ON 2-COLOR PARTITIONS IN WHICH THE PARTS OF ONE COLOR ARE RESTRICTED TO MULTIPLES OF 7^k

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Abstract
In this work, we investigate the arithmetic properties of $p_{1,7^k}(n)$, which counts 2-color partitions of n where one of the colors appears only in parts that are multiples of 7^k. By constructing generating functions for $p_{1,7^k}(n)$ across specific arithmetic progressions, we establish a set of Ramanujan-type infinite families of congruences modulo powers of 7.

1. Introduction

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum equals n. Let $p(n)$ denote the number of partitions of n, with the generating function given by

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{f_1},$$

where, throughout this paper, we define

$$f_r := (q^r;q^r)_\infty = \prod_{m=1}^{\infty}(1-q^{rm}).$$

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Ramanujan [7] conjectured that if $\ell \in \{5, 7, 11\}$ and $0 < \delta_{\ell,\beta} < \ell^{\beta}$ satisfying $24\delta_{\ell,\beta} \equiv 1 \pmod{\ell^{\beta}}$, the partition function satisfies the congruence

$$p\left(\ell^{\beta} n + \delta_{\ell,\beta}\right) \equiv 0 \pmod{\ell^{\beta}}. \tag{1}$$

Watson [10] later proved this conjecture for $\ell = 5$. However, Chowla [4] observed that since $p(243)$ is not divisible by 7^3, this contradicts the conjecture for $\ell = 7$ and $\beta = 3$. The conjecture was subsequently refined and proved by Watson [10], establishing that for $k \geq 1$,

$$p(7^{2k-1}n + \lambda_{2k-1}) \equiv 0 \pmod{7^k}$$

and

$$p(7^{2k}n + \lambda_{2k}) \equiv 0 \pmod{7^{k+1}},$$

where λ_k is the modular inverse of 24 modulo 7^k.

Using the classical identities of Euler and Jacobi, Hirschhorn and Hunt [5] provided a simple proof of (1) for $\ell = 5$. Later, Garvan [6] extended these results to $\ell = 7$ by deriving the generating functions

$$\sum_{n\geq 0} p\left(7^{2k-1}n + \lambda_{2k-1}\right) q^n = \sum_{j\geq 1} x_{2k-1,j}\, q^{j-1} \frac{f_7^{4j-1}}{f_1^{4j}} \tag{2}$$

and

$$\sum_{n\geq 0} p\left(7^{2k}n + \lambda_{2k}\right) q^n = \sum_{j\geq 1} x_{2k,j}\, q^{j-1} \frac{f_7^{4j}}{f_1^{4j+1}},$$

where the coefficient vectors $\mathbf{x}_k = (x_{k,1}, x_{k,2}, \dots)$ are defined recursively by

$$\mathbf{x}_1 = (x_{1,1}, x_{1,2}, \dots) = \left(7, 7^2, 0, \dots\right),$$

and for $k \geq 1$,

$$x_{k+1,i} = \begin{cases} \displaystyle\sum_{j\geq 1} x_{k,j}\, m_{4j,j+i}, & \text{if } k \text{ is odd}, \\ \displaystyle\sum_{j\geq 1} x_{k,j}\, m_{4j+1,j+i}, & \text{if } k \text{ is even}, \end{cases}$$

where the first seven rows of $M = \left(m_{i,j}\right)_{i,j\geq 1}$ are given in Table 1 and Table 2. Moreover, for $i \geq 4$, $m_{i,1} = 0$; for $i \geq 8$, $m_{i,2} = 0$; and for $i \geq 8$, $j \geq 3$,

$$\begin{aligned} m_{i,j} &= 7m_{i-3,j-1} + 35m_{i-2,j-1} + 49m_{i-1,j-1} + m_{i-7,j-2} + 7m_{i-6,j-2} \\ &\quad + 21m_{i-5,j-2} + 49m_{i-4,j-2} + 147m_{i-3,j-2} + 343m_{i-2,j-2} + 343m_{i-1,j-2}. \end{aligned}$$

$i \backslash j$	1	2	3	4	5	6	7
1	7	7^2	0	0	0	0	0
2	10	9×7^2	2×7^4	7^5	0	0	0
3	3	114×7	85×7^3	24×7^5	3×7^7	7^8	0
4	0	82×7	176×7^3	845×7^4	272×7^6	46×7^8	4×7^{10}
5	0	190	1265×7^2	1895×7^4	1233×7^6	3025×7^7	620×7^9
6	0	27	736×7^2	16782×7^3	20424×7^5	12825×7^7	4770×7^9
7	0	1	253×7^2	1902×7^4	4246×7^6	31540×7^7	19302×7^9

Table 1: Values of $m_{i,j}$, where $1 \leq i \leq 7$ and $1 \leq j \leq 7$.

