)((k+1)a+k)-2k^2-3k\\&=(2k+1)a-k\geq k+1>0,\end{aligned}$$

as desired. □

Lemma 4.5. *For an irreducible fraction $t\in(0,1)$, the following equality holds:*

$$C_t(k,-k)=\begin{bmatrix}u_t & m_t\\ * & (3+3k)m_t-u_t-k\end{bmatrix}.$$

Proof. We can check the statement for $t=\dfrac{1}{2}$ directly. We set

$$C_t(k,-k)=\begin{bmatrix}a_t & m_t\\ * & (3+3k)m_t-a_t-k\end{bmatrix},$$

and we will prove $a_t=u_t$. Let $(C_r(k,-k),C_t(k,-k),C_s(k,-k))$ be a k-generalized Cohn triple in $\mathrm{LGC}\mathbb{T}(k,-k)$. Since $C_s(k,-k)=C_r(k,-k)^{-1}(C_t(k,-k)+S)$, by

comparing $(1,2)$-entries, we have $m_s = a_t m_r - (a_r + k)m_t$. Therefore, we have $m_r a_t \equiv m_s \pmod{m_t}$, which implies $a_t \equiv \pm u_t \pmod{m_t}$. By the monotonically increasing property, the inequalities $I_{\frac{0}{1}} < I_t < I_{\frac{1}{1}}$ hold, and we have $-k < \dfrac{a_t}{m_t} < \dfrac{1}{k+2}$. By Lemma 4.4 and the previous inequality, we have $0 < a_t < \dfrac{m_t}{k+2} \leq \dfrac{m_t}{2}$. Therefore, by the definition of u_t, we have $a_t = u_t$. □

In the proof of Lemma 4.5, we have the following corollary, which is a generalization of [1, Remark 4.14].

Corollary 4.6. *The characteristic number u_t is a solution of $m_r x \equiv m_s \pmod{m_t}$.*

Moreover, we have the following property for u_t.

Lemma 4.7. *The characteristic number u_t is a solution to $x^2 + kx + 1 \equiv 0 \pmod{m_t}$.*

Proof. By $m_r^2 + k m_r m_s + m_s^2 \equiv 0 \pmod{m_t}$ and $m_r u_t \equiv m_s \pmod{m_t}$, we have

$$m_r^2 u_t^2 + k m_r^2 u_t + m_r^2 \equiv 0 \pmod{m_t}.$$

Since m_r and m_t are relatively prime, we have

$$u_t^2 + k u_t + 1 \equiv 0 \pmod{m_t},$$

as desired. □

Lemma 4.8. *Let $k \in \mathbb{Z}_{\geq 0}$. For an irreducible fraction $t \in (0,1)$, the inequality $u_t + k < \dfrac{m_t}{2}$ holds.*

Proof. Since $m_t \geq 2k^2 + 6k + 5$ and $u_t < \dfrac{m_t}{k+2}$ by the monotonically increasing property of the index, we have

$$\frac{m_t}{2} - u_t > \left(\frac{1}{2} - \frac{1}{k+2}\right) m_t \geq \frac{k(2k^2+6k+5)}{2(k+2)} > k,$$

as desired. □

Now, we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. By Proposition 4.1, it suffices to show that a fraction $t \in (0,1)$ which satisfies $b = m_t$ is unique. Let t and τ be irreducible fractions in $(0,1)$ satisfying $b = m_t = m_\tau$. We will prove $t = \tau$. By Lemma 4.7, u_t and u_τ are solutions to $x^2 + kx + 1 \equiv 0 \pmod{b}$. By Lemma 4.8, we have $u_t, u_\tau < \dfrac{b}{2} - k$. Let $\rho \in \{t, \tau\}$. Now, $b - (u_\rho + k)$ is also a solution to $x^2 + kx + 1 \equiv 0 \pmod{b}$, and

$b-(u_\rho+k)\geq \dfrac{b}{2}$. By the assumption that the number of solutions is at most two, u_t and u_τ are the unique solutions of

$$x^2+kx+1\equiv 0 \pmod{b} \text{ and } 0<x<\frac{b}{2}. \tag{4.2}$$

Therefore, we have $u_t = u_\tau$. Moreover, by Lemma 4.5, we have $C_t(k,-k) = C_\tau(k,-k)$. By Theorem 3.13, we have $t=\tau$. □

4.2. Uniqueness Theorem for Prime Powers and Twice Prime Powers

Using Theorem 1.5, we can prove Theorem 1.6. To use Theorem 1.5, the following lemma is essential.

