ON FORMULAS, RECURRENCES, AND CONGRUENCES FOR REPRESENTATIONS OF INTEGERS AS SUMS OF SQUARES AND TRIANGULAR NUMBERS

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Abstract

In this paper, we present explicit formulas expressed as composition sums for the functions $r_k(n)$ and $t_k(n)$, which count the number of ways a non-negative integer n can be represented as a sum of k squares and k triangular numbers, respectively. These formulas show that $r_k(n)$ and $t_k(n)$ are polynomials in k of degree n. We leverage this property to derive recurrence relations and congruences.

1. Introduction and Background

The study of representations of positive integers as sums of squares has a rich history, with foundational contributions spanning several centuries. The Brahmagupta–Fibonacci identity expresses the product of two sums of two squares as a sum of two squares in two distinct ways. This identity, first proved by Diophantus of Alexandria in the 3rd century, was later recognized as a special case of Euler's four-square and Lagrange's identities in the 18th century.

Fermat's theorem on sums of two squares, Legendre's three-square theorem, and Jacobi's four-square theorem exemplify the search for explicit formulas for the sequence $r_k(n)$, where $r_k(n)$ denotes the number of representations of n as a sum of k squares. For instance,

$$r_2(n) = 4 \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}}, \quad r_4(n) = 8 \sum_{\substack{d|n \\ 4\nmid d}} d,$$

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which may be derived by extracting coefficients from the identities

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}},$$

and

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4 = 1 + 8\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 8\sum_{n=1}^{\infty} \frac{4nq^{4n}}{1-q^{4n}},$$

both due to Jacobi [20]. The formula for $r_6(n)$ is more involved and can be proved using Ramanujan's ${}_1\psi_1$ summation formula; see [20, 12] for proofs.

In Chapter 6 of [18], Hardy and Wright derive the formulas for $r_4(n)$ and $r_8(n)$ using the theory of elliptic functions. Subsequently, L. Carlitz [13] employed the following identity, originally established by W. N. Bailey [11], to derive expressions for $r_4(n)$ and $r_6(n)$:

$$\sum_{n=-\infty}^{\infty} \left\{ \frac{xq^n}{(1-xq^n)^2} - \frac{yq^n}{(1-yq^n)^2} \right\} = \frac{(xyq)_\infty (q/(xy))_\infty (xq/y)_\infty (yq/x)_\infty (q)_\infty^4}{(xq)_\infty^2 (q/x)_\infty^2 (yq)_\infty^2 (q/y)_\infty^2}.$$

Here the q-Pochhammer symbol is defined by

$$(a;q)_n = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{if } n > 0, \\ 1, & \text{if } n = 0. \end{cases}$$

Taking the limit as $n \to \infty$, we obtain

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

For simplicity, we shall use the notation $(a;q)_\infty = (a)_\infty$.

The problem of expressing a positive integer as a sum of triangular numbers has been extensively studied from antiquity to the present. Let $t_k(n)$ denote the number of representations of n as a sum of k triangular numbers. Several explicit formulas have been established for specific cases of $t_k(n)$. For example,

$$t_4(n) = \sigma_1(2n+1), \quad t_6(n) = \frac{1}{8} \sum_{d \mid 4n+3} (-1)^{(d+1)/2} d^2.$$

It is not known who first proved the formula for $t_6(n)$. A formulation appears in an unpublished manuscript of Ramanujan [23, 9]. The earliest known published proof is due to Ono, Robins, and Wahl [22].

In [16], Cooper and Lam employed Ramanujan's ψ_1 summation formula to derive explicit formulas for the number of representations of an integer as sums of k squares and sums of k triangular numbers, for $k = 2, 4, 6, 8$.

To obtain the main results of this paper, we use the functions $\varphi(q)$ and $\psi(q)$, which originate from Ramanujan's theta functions, defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty \tag{1}$$

for $|ab| < 1$.

For $\varphi = f(q,q)$ and $\psi(q) = f(q,q^3)$, these well-known identities are useful:

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

To present the main results of this paper, we recall the following definition.

Definition 1. A *composition* of a positive integer n is an ordered sequence of positive integers whose sum is n. The set of all such compositions of n is denoted by $C(n)$.

From the above representation of $\varphi(q)$, it follows that

$$\varphi^k(q) = \sum_{n=0}^{\infty} r_k(n) q^n.$$

This observation leads to the following theorem.