$i \backslash j$	8	9	10	11	12	13	14
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	7^{11}	0	0	0	0	0	0
5	75×7^{11}	5×7^{13}	7^{14}	0	0	0	0
6	7830×7^{10}	1178×7^{12}	111×7^{14}	6×7^{16}	7^{17}	0	0
7	7501×7^{11}	1944×7^{13}	2397×7^{14}	285×7^{16}	22×7^{18}	7^{20}	7^{20}

Table 2: Values of $m_{i,j}$, where $1 \leq i \leq 7$ and $8 \leq j \leq 14$.

Let $p_{1,\ell}(n)$ be the number of 2-color partitions of n where one of the colors appears only in parts that are multiples of ℓ; its generating function is given by

$$\sum_{n \geq 0} p_{1,\ell}(n) q^n = \frac{1}{f_1 f_\ell}. \tag{3}$$

Ahmed, Baruah, and Dastidar [1] discovered new congruences modulo 5, proving that for certain values of t

$$p_{1,\ell}(25n + t) \equiv 0 \pmod{5},$$

where $\ell \in \{0, 1, 2, 3, 4, 5, 10, 15, 20\}$ and $\ell + t = 24$. They further conjectured similar congruences for $\ell = 7, 8, 17$ which were confirmed by Chern [3]. Additionally, Chern

[3] established a new family of congruences:

$$p_{1,4}\,(49n+t) \equiv 0 \pmod{7}, \text{ for } t \in \{11, 25, 32, 39\}.$$

Further generalizations were made by Wang [9], who derived congruences for $p_{1,5}(n)$, strengthening prior results. Ranganatha [8] extended these findings to $p_{1,25}(n)$ and $p_{1,49}(n)$, proving that for all $n, \beta \geq 0$,

$$p_{1,25}\left(5^{2\beta+1}n + \frac{7 \times 5^{2\beta+1} + 13}{12}\right) \equiv 0 \pmod{5^{\beta+1}},$$
$$p_{1,25}\left(5^{2\beta+2}n + \frac{11 \times 5^{2\beta+2} + 13}{12}\right) \equiv 0 \pmod{5^{\beta+2}}$$

and

$$p_{1,49}\left(7^{2\beta+1}n + \frac{5 \times 7^{2\beta+1} + 25}{12}\right) \equiv 0 \pmod{7^{\beta+1}}, \tag{4}$$
$$p_{1,49}\left(7^{2\beta+2}n + \frac{11 \times 7^{2\beta+2} + 25}{12}\right) \equiv 0 \pmod{7^{\beta+2}}. \tag{5}$$

In this paper, we establish congruences for $p_{1,7^k}$ for all positive integers k. The results presented here are generalized and extend those of Ranganatha [8].

The main results are as follows.

Theorem 1. *For each $n, \beta \geq 0$ and $k \geq 1$, we have*

$$p_{1,7^{2k-1}}\left(7^{2k+\beta-1}n + \frac{16 \cdot 7^{2k+\beta-1} + 7^{2k-1} + 1}{24}\right) \equiv 0 \pmod{7^k}, \tag{6}$$
$$p_{1,7^{2k}}\left(7^{2k+2\beta-1}n + \frac{10 \cdot 7^{2k+2\beta-1} + 7^{2k} + 1}{24}\right) \equiv 0 \pmod{7^{k+\beta}}, \tag{7}$$
$$p_{1,7^{2k}}\left(7^{2k+2\beta}n + \frac{22 \cdot 7^{2k+2\beta} + 7^{2k} + 1}{24}\right) \equiv 0 \pmod{7^{k+\beta+1}}, \tag{8}$$

and

$$p_{1,7^{2k}}\left(7^{2k+2\beta+1}n + \frac{(24r+22) \cdot 7^{2k+2\beta} + 7^{2k} + 1}{24}\right) \equiv 0 \pmod{7^{k+\beta+2}}, \tag{9}$$

where $r \in \{3, 4, 6\}$.

It is worth noting that the congruences (4) and (5) correspond to the case $k = 1$ of the congruences (7) and (8), respectively.