Lemma 4.9. *Let p be a prime. The number of solutions to $x^2+kx+1\equiv 0 \pmod{p}$ is at most two.*

Proof. Let x_1, x_2 be solutions to $x^2+kx+1\equiv 0 \pmod{p}$ such that $x_1\not\equiv x_2$ holds. Since $(x_1-x_2)(x_1+x_2+k)\equiv 0 \pmod{p}$, we have $x_1+x_2+k\equiv 0 \pmod{p}$. Let x_3 be another solution to $x^2+kx+1\equiv 0 \pmod{p}$ such that $x_1\not\equiv x_3$ holds. Then, we have $x_1+x_3+k\equiv 0 \pmod{p}$. Therefore, we have $x_2-x_3\equiv 0 \pmod{p}$. □

Proof of Theorem 1.6. Since $b=p$ or $b=2p$ for a prime p, we have $b\neq 1$. If $b=k+2$, then the desired uniqueness follows from Proposition 4.1. Thus we may assume that $b\notin\{1,k+2\}$.

When $b=p$, Lemma 4.9 shows that the congruence $x^2+kx+1\equiv 0 \pmod{b}$ has at most two solutions. Therefore, Theorem 1.5 implies the desired uniqueness for b.

We assume $b=2p$. If $p=2$, then $b=4$. By the construction of the fraction labeling, for any irreducible fraction $t\in(0,1)$, we have $m_t\geq 2k^2+6k+5>4$. Hence, if 4 is a k-generalized Markov number, then it must be equal to the exceptional value $m_{\frac{1}{1}}=k+2$, and so $k=2$. This is impossible under the present assumption $b\notin\{1,k+2\}$. Therefore, p is odd. Clearly, $x^2+kx+1\equiv 0 \pmod{2}$ has at most one solution. Since 2 and p are relatively prime, by using Lemma 4.9 and the Chinese remainder theorem, $x^2+kx+1\equiv 0 \pmod{2p}$ has at most two solutions. Therefore, applying Theorem 1.5, we obtain the desired uniqueness for b. □

Next, we consider the case where $b=p^m$ or $2p^m$. In this case, using Theorem 1.5 requires some constraints regarding k and p.

Theorem 4.10. *Let $k\in\mathbb{Z}_{\geq 0}$. For a k-generalized Markov number b, if there exists an odd prime p and $m\in\mathbb{Z}_{\geq 2}$ such that $b=p^m$ or $2p^m$ and $k^2-4\not\equiv 0 \pmod{p}$, then there is a unique k-generalized Markov triple (a,b,c) up to order such that $\max\{a,b,c\}=b$.*

Theorem 4.10 follows from the following lemma and Theorem 1.5.

Lemma 4.11. *Let p be an odd prime and $m \in \mathbb{Z}_{\geq 2}$. If $k^2 - 4 \not\equiv 0 \pmod{p}$ holds, then the number of solutions to $x^2 + kx + 1 \equiv 0 \pmod{p^m}$ is at most two.*

Proof. Let x_1, x_2 be solutions to $x^2 + kx + 1 \equiv 0 \pmod{p^m}$ such that $x_1 \not\equiv x_2$ holds. Then, we have $(x_1 - x_2)(x_1 + x_2 + k) \equiv 0 \pmod{p^m}$. We assume that $(x_1 - x_2) \equiv (x_1 + x_2 + k) \equiv 0 \pmod{p}$. Then, we have $2x_1 + k \equiv 0 \pmod{p}$. If $2x_1 + k$ is even, there exists $n \in \mathbb{Z}$ such that $2x_1 + k = 2np$. Therefore, we have $x_1 = np - \dfrac{k}{2}$. However, we have $-1 \equiv x_1(x_1 + k) \equiv -\dfrac{k^2}{4} \pmod{p}$, which conflicts with the assumption. If $2x_1 + k$ is odd, then there exists $n \in \mathbb{Z}$ such that $2x_1 + k = (2n + 1)p$. Therefore, we have $x_1 = np + \dfrac{p - k}{2}$. However, we have $-1 \equiv x_1(x_1 + k) \equiv \dfrac{p^2 - k^2}{4} \pmod{p}$, which conflicts with the assumption. Therefore, we have $x_1 + x_2 + k \equiv 0 \pmod{p^m}$. Let x_3 be another solution to $x^2 + kx + 1 \equiv 0 \pmod{p^m}$ such that $x_1 \not\equiv x_3$ holds. Then, we have $x_1 + x_3 + k \equiv 0 \pmod{p^m}$. Therefore, we have $x_2 - x_3 \equiv 0 \pmod{p^m}$. □