Theorem 1. *For positive integers k, n, we have*

$$r_k(n) = \sum_{l=1}^{n} \frac{k^l}{l!} \sum_{(w_1,w_2,\ldots,w_l)\in C_n} a_{w_1} a_{w_2} \cdots a_{w_l}, \tag{2}$$

where the coefficients are given by

$$a_n = 2\sum_{d|n} (-1)^{d+1} \frac{1}{d} - \sum_{\substack{d|n \\ d\equiv 0 \pmod 2}} \frac{1}{d}.$$

Similarly, we have

$$\psi^k(q) = \sum_{n=0}^{\infty} t_k(n) q^n,$$

which leads to the following theorem.

Theorem 2. *For positive integers k, n, we have*

$$t_k(n) = \sum_{l=1}^{n} \frac{k^l}{l!} \sum_{(w_1,w_2,\ldots,w_l)\in C_n} b_{w_1} b_{w_2} \cdots b_{w_l}, \tag{3}$$

where

$$b_n = \sum_{\substack{d|n \\ d\equiv 1 \pmod 2}} \frac{1}{d} - \sum_{\substack{d|n \\ d\equiv 0 \pmod 2}} \frac{1}{d}.$$

Our approach to deriving formulas (2) and (3) is analogous to the method employed by Alegri in [1, 2] for obtaining formulas for $P_n(z)$. Considering

$$\left(q^{-\frac{1}{24}}\eta(\tau)\right)^{-z} = \prod_{n=1}^{\infty}(1-q^n)^{-z}, \tag{4}$$

for $\tau \in H = \{b \in \mathbb{C} \mid \mathrm{Im}(b) > 0\}$, $z \in \mathbb{C}$, and the Dedekind η-function as given in Ono [21], we have the Fourier expansion of (4):

$$\left(q^{-\frac{1}{24}}\eta(\tau)\right)^{-z} = \sum_{n=0}^{\infty} P_n(z) q^n,$$

where

$$P_n(z) = \sum_{l=1}^{n} \frac{z^l}{l!} \sum_{w_1+w_2+\cdots+w_l\in C(n)} \frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1 w_2 \cdots w_l}.$$

In particular, for $z = k$ a positive integer, we obtain $P_n(k) = p_k(n)$, the number of k-colored integer partitions[1] of n. The expressions for $r_k(n)$ and $t_k(n)$ given in (2) and (3) serve as a foundation for deriving recurrence relations and investigating congruence properties.

2. Proofs of Theorems 1 and 2 and Recurrences

Proof of Theorem 1. Since

$$\varphi^k(q) = (-q;q)_\infty^{2k}(q^2;q^2)_\infty^k,$$

for $|q| < 1$, taking logarithms yields

$$\ln(\varphi^k(q)) = 2k\sum_{m=1}^{\infty}\log(1+q^m) + k\sum_{m=1}^{\infty}\log(1-q^{2m}).$$

[1]Several results on partitions and colored partitions can be found in Andrews and Eriksson [8] and Fu and Tang [17].

Expanding the logarithms into power series, we have

$$\ln(\varphi^k(q)) = k\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} 2(-1)^{m+1}\frac{q^{mn}}{m} - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q^{2mn}}{m}\right].$$

The coefficient of q^n in this series is exactly a_n as defined, so

$$\varphi^k(q) = \exp\left(k\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{l=0}^{\infty}\frac{k^l}{l!}\left(\sum_{n=1}^{\infty} a_n q^n\right)^l.$$

The coefficient of q^n in $\left(\sum_{n=1}^{\infty} a_n q^n\right)^l$ can be written as a sum over compositions of n of length l, which gives the formula for $r_k(n)$. □

Proof of Theorem 2. Similarly, for

$$\psi^k(q) = \frac{(q^2;q^2)_\infty^k}{(q;q^2)_\infty^k},$$

taking logarithms yields

$$\ln(\psi^k(q)) = k\left[\sum_{m=1}^{\infty}\log(1-q^{2m}) - \sum_{m=1}^{\infty}\log(1-q^{2m-1})\right].$$

Expanding into series, the coefficient of q^n is b_n. Applying the exponential generating function method as in Theorem 1 completes the proof. □