2. Preliminaries

In this section, we state some lemmas which play a vital role in proving our results. Let H be the "huffing" operator modulo 7, that is,

$$H\left(\sum p_n q^n\right) = \sum p_{7n} q^{7n}.$$

Lemma 1. *([6, Lemma 3.1 and 3.4]) If $\xi = \frac{f_1}{q^2 f_{49}}$ and $T = \frac{f_7^4}{q^7 f_{49}^4}$, then*

$$H\left(\xi^{-i}\right) = \sum_{j\geq 1} m_{i,j} T^{-j} \tag{10}$$

and

$$H\left(\xi^{-4i}\right) = \sum_{j\geq 1} m_{4i,i+j} T^{-i-j}.$$

Lemma 2. *For all $i \geq 1$, we have*

$$H\left(q^{i+2}\frac{f_7^{4i}}{f_1^{4i+1}}\right) = \sum_{j\geq 1} m_{4i+1,i+j} q^{7j}\frac{f_{49}^{4j-1}}{f_7^{4j}}, \tag{11}$$

$$H\left(q^{i+4}\frac{f_7^{4i}}{f_1^{4i+2}}\right) = \sum_{j\geq 1} m_{4i+2,i+j}\, q^{7j}\frac{f_{49}^{4j-2}}{f_7^{4j}}, \tag{12}$$

and

$$H\left(q^{i}\frac{f_7^{4i}}{f_1^{4i}}\right) = \sum_{j\geq 1} m_{4i,i+j}\, q^{7j}\frac{f_{49}^{4j}}{f_7^{4j}}. \tag{13}$$

Proof. We can rewrite (10) as

$$H\left(q^{2i}\frac{f_{49}^{i}}{f_1^{i}}\right) = \sum_{j\geq 1} m_{i,j}\, q^{7j}\frac{f_{49}^{4j}}{f_7^{4j}}. \tag{14}$$

From (14) and the fact that $m_{4i+1,j} = 0$ for $1 \leq j \leq i$, we have

$$H\left(\left(q^2\frac{f_{49}}{f_1}\right)^{4i+1}\right) = \sum_{j=i+1}^{4i+1} m_{4i+1,j}\, q^{7j}\frac{f_{49}^{4j}}{f_7^{4j}}. = \sum_{j\geq 1} m_{4i+1,i+j}\, q^{7i+7j}\frac{f_{49}^{4i+4j}}{f_7^{4i+4j}},$$

which yields (11). Similarly, we can prove (12) and (13). □

3. Generating Functions

In this section, we establish generating functions for $p_{1,7^\ell}(n)$ within specific arithmetic progressions.

Theorem 2. *For each $\beta \geq 0$ and $k \geq 1$, we have*

$$\sum_{n\geq 0} p_{1,7^{2k-1}}\left(7^{2k+\beta-1}n + \frac{16\cdot 7^{2k+\beta-1} + 7^{2k-1} + 1}{24}\right) q^n = \sum_{i\geq 1} y^{(2k-1)}_{\beta+1,i}\, q^{i-1} \frac{f_7^{4i-1}}{f_1^{4i+1}}, \tag{15}$$

where the coefficient vectors are defined as follows:

$$y^{(2k-1)}_{1,j} = x_{2k-1,j} \quad \text{and} \quad y^{(2k-1)}_{\beta+1,j} = \sum_{i\geq 1} y^{(2k-1)}_{\beta,i} m_{4i+1,i+j}$$

for all $\beta, j \geq 1$.

Proof. From (3), we have

$$\sum_{n\geq 0} p_{1,7^{2k-1}}(n) q^n = \frac{1}{f_{7^{2k-1}}} \sum_{n\geq 0} p(n) q^n. \tag{16}$$

Extracting the terms involving $q^{7^{2k-1}n+\lambda_{2k-1}}$ on both sides of (16) and dividing throughout by $q^{\lambda_{2k-1}}$, we obtain

$$\sum_{n\geq 0} p_{1,7^{2k-1}}(7^{2k-1}n + \lambda_{2k-1}) q^{7^{2k-1}n} = \frac{1}{f_{7^{2k-1}}} \sum_{n\geq 0} p(7^{2k-1}n + \lambda_{2k-1}) q^{7^{2k-1}n}.$$

If we replace $q^{7^{2k-1}}$ by q and use (2), we get

$$\sum_{n\geq 0} p_{1,7^{2k-1}}(7^{2k-1}n + \lambda_{2k-1}) q^n = \sum_{j\geq 1} x_{2k-1,j}\, q^{j-1} \frac{f_7^{4j-1}}{f_1^{4j+1}},$$

which is the case $\beta = 0$ of (15).

