# BONSE'S INEQUALITY ALONG SUBSEQUENCES OF PRIMES

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**Abstract**

Bonse's inequality states that the product of the first $n$ primes exceeds the square of the next prime for all $n \geq 4$. In this note, we extend this question to general increasing sequences $\mathfrak{a} = (a_n)_{n\geq 1}$ of indices, determining growth conditions under which the corresponding subsequence of primes satisfies $p_{a_1} p_{a_2} \cdots p_{a_n} > p_{a_{n+1}}^2$ for all sufficiently large $n$.

## 1. Introduction

Let $(p_n)_{n\geq 1}$ be the sequence of prime numbers. In 1907, Bonse [1] proved the striking inequality

$$p_1 p_2 \cdots p_n > p_{n+1}^2 \qquad (n \geq 4). \tag{1}$$

More generally, one may ask when the primorial

$$P_n := p_1 p_2 \cdots p_n$$

dominates powers of the next prime. Bonse proved, in particular, the cases corresponding to the exponents 2 and 3. Later, Rademacher and Toeplitz [6, Chap. 28] discussed related extensions, and Suzuki [2] proved that, for every integer $k \geq 2$, there exists $n_k$ such that

$$P_n > p_{n+1}^k \qquad (n \geq n_k).$$

Bonse's inequality is simple to state, but it has generated a substantial literature. Classical estimates for primorials and Chebyshev functions go back to Rosser [4], Rosser and Schoenfeld [9], and Hanson [8]. Other proofs, refinements, and variants of Bonse-type inequalities have also appeared in several directions; see, for example, [3, 5, 7, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 23]. Some of these works sharpen

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classical bounds, replace the lower term by other functions of primes, or transport Bonse-type ideas to related settings.

Recent developments include inequalities for inert primes and applications [18], a generalization of Bonse's inequality [21], and a generalization of Schatunowsky's theorem [22]. These works show that Bonse-type inequalities remain a useful and flexible tool in problems involving prime numbers and related arithmetic structures.

The purpose of the present note is complementary to these directions. Instead of considering the complete sequence of primes, we study Bonse's inequality along arbitrary subsequences of primes. Let

$$\mathfrak{a} = (a_n)_{n\geq 1}$$

be a strictly increasing sequence of positive integers. We are interested in the asymptotic validity of

$$p_{a_1}p_{a_2}\cdots p_{a_n} > p_{a_{n+1}}^2. \tag{2}$$

We say that $\mathfrak{a}$ is a *Bonse sequence* if (2) holds for all sufficiently large $n$.

The classical inequality (1) says precisely that $a_n = n$ is a Bonse sequence. The main question is therefore to determine which increasing sequences of indices retain this Bonse-type behavior.

The answer is governed by the growth of

$$A_n := \prod_{k=1}^{n} a_k$$

relative to $a_{n+1}^2$. This is natural in view of the prime number theorem, which gives $p_n \sim n\log n$. Our first result confirms this heuristic by showing that a Bonse-type inequality for the sequence of indices implies the corresponding inequality for the associated subsequence of primes.

**Theorem 1.** *Let $\mathfrak{a} = (a_n)_{n\geq 1}$ be a strictly increasing sequence of positive integers. If*

$$\liminf_{n\to\infty} \frac{\log A_n}{\log a_{n+1}} > 2, \tag{3}$$

*then $\mathfrak{a}$ is a Bonse sequence.*

The condition (3) holds in many natural situations. For example, it is satisfied whenever $\mathfrak{a} = (a_n)$ grows at most exponentially. Indeed, if $a_n \leq \lambda^n$ for some $\lambda > 1$ and all sufficiently large $n$, then

$$\log a_{n+1} = O(n),$$

whereas, since $\mathfrak{a}$ is strictly increasing, $a_k \geq k$ and therefore

$$\log A_n \geq \sum_{k=1}^{n} \log k = \log(n!).$$

Thus $\log A_n/\log a_{n+1} \to \infty$.

On the other hand, there are sequences for which the Bonse-type inequality fails. Our second result gives a simple criterion in this direction.