Corollary 1. *The functions $r_k(n)$ and $t_k(n)$ satisfy the following recurrences for $n \geq 0$:*

$$r_k(n+1) = \sum_{j=0}^{n}\frac{k(j+1)a_{j+1}}{n+1}\, r_k(n-j),$$

$$t_k(n+1) = \sum_{j=0}^{n}\frac{k(j+1)b_{j+1}}{n+1}\, t_k(n-j).$$

Proof. In the proofs of Theorems 1 and 2, we observe that

$$\varphi^k(q) = \exp\left(\sum_{n=1}^{\infty} k a_n q^n\right) \quad \text{and} \quad \psi^k(q) = \exp\left(\sum_{n=1}^{\infty} k b_n q^n\right).$$

Since the Bell polynomial $B_n(x_1,\ldots,x_n)$, as presented in Chapter 11 of Charalambides [14], has the exponential generating function

$$\exp\left(\sum_{j=1}^{\infty} x_j\frac{t^j}{j!}\right),$$

by selecting suitable values for x_j $(j = 1, 2, \ldots, n)$, we can express $r_k(n)$ and $t_k(n)$ in terms of Bell polynomials.

Indeed,

$$\sum_{n=0}^{\infty} B_n(ka_1,\, k2!a_2,\, \ldots,\, kn!a_n)\frac{q^n}{n!} = \varphi^k(q),$$

and

$$\sum_{n=0}^{\infty} B_n(kb_1,\, k2!b_2,\, \ldots,\, kn!b_n)\frac{q^n}{n!} = \psi^k(q).$$

A well-known property of Bell polynomials is the recurrence

$$B_{n+1} = \sum_{j=0}^{n} \binom{n}{j} x_{j+1} B_{n-j}.$$

Applying this recurrence with $x_j = kj!a_j$ (or $x_j = kj!b_j$) yields the stated formulas for $r_k(n)$ and $t_k(n)$. □

Remark 1. The above corollary is analogous to results appearing as Corollaries 2 and 4 in Andrews, Jha, and López-Bonilla [6], namely:

$$r_k(n) = -\frac{2k}{n}\sum_{j=1}^{n}(-1)^j jD(j)r_k(n-j),$$

$$t_k(n) = -\frac{k}{n}\sum_{j=1}^{n} jT(j)t_k(n-j),$$

where

$$D(n) = \sum_{\substack{d|n \\ d \text{ odd}}} \frac{1}{d}, \quad \text{and} \quad T(n) = \sum_{d|n} \frac{1+2(-1)^d}{d} = \frac{1}{n}\sum_{d|n}(-1)^d d.$$

3. Congruences and Representability Criteria

3.1. Congruences

The polynomial forms of $r_k(n)$ and $t_k(n)$ allow us to deduce various congruences and criteria for representability, applying results from [24].

Theorem 3. *Let n and k be positive integers and p be a prime number such that $p \geq n+1$. If*

$$r_k(n) \equiv 0 \pmod{p},$$

then

$$r_{k+rp}(n) \equiv 0 \pmod{p}, \quad \forall r \in \mathbb{Z}.$$

Theorem 4. *Let n and k be positive integers and p be a prime number such that $p \geq n+1$. If $t_k(n) \equiv 0 \pmod{p}$, then*

$$t_{k+rp}(n) \equiv 0 \pmod{p}, \quad \forall r \in \mathbb{Z}.$$

In view of Theorem 2.8 of [24], we obtain the following results.

Theorem 5. *Let n be a positive integer and p be an odd prime number such that $p-1 \geq n+1$. Then for every integer $s \geq p$,*

$$\sum_{i=0}^{p-1} r_{s-i}(n) \equiv 0 \pmod{p}.$$

Theorem 6. *Let n be a positive integer and p be an odd prime number such that $p-1 \geq n+1$. Then for every integer $s \geq p$,*

$$\sum_{i=0}^{p-1} t_{s-i}(n) \equiv 0 \pmod{p}.$$

3.2. Representability Criteria

The above two results provide a criterion for representing a positive integer n as a sum of k squares and/or triangular numbers.

Theorem 7 (Representability criterion)**.** *Let n be a positive integer and p an odd prime such that $p-1 \geq n+1$. If $r_{s-(p-1)}(n) + \cdots + r_{s-1}(n) \not\equiv 0 \pmod{p}$, then $r_s(n) \neq 0$.*

The same criterion holds for $t_s(n)$.

Theorem 8. *Let n be a positive integer and p an odd prime such that $p-1 \geq n+1$. If $t_{s-(p-1)}(n) + \cdots + t_{s-1}(n) \not\equiv 0 \pmod{p}$, then $t_s(n) \neq 0$.*

In what follows, we derive criteria for the existence of integral roots of the polynomials $r_k(n)$ and $t_k(n)$, following the framework in [24]. If $r_k(n)$ has an integral root $k = s$ for fixed n, then n cannot be expressed as a sum of s squares; a similar conclusion holds for $t_k(n)$ in the case of triangular number representations.