We now assume that (15) is true for some integer $\beta \geq 0$. Multiplying by q^3 and then applying the operator H to both sides, by (11), we have

$$\begin{aligned}
&\sum_{n\geq 0} p_{1,7^{2k-1}}\left(7^{2k+\beta}n + \frac{16\cdot 7^{2k+\beta} + 7^{2k-1} + 1}{24}\right) q^{7n+7} \\
&= \sum_{i\geq 1} y^{(2k-1)}_{\beta+1,i}\, H\left(q^{i+2} \frac{f_7^{4i}}{f_1^{4i+1}}\right) \times \frac{1}{f_7} \\
&= \sum_{i\geq 1} y^{(2k-1)}_{\beta+1,i} \sum_{j=1}^{4i+1} m_{4i+1,i+j}\, q^{7j} \frac{f_{49}^{4j-1}}{f_7^{4j+1}} \\
&= \sum_{j\geq 1} \left(\sum_{i\geq 1} y^{(2k-1)}_{\beta+1,i}\, m_{4i+1,i+j}\right) q^{7j} \frac{f_{49}^{4j-1}}{f_7^{4j+1}} \\
&= \sum_{j\geq 1} y^{(2k-1)}_{\beta+2,j}\, q^{7j} \frac{f_{49}^{4j-1}}{f_7^{4j+1}}.
\end{aligned}$$

That is,

$$\sum_{n\geq 0} p_{1,7^{2k-1}}\left(7^{2k+\beta}n+\frac{16\cdot 7^{2k+\beta}+7^{2k-1}+1}{24}\right)q^n=\sum_{j\geq 1} y^{(2k-1)}_{\beta+2,j}\,q^{j-1}\frac{f_7^{4j-1}}{f_1^{4j+1}}.$$

So we obtain (15) with β replaced by $\beta+1$. $\square$

Theorem 3. *For each $\beta\geq 0$, and $k\geq 1$, we have*

$$\sum_{n\geq 0} p_{1,7^{2k}}\left(7^{2k+2\beta-1}n+\frac{10\cdot 7^{2k+2\beta-1}+7^{2k}+1}{24}\right)q^n=\sum_{i\geq 1} y^{(2k)}_{2\beta+1,i}\,q^{i-1}\frac{f_7^{4i-2}}{f_1^{4i}} \quad (17)$$

and

$$\sum_{n\geq 0} p_{1,7^{2k}}\left(7^{2k+2\beta}n+\frac{22\cdot 7^{2k+2\beta}+7^{2k}+1}{24}\right)q^n=\sum_{j\geq 1} y^{(2k)}_{2\beta+2,j}q^{j-1}\frac{f_7^{4j}}{f_1^{4j+2}}, \quad (18)$$

where coefficient vectors are defined as follow:

$$y^{(2k)}_{1,j}=x_{2k-1,j}$$

and

$$y^{(2k)}_{\beta+1,j}=\begin{cases}\sum_{i\geq 1} y^{(2k)}_{\beta,i}m_{4i,i+j} & \text{if } \beta \text{ is odd,}\\ \sum_{i\geq 1} y^{(2k)}_{\beta,i}m_{4i+2,i+j} & \text{if } \beta \text{ is even,}\end{cases}$$

for all $\beta, j\geq 1$.

Proof. Equation (3) can be written as

$$\sum_{n\geq 0} p_{1,7^{2k}}(n)q^n=\frac{1}{f_{7^{2k}}}\sum_{n\geq 0} p(n)q^n.$$

Extracting the terms involving $q^{7^{2k-1}n+\lambda_{2k-1}}$ on both sides of the above equation and dividing throughout by $q^{\lambda_{2k-1}}$, we obtain

$$\sum_{n\geq 0} p_{1,7^{2k}}(7^{2k-1}n+\lambda_{2k-1})q^{7^{2k-1}n}=\frac{1}{f_{7^{2k}}}\sum_{n\geq 0} p(7^{2k-1}n+\lambda_{2k-1})q^{7^{2k-1}n}.$$

If we replace $q^{7^{2k-1}}$ by q and use (2), we get

$$\sum_{n\geq 0} p_{1,7^{2k}}(7^{2k-1}n+\lambda_{2k-1})q^n=\sum_{j\geq 1} x_{2k-1,j}\,q^{j-1}\frac{f_7^{4j-2}}{f_1^{4j}},$$

which is the case $\beta=0$ of (17).