**Theorem 2.** *Let $\mathfrak{a} = (a_n)_{n\geq 1}$ be a strictly increasing sequence of positive integers. If*

$$\lim_{n\to\infty} \frac{n \log\log a_n}{\log a_{n+1}} = 0 \tag{4}$$

*and*

$$\liminf_{n\to\infty} \frac{\log A_n}{\log a_{n+1}} < 2, \tag{5}$$

*then $\mathfrak{a}$ is not a Bonse sequence.*

The regularity condition (4) ensures that the cumulative contribution of the secondary terms $\log\log a_k$ is negligible relative to $\log a_{n+1}$. In particular, it prevents these secondary terms from dominating the main comparison between $\log A_n$ and $2\log a_{n+1}$.

As an illustration, consider the double-exponential sequence

$$a_n = 2^{2^n} \qquad (n \geq 1).$$

Then

$$\frac{\log A_n}{\log a_{n+1}} \to 1,$$

and condition (4) is easily checked. Hence this sequence is not a Bonse sequence.

We employ standard asymptotic notation. For functions $f$ and $g$, we write $f(n) = O(g(n))$ if $|f(n)| \leq C|g(n)|$ for some constant $C > 0$ and all sufficiently large $n$. We write $f(n) = o(g(n))$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0,$$

and $f(n) \sim g(n)$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

All implied constants may depend on fixed parameters unless otherwise stated.

## 2. Proof of Theorem 1

We now present a proof of Theorem 1.

*Proof of Theorem 1.* To establish that $\mathfrak{a}$ is a Bonse sequence, it suffices to prove that

$$\sum_{k=1}^{n} \log p_{a_k} > 2\log p_{a_{n+1}}$$

for all sufficiently large $n$.

By (3), there exists $\varepsilon > 0$ such that, for all sufficiently large $n$,

$$\log A_n > (2+\varepsilon)\log a_{n+1}. \tag{6}$$

The prime number theorem implies $p_m \sim m \log m$, and hence

$$\log p_m = \log m + \log\log m + O(1).$$

Taking $m = a_{n+1}$, we obtain

$$\log p_{a_{n+1}} = \log a_{n+1} + \log\log a_{n+1} + O(1).$$

Since $a_n \to \infty$, we have

$$\log\log a_{n+1} \le \frac{\varepsilon}{4}\log a_{n+1}$$

for all sufficiently large $n$. Therefore,

$$2\log p_{a_{n+1}} \le \left(2+\frac{\varepsilon}{2}\right)\log a_{n+1} + O(1).$$

On the other hand, using $p_m \ge m$ and (6), we get

$$\begin{aligned}\sum_{k=1}^{n}\log p_{a_k} &\ge \sum_{k=1}^{n}\log a_k\\ &= \log A_n\\ &> (2+\varepsilon)\log a_{n+1}.\end{aligned}$$

For all sufficiently large $n$, this last quantity is greater than $2\log p_{a_{n+1}}$. Hence

$$p_{a_1}p_{a_2}\cdots p_{a_n} > p_{a_{n+1}}^2$$

for all large $n$, as required. $\square$

## 3. Proof of Theorem 2

We now present a proof of Theorem 2.

*Proof of Theorem 2.* By (5), there exist $\varepsilon > 0$ and an infinite set $\mathcal{N} \subseteq \mathbb{N}$ such that

$$\log A_n \le (2-\varepsilon)\log a_{n+1} \tag{7}$$

for all $n \in \mathcal{N}$.

We shall show that (2) fails for all sufficiently large $n \in \mathcal{N}$. As before, the prime number theorem gives

$$\log p_m = \log m + \log\log m + O(1).$$

Thus,

$$\sum_{k=1}^{n} \log p_{a_k} = \sum_{k=1}^{n} \left(\log a_k + \log\log a_k + O(1)\right).$$

Since $\mathfrak{a}$ is increasing, we have

$$\sum_{k=1}^{n} \log\log a_k \le n \log\log a_n = o(\log a_{n+1})$$

by (4). Also, (4) implies $n = o(\log a_{n+1})$, and therefore the total contribution of the $O(1)$ terms is $o(\log a_{n+1})$.