Proof of Theorem 4.10. The cases where $b = 1$ and $b = k + 2$ are clear. We assume that $b \notin \{1, k+2\}$. The statement for $b = p^m$ follows from Theorem 1.5 and Lemma 4.11. We assume $b = 2p^m$. We note that $x^2 + kx + 1 \equiv 0 \pmod{2}$ has at most one solution. Since 2 and p^m are relatively prime, by the Chinese remainder theorem and Lemma 4.11, $x^2 + kx + 1 \equiv 0 \pmod{2p^m}$ has at most two solutions. Therefore, applying Theorem 1.5, we have the conclusion. □

Remark 4.12. Theorem 4.10 is also proved by using Hensel's lifting lemma. The condition $k^2 - 4 \not\equiv 0 \pmod{p}$ in Theorem 4.10 is derived from the existence conditions of x satisfying $f(x) = x^2 + kx + 1 \equiv 0 \pmod{p^m}$ and $f'(x) = 2x + k \not\equiv 0 \pmod{p}$. Therefore, we cannot remove this condition even if we use Hensel's lifting lemma.

In Theorem 4.10, we only consider the case where p is odd. There are no k-generalized Markov numbers of the form 2^m if $k^2 - 4 \not\equiv 0 \pmod{2}$. More strongly, the following lemma holds.

Lemma 4.13. *For $k \in \mathbb{Z}_{\geq 0}$, if there exists an even k-generalized Markov number, then k is even.*

Proof. We will prove that if k is odd, all k-generalized Markov numbers are odd. Clearly, $(1, 1, 1)$ is a k-generalized Markov triple whose elements are odd. If all of (a, b, c) are odd, then so are $(3 + 3k)ab - c - k(a + b)$ and $(3 + 3k)bc - a - k(b + c)$. Therefore, all elements of each k-generalized Markov triple of $\mathrm{WM}\mathbb{T}(k)$ are odd. By Proposition 2.1, we have the conclusion. □

Furthermore, weakening the constraint on k to $\frac{k^2}{4} - 1 \not\equiv 0 \pmod 2$ does not yield any additional properties.

Proposition 4.14. *If there exists a k-generalized Markov number b which is a multiple of 4, then $k \equiv 2 \pmod 4$ holds. In particular, $\frac{k^2}{4} - 1 \equiv 0 \pmod 2$ holds.*

Proof. Since b is even, we have $k \not\equiv 1$ or $3 \pmod 4$ by Lemma 4.13. Therefore, it suffices to show that $k \not\equiv 0 \pmod 4$. We assume that $k = 4n$. We set $b = 4m$. We consider a k-generalized Cohn matrix corresponding to $4m$,

$$C = \begin{bmatrix} a & 4m \\ c & (3 + 3(4n)) \cdot 4m - a - 4n \end{bmatrix}.$$

Since $\det C = 1$, we have

$$1 \equiv a(-a - 4n) \pmod{4m},$$

If $a = 2\ell$, then $a(-a-4n)$ is even, and thus $1 \not\equiv a(-a-4n) \pmod{4m}$. If $a = 2\ell+1$, then we have $a(-a - 4n) \equiv -1 \pmod 4$. Therefore, such a k-generalized Cohn matrix would not exist, which is a contradiction. □

We provide a sufficient condition about p that is simpler than Theorem 4.10.

Proposition 4.15. *Let $k \in \mathbb{Z}_{\geq 0}$ and p be an odd prime. If k and p satisfy either of the following two conditions, then the condition $k^2 - 4 \not\equiv 0 \pmod p$ in Theorem 4.10 holds:*

(1) $k \geq 4$ is even and $p > \frac{k}{2} + 1$.