We now assume that (17) is true for some integer $\beta \geq 0$. Multiplying by q and then applying the operator H to both sides, by (13), we have

$$\begin{aligned}
&\sum_{n\geq 0} p_{1,7^{2k}}\left(7^{2k+2\beta}n + \frac{22\cdot 7^{2k+2\beta}+7^{2k}+1}{24}\right) q^{7n+7}\\
&= \sum_{i\geq 1} y^{(2k)}_{2\beta+1,i}\, H\left(q^i \frac{f_7^{4i}}{f_1^{4i}}\right) \times \frac{1}{f_7^2}\\
&= \sum_{i\geq 1} y^{(2k)}_{2\beta+1,i} \sum_{j\geq 1} m_{4i,i+j}\, q^{7j}\, \frac{f_{49}^{4j}}{f_7^{4j+2}}\\
&= \sum_{j\geq 1}\left(\sum_{i\geq 1} y^{(2k)}_{2\beta+1,i}\, m_{4i,i+j}\right) q^{7j}\, \frac{f_{49}^{4j}}{f_7^{4j+2}}\\
&= \sum_{j\geq 1} y^{(2k)}_{2\beta+2,j}\, q^{7j}\, \frac{f_{49}^{4j}}{f_7^{4j+2}},
\end{aligned}$$

which yields

$$\sum_{n\geq 0} p_{1,7^{2k}}\left(7^{2k+2\beta}n + \frac{22\cdot 7^{2k+2\beta}+7^{2k}+1}{24}\right) q^{n} = \sum_{j\geq 1} y^{(2k)}_{2\beta+2,j} q^{j-1} \frac{f_7^{4j}}{f_1^{4j+2}}.$$

Hence, if (17) is true for some integer $\beta \geq 0$, then (18) is true for β.

Suppose that (18) is true for some integer $\beta \geq 0$. Multiplying by q^5 and then applying the operator H to both sides, by (12), we obtain

$$\begin{aligned}
&\sum_{n\geq 0} p_{1,7^{2k}}\left(7^{2k+2\beta+1}n + \frac{10\cdot 7^{2k+2\beta+1}+7^{2k}+1}{24}\right) q^{7n+7}\\
&= \sum_{i\geq 1} y^{(2k)}_{2\beta+2,i} \sum_{j\geq 1} m_{4i+2,i+j} q^{7j}\, \frac{f_{49}^{4j-2}}{f_7^{4j}}\\
&= \sum_{j\geq 1}\left(\sum_{i\geq 1} y^{(2k)}_{2\beta+2,i}\, m_{4i+2,i+j}\right) q^{7j}\, \frac{f_{49}^{4j-2}}{f_7^{4j}}\\
&= \sum_{j\geq 1} y^{(2k)}_{2\beta+3,j}\, q^{7j}\, \frac{f_{49}^{4j-2}}{f_7^{4j}},
\end{aligned}$$

which yields

$$\sum_{n\geq 0} p_{1,7^{2k}}\left(7^{2k+2\beta+1}n + \frac{10\cdot 7^{2k+2\beta+1}+7^{2k}+1}{24}\right) q^{n} = \sum_{j\geq 1} y^{(2k)}_{2\beta+3,j}\, q^{j-1} \frac{f_7^{4j-2}}{f_1^{4j}}.$$

This is (17) with β replaced by $\beta+1$. This completes the proof. $\square$

4. Proof of Congruences

For a positive integer n, let $\pi(n)$ be the highest power of 7 that divides n, and define $\pi(0) = +\infty$.

Lemma 3 ([6], Lemma 5.1)**.** *For each $i, j \geq 1$, we have*

$$\pi\left(m_{i,j}\right) \geq \left[\frac{7j-2i-1}{4}\right].$$

In particular,

$$\pi\left(m_{4i,i+j}\right) \geq \left[\frac{7(i+j)-2(4i)-1}{4}\right] = \left[\frac{7j-i-1}{4}\right],$$
$$\pi\left(m_{4i+1,i+j}\right) \geq \left[\frac{7(i+j)-2(4i+1)-1}{4}\right] = \left[\frac{7j-i-3}{4}\right],$$

and

$$\pi\left(m_{4i+2,i+j}\right) \geq \left[\frac{7(i+j)-2(4i+2)-1}{4}\right] = \left[\frac{7j-i-5}{4}\right].$$