This viewpoint aligns with classical additive number theory, particularly the study of representing integers as sums of a fixed number of nonzero squares. Rouse [3] provided explicit bounds using modular form techniques, while Kim and Oh [4] established general lower bounds for such representations by squares in integral quadratic forms. Extending these ideas, Thompson [5] derived formulas for representations in real quadratic number fields.

Thus, the vanishing of $r_k(n)$ and $t_k(n)$ signifies fundamental representational constraints and offers an algebraic viewpoint on classical problems concerning the expression of integers as sums of squares and triangular numbers.

To establish these criteria, we first note that the scaled polynomials $n!r_k(n)$ and $n!t_k(n)$ belong to $\mathbb{Z}[k]$, enabling a precise analysis of their integral roots.
Define, for $z \in \mathbb{C}$,

$$\varphi^z(q) = \sum_{n=0}^{\infty} r_z(n) q^n.$$

Taking the logarithmic derivative with respect to q and multiplying by z, we obtain

$$z \cdot \frac{\varphi'(q)}{\varphi(q)} = z \sum_{n=1}^{\infty} n a_n q^{n-1}.$$

Extracting the coefficients yields the recurrence

$$(n+1) r_z(n+1) = z \sum_{k=1}^{n+1} k a_k \cdot r_z(n+1-k).$$

An inductive argument, as in Lemma 1.1 of [24], then establishes that $n!r_z(n)$ is a polynomial in z of degree n. A similar argument applies to $t_z(n)$.

Define

$$n!r_z(n) = r_{n,0} + r_{n,1} z + \cdots + r_{n,n} z^n,$$

and

$$n!t_z(n) = t_{n,0} + t_{n,1} z + \cdots + t_{n,n} z^n.$$

Furthermore, define

$$R_n = \max_{0 \le i \le n-1} |r_{n,i}|^{\frac{1}{n-i}}, \quad T_n = \max_{0 \le i \le n-1} |t_{n,i}|^{\frac{1}{n-i}}.$$

We are now in a position to state the criteria, as outlined in the proof of Theorem 3.2 of [24].

Theorem 9. *Let n be a positive integer. Suppose $r_k(n) = 0$. Then*

$$k \in \{ s \in \mathbb{Z} : s \mid r_{n,1} \text{ and } |s| < 2R_n \}.$$

Theorem 10. *Let n be a positive integer. Suppose $t_k(n) = 0$. Then*

$$k \in \{ s \in \mathbb{Z} : s \mid t_{n,1} \text{ and } |s| < 2T_n \}.$$

3.3. $n!r_k(n)$, $n!t_k(n) \in \mathbb{Z}[k]$ Implies Certain Congruences

Furthermore, we present several congruences derived from the fact that both $n!r_k(n)$ and $n!t_k(n)$ lie in $\mathbb{Z}[k]$. Our derivation relies on the classical result known as *Bertrand's postulate*, which asserts that for every integer $n > 2$, there exists at least one prime p such that $\frac{n}{2} < p < n$. A proof of this postulate was first given by Chebyshev in 1850 (see [10]).

Now we state the congruences.

Theorem 11. *Let p be an odd prime number. Let q be a prime number such that $\frac{p}{2} < q < p$. We have*

$$r_q(p) \equiv 0 \pmod{q} \tag{5}$$

and

$$t_q(p) \equiv 0 \pmod{q}. \tag{6}$$

Proof. Since $p!r_t(p) \in \mathbb{Z}[t]$, we have $p!r_q(p) \equiv qa_p \pmod{q^2}$. From the definition of a_n, we have $a_p = 2\left(1 + \frac{1}{p}\right)$. Also, since $q > \frac{p}{2}$ we obtain that $q \mid p!$ and $q^2 \nmid p!$. These observations lead to the conclusion $r_q(p) \equiv 0 \pmod{q}$. A similar argument proves the second congruence. □

4. Concluding Remarks

We initiated our study with the polynomial structures of $r_k(n)$ and $t_k(n)$, with the aim of identifying congruences and recurrences extending beyond those already established. The investigation of further congruences of the type presented in Section 3 remains a promising direction for future research. In this regard, the properties of the polynomials $P_n(k)$, as examined in [1, 2, 24], may provide effective tools for future progress.

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