Combining this with (7), we obtain, for $n \in \mathcal{N}$,

$$\sum_{k=1}^{n} \log p_{a_k} \le (2-\varepsilon)\log a_{n+1} + o(\log a_{n+1}).$$

Hence, for all sufficiently large $n \in \mathcal{N}$,

$$\sum_{k=1}^{n} \log p_{a_k} \le \left(2 - \frac{\varepsilon}{2}\right) \log a_{n+1}. \tag{8}$$

On the other hand,

$$2\log p_{a_{n+1}} = 2\log a_{n+1} + o(\log a_{n+1}),$$

so, for all sufficiently large $n$,

$$2\log p_{a_{n+1}} \ge \left(2 - \frac{\varepsilon}{3}\right) \log a_{n+1}. \tag{9}$$

The inequalities (8) and (9) show that

$$\sum_{k=1}^{n} \log p_{a_k} \le 2\log p_{a_{n+1}}$$

for infinitely many $n$. Therefore (2) cannot hold for all sufficiently large $n$, and $\mathfrak{a}$ is not a Bonse sequence. $\square$

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## References


[1] H. Bonse, Über eine bekannte Eigenschaft der Zahl 30 und ihre Verallgemeinerung, *Arch. Math. Phys.* **12** (1907), 292–295.

[2] T. Suzuki, A theorem on the series of prime numbers, *Tohoku Math. J.* **3** (1913), 83–86.

[3] E. Zermelo, Elementare Betrachtungen zur Theorie der Primzahlen, *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. I, N. F.* **1** (1934), 43–46.

[4] B. Rosser, Explicit bounds for some functions of prime numbers, *Amer. J. Math.* **63** (1941), 211–232.

[5] S. Kaniel, Extension of the method of Bonse-Zermelo, *Scripta Math.* **21** (1956), 269–272.

[6] H. Rademacher and O. Toeplitz, *The Enjoyment of Mathematics. Selections from Mathematics for the Amateur*, translated by H. Zuckerman, Princeton University Press, Princeton, 1957.

[7] L. Panaitopol, On an inequality by Bonse, *Gaz. Mat., Bucur., Ser. A* **76** (1971), 100–102.

[8] D. Hanson, On the product of the primes, *Canad. Math. Bull.* **15** (1972), 33–37.

[9] J. B. Rosser and L. Schoenfeld, Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$, *Math. Comp.* **29** (1975), 243–269.

[10] H. Iwata, On Bonse's theorem, *Math. Rep. Toyama Univ.* **7** (1984), 115–117.

[11] J. Sándor, Über die Folge der Primzahlen, *Math. Rev. Anal. Numér. Théor. Approximation* **30** (1988), 67–74.

[12] L. Cseh, Generalized integers and Bonse's theorem, *Stud. Univ. Babeş-Bolyai Math.* **34** (1989), 3–6.

[13] L. Panaitopol, An inequality involving prime numbers, *Publ. Elektrotehn. Fak. Univ. Beograd Ser. Mat.* **11** (2000), 33–35.

[14] M. Hassani, A remark on the Mandl's inequality, preprint, arXiv:math/0606765 [math.NT], 2006.

[15] R. J. Betts, Using Bonse's inequality to find upper bounds on prime gaps, *J. Integer Seq.* **10** (2007), Article 07.3.8, 7 pp.

[16] C. Sanna, A new elementary proof of the inequality $\varphi(n) > \pi(n)$, *Notes Number Theory Discrete Math.* **18** (2012), 35–37.

[17] V. Pambuccian, Schatunowsky's theorem, Bonse's inequality, and Chebyshev's theorem in weak fragments of Peano arithmetic, *Math. Log. Q.* **61** (2015), 230–235.

[18] Z. He, Inequalities for inert primes and their applications, *Int. J. Number Theory* **16** (2020), 1819–1832.

[19] S. Yang and Q. Liao, On the product of continuous prime numbers, *J. Math. Inequal.* **15** (2021), 239–247.

[20] D. Marques and P. Trojovský, Error estimates for a class of continuous Bonse-type inequalities, *Math. Comp.* **91** (2022), 2335–2345.

[21] R. Farhadian and V. Ponomarenko, A generalization of Bonse's inequality, *Amer. Math. Monthly* **132** (2025), 14.

[22] Y. Kaneko and H. Nakai, A generalization of Schatunowsky's theorem, *Amer. Math. Monthly* **132** (2025), 443–447.

[23] D. Marques and P. Trojovsky, Asymptotic error terms in Bonse-type inequalities, preprint, arXiv:2511.13691.