Lemma 4 ([6], Lemma 5.3)**.** *For all $k, j \geq 1$, we have*

$$\pi(x_{1,1}) = 1, \;\; \pi(x_{1,2}) = 2, \;\; \pi\left(x_{2k+1,j}\right) \geq k+1+\left[\frac{7j-4}{4}\right] \tag{19}$$

and

$$\pi\left(x_{2k,j}\right) \geq k+1+\left[\frac{7j-6}{4}\right].$$

Lemma 5. *For all $j, k \geq 1$ and $\beta \geq 0$, we have*

$$\pi\left(y^{(2k)}_{2\beta+1,j}\right) \geq k+\beta+\left[\frac{7j-7}{4}\right] \tag{20}$$

and

$$\pi\left(y^{(2k)}_{2\beta+2,j}\right) \geq k+\beta+1+\left[\frac{7j-7}{4}\right]. \tag{21}$$

Proof. In view of Theorem 3, we have

$$y^{(2k)}_{1,j} = x_{2k-1,j}.$$

From (19), we can see that

$$\pi\left(x_{2k-1,j}\right) \geq k+\left[\frac{7j-7}{4}\right],$$

this shows that (20) holds when $\beta = 0$.

We now assume that (20) is true for some $\beta \geq 0$. Then

$$\begin{aligned}\pi\left(y_{2\beta+2,j}^{(2k)}\right) &\geq \min_{i\geq 1}\left\{\pi\left(y_{2\beta+1,i}^{(2k)}\right)+\pi\left(m_{4i,i+j}\right)\right\}\\ &\geq \min_{i\geq 1}\left\{k+\beta+\left\lceil\frac{7i-7}{4}\right\rceil+\left\lceil\frac{7j-i-1}{4}\right\rceil\right\}\\ &\geq k+\beta+\left\lceil\frac{7j-2}{4}\right\rceil\\ &\geq k+\beta+1+\left\lceil\frac{7j-7}{4}\right\rceil,\end{aligned}$$

which is (21). Now suppose (21) holds for some $\beta \geq 0$. Then,

$$\begin{aligned}\pi\left(y_{2\beta+3,j}^{(2k)}\right) &\geq \min_{i\geq 1}\left\{\pi\left(y_{2\beta+2,i}^{(2k)}\right)+\pi\left(m_{4i+2,i+j}\right)\right\}\\ &\geq \min_{i\geq 1}\left\{k+\beta+1+\left\lceil\frac{7i-7}{4}\right\rceil+\left\lceil\frac{7j-i-5}{4}\right\rceil\right\}\\ &\geq k+\beta+1+\left\lceil\frac{7j-7}{4}\right\rceil,\end{aligned}$$

which is (20) with $\beta+1$ for β. This completes the proof. □

Lemma 6. *For all $j, k \geq 1$ and $\beta \geq 0$, we have*

$$\pi\left(y_{\beta+1,j}^{(2k-1)}\right) \geq k+\left\lceil\frac{7j-7}{4}\right\rceil. \tag{22}$$

Proof. The proof is similar to that of the previous lemma, so we omit the details. □

We now present the proof of Theorem 1.

Proof of Theorem 1. By (22), we see that $y_{\beta+1,j}^{(2k-1)} \equiv 0 \pmod{7^k}$. So, (6) follows from (15). Similarly, congruence (7) follows from (17) and (20), and congruence (8) follows from (18) together with (21). In view of (21), we can express (18) as

$$\begin{aligned}\sum_{n\geq 0} p_{1,7^{2k}}\left(7^{2k+2\beta}n+\frac{22\cdot 7^{2k+2\beta}+7^{2k}+1}{24}\right)q^n &\equiv y_{2\beta+2,1}^{(2k)}\frac{f_7^4}{f_1^6} \pmod{7^{k+\beta+2}}\\ &\equiv y_{2\beta+2,1}^{(2k)}f_7^3 f_1 \pmod{7^{k+\beta+2}}.\end{aligned} \tag{23}$$

From [2, p. 303, Entry 17(v)], we recall that

$$f_1 = f_{49}\times\left(\frac{B(q^7)}{C(q^7)}-q\frac{A(q^7)}{B(q^7)}-q^2+q^5\frac{C(q^7)}{A(q^7)}\right), \tag{24}$$

where

$$A(q) := \frac{f(-q^3;-q^4)}{f(-q^2)}, \quad B(q) := \frac{f(-q^2;-q^5)}{f(-q^2)} \quad \text{and} \quad C(q) := \frac{f(-q;-q^6)}{f(-q^2)}.$$

Using (24) in (23) and then comparing the coefficients of q^{7n+r} with $r \in \{3,4,6\}$, we arrive at (9). $\square$

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