(2) k is odd and $p > k + 2$.

Proof. Assume that Condition (1) holds. Then we have $\frac{k}{2} + 1 \not\equiv 0 \pmod p$ and $\frac{k}{2} - 1 \not\equiv 0 \pmod p$. Therefore, we have $\frac{k^2}{4} - 1 \not\equiv 0 \pmod p$. Since p is an odd prime, it is equivalent to $k^2 - 4 \not\equiv 0 \pmod p$. The case of Condition (2) is proved in the same way. □

Proposition 4.15 implies that if p is large enough, we can apply Theorem 4.10 to p^m and $2p^m$. More strongly, we have the following proposition.

Proposition 4.16. *Let $k \in \mathbb{Z}_{\geq 0}$ and p be an odd prime. If k and p satisfy either of the following two conditions, then the condition $k^2 - 4 \not\equiv 0 \pmod p$ in Theorem 4.10 holds:*

(1) $k \geq 4$ is even, and both $\frac{k}{2} + 1$ and $\frac{k}{2} - 1$ are not divided by p^2,

(2) k is odd, and both $k+2$ and $k-2$ are not divided by p^2.

Proof. We will prove the case where $b = p^m$ (the case $b = 2p^m$ is shown in exactly the same way). First, we assume that k is even and $k \geq 4$. We will prove that $k^2 - 4 \equiv 0 \pmod{p}$ implies $\frac{k}{2} + 1 \equiv 0 \pmod{p^2}$ or $\frac{k}{2} - 1 \equiv 0 \pmod{p^2}$. Since p is odd, we have $\frac{k}{2} + 1 \equiv 0 \pmod{p}$ or $\frac{k}{2} - 1 \equiv 0 \pmod{p}$. If $\frac{k}{2} + 1 \equiv 0 \pmod{p}$, there exists $n \in \mathbb{Z}$ such that $\frac{k}{2} + 1 = np$. We consider the k-generalized Cohn matrix associated with p^m, $C = \begin{bmatrix} a & p^m \\ c & (3+3k)p^m - a - k \end{bmatrix}$. Since $\det C = 1$, we have $a(-a-k) \equiv 1 \pmod{p}$. Since $k = 2np - 2$, we have $-a^2 - 2a \equiv 1 \pmod{p}$. Therefore, we have $(a-1)^2 \equiv 0 \pmod{p}$, and $a - 1 \equiv 0 \pmod{p}$. Let $a = n'p + 1$. By calculating $\det C$ modulo p^2, we have

$$1 \equiv -a(a+k) \equiv -(n'p+1)(n'p + 2(np-1) + 1) \equiv -2np + 1 \pmod{p^2}.$$

Hence we have $2n \equiv 0 \pmod{p}$, and thus $n \equiv 0 \pmod{p}$. Therefore, we have $\frac{k}{2} + 1 \equiv 0 \pmod{p^2}$. Similarly, if $\frac{k}{2} - 1 \equiv 0 \pmod{p}$, we have $\frac{k}{2} - 1 \equiv 0 \pmod{p^2}$. We can prove (2) in the same way. □

We use the congruence argument in the proof of Proposition 4.16 to prove Theorem 1.7. We first state the result and observation needed for the proof.

Proposition 4.17 ([9, Theorem 2.9])**.** *If (a, b, c) is a Markov triple, then (a^2, b^2, c^2) is a 2-generalized Markov triple. Conversely, if (A, B, C) is a 2-generalized Markov triple, then $(\sqrt{A}, \sqrt{B}, \sqrt{C})$ is a Markov triple.*

Remark 4.18. By Proposition 4.17, any 2-generalized Markov number cannot be of the form $2p^m$, where p is an odd prime.

Proof of Theorem 1.7. To prove Statement (1) of Theorem 1.7, suppose that $k = 2$. By Proposition 4.17, we have $\sqrt{b} = p^{m'}$, and it is a Markov number. By Theorem 1.3, there is a unique Markov triple $(a', \sqrt{b}, c')$ such that $\sqrt{b} \geq \max\{a', c'\}$. By Proposition 4.17 again, (a'^2, b, c'^2) is the unique 2-generalized Markov triple such that $b \geq \max\{a'^2, c'^2\}$.

To prove Statements (2) and (3) of Theorem 1.7, the cases where $b = 1$ and $b = k + 2$ are clear. We assume that $b \notin \{1, k+2\}$. First, we prove the case where p is an odd prime. Let b be a k-generalized Markov number of the form $b = p^m$ or $2p^m$. By the assumption and Proposition 4.16, we have $k^2 - 4 \not\equiv 0 \pmod{p}$ for all odd primes p. Therefore, the statement follows from Theorem 4.10. Second, we prove that there is no k-generalized Markov number b of the form $b = 2^m$ under Statement (2) or Statement (3) of Theorem 1.7. We assume that there is a k-generalized Markov number b of the form $b = 2^m$. By Proposition 4.14, we have

$k = 4n+2$. Then, either $\dfrac{k}{2}-1$ or $\dfrac{k}{2}+1$ is divided by 4. Therefore, k satisfies neither Statement (2) nor Statement (3) of Theorem 1.7, and thus there is no k-generalized Markov number b of the form $b = 2^m$. □

Remark 4.19. Although Theorem 4.10 does not cover all k and p, we have yet to find examples of k-generalized Markov numbers p^m (or $2p^m$) to which Theorem 1.5 cannot be applied, i.e., k and p satisfy $p^m \notin \{1, k+2\}$, and the equation $x^2 + kx + 1 \equiv 0 \pmod{p^m}$ has more than two solutions. Even after removing the last condition, we have yet to find such an example. However, it is easy to find an example where the first condition is removed. Indeed, when we set $k = 7$, $p = 3$ and $m = 2$ (note that $k^2 - 4 \equiv 0 \pmod 3$ holds), $9 = 3^2$ is the second smallest 7-generalized Markov number, and the equation $x^2 + 7x + 1 \equiv 0 \pmod 9$ has three solutions $x = 1, 4$, and 7.

From the discussion in this section, the following theorem for general k-generalized Markov numbers, which is a generalization of [1, Theorem 3.19], is proved.

Theorem 4.20. *Let $k \in \mathbb{Z}_{\geq 0}$. Assume that either of the following two conditions is satisfied:*

(1) $k \geq 4$ is even, and both $\dfrac{k}{2}+1$ and $\dfrac{k}{2}-1$ are not divided by a squared prime,

(2) k is odd, and both $k+2$ and $k-2$ are not divided by a squared prime.

For a k-generalized Markov number $b = p_1^{a_1} \cdots p_n^{a_n}$ or $2p_1^{a_1} \cdots p_n^{a_n}$, where p_i are odd primes, b is the maximum of at most 2^{n-1} k-generalized Markov triples.

Proof. We consider how many possible fractions t satisfy $b = m_t$. To do so, we consider how many possible values u_t, which is uniquely determined from t, can take. First, we prove the case where $b = p_1^{a_1} \cdots p_n^{a_n}$. The value u_t is a solution to $x^2 + kx + 1 \equiv 0$. When u_t satisfies $x^2 + kx + 1 \equiv 0 \pmod b$, it must be a solution of

$$x^2 + kx + 1 \equiv 0 \pmod{p_i^{a_i}} \tag{4.3}$$

for every $p_i^{a_i}$. By Proposition 4.16, for each $p_i^{a_i}$, the number of solutions to (4.3) is at most 2. Since any two elements in $\{p_i^{a_i} \mid 1 \leq i \leq n\}$ are relatively prime, by using the Chinese remainder theorem, the number of solutions to $x^2 + kx + 1 \equiv 0 \pmod b$ is at most 2^n. Of these solutions, at most 2^{n-1} solutions can be u_t for some t. Otherwise, since $0 < u_t < \dfrac{b}{2} - k$ and $b - (u_t + k)$ is also a solution to this equation, the number of solutions is more than 2^n, which is a contradiction. Next, we will prove the case where $b = 2p_1^{a_1} \cdots p_n^{a_n}$. Since the number of solutions to $x^2 + kx + 1 \equiv 0 \pmod 2$ is at most 1, we can apply the same argument to the case $b = 2p_1^{a_1} \cdots p_n^{a_n}$, and we have the conclusion. □

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