ON GRAPH AUTOMORPHISMS RELATED TO SNORT

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Abstract

We study the outcomes of various positions of the game SNORT. When played on graphs admitting an automorphism of order two that maps vertices outside of their closed neighborhoods (called *opposable* graphs), the second player has a winning strategy. We give a necessary and sufficient condition for a graph to be opposable and prove that the property of being opposable is preserved by several graph products. We provide examples showing that a graph being second-player win does not imply that the graph is opposable, which answers a conjecture of Kakihara. We give an analogous definition to opposability, which gives a first-player winning strategy; we prove a necessary condition for this property to be preserved by the Cartesian and strong products. As an application of our results, we determine the outcome of SNORT when played on various $n \times m$ chess graphs.

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1. Introduction

Two farmers share a number of fields. One farmer would like to use the fields for their bulls, and the other for their cows. They both agree that no bulls should be adjacent to any cows. They decide that they will take turns in picking fields. This is the game of SNORT, devised by Simon Norton, and first written about in [7, pp. 91–92] and [4, pp. 145–147] in roughly the same way we have just described. Being competitive farmers (or perhaps unfriendly), they each want to make sure that they have the most fields, restricting the choice of the other farmer. As such, if a farmer, on their turn, has no field to pick (either because every field has been claimed already, or because picking an unclaimed field would place bulls next to cows), then they are said to lose. To the initiated reader, this is the *normal play* convention in combinatorial game theory. This is evidently a two-player, combinatorial game.

We will not talk of bulls, cows, and fields when we play this game. Instead, as is typically done, we will consider a game of SNORT to be played on a finite, simple graph. Enforcing the graph to be simple is reasonable, since adding loops or multiple edges would not affect the play of the game in any way, as the reader may confirm to themselves. Enforcing finiteness, however, does change things; the brave reader may wish to investigate the transfinite case.

The two players, Left and Right, will alternate between coloring uncolored vertices blue and red, respectively. We say that a vertex is *tinted* blue (respectively, red) if it is adjacent to a blue (respectively, red) vertex but is not itself colored. Since Left colors vertices blue, Left is never allowed to color a vertex that is tinted red. Similarly, Right is never allowed to color a vertex that is tinted blue. The first player who is unable to color a vertex on their turn loses the game. Note that we assume all vertices of a graph start untinted unless otherwise specified.

The attentive reader may ask whether the graph should be planar, given that this would be the layout of the fields in the original description, and indeed this is how the game was first played and analyzed (see, for example, [7, p. 91]). As such, the ruleset we have just described should be considered a generalization of SNORT, but it is a standard one.

In this paper, we analyze the outcomes of SNORT, starting with graphs in which the second player has a winning strategy. To do this, we make use of a structural property of graphs, introduced by Kakihara [11], to give a player a strategy in which they can "mirror" their opponent's previous move (much in the same vein as a typical *Tweedledum and Tweedledee* argument [4, p. 3]). A graph G is *opposable* if it contains an automorphism f of order two such that $f(v) \notin N[v]$ for all $v \in V(G)$. We will say that a graph is *non-opposable* if it is not opposable. Such an automorphism f is called an *opposition* of G. It is known that the second player wins SNORT on an opposable graph (see [11, Proposition 3.6 on p. 24] or Section 2 for a proof). Kakihara [11] and Arroyo [1] both independently studied such graphs, finding various nice families. Among his results, Arroyo [1] showed that the property of being opposable is preserved by the Cartesian product. In Section 2, we further develop Arroyo and Kakihara's works by finding new properties and behaviors of opposable graphs, including a generalization of Arroyo's graph product result.

Although opposability necessarily implies that the second player wins, it is unknown exactly how having a winning strategy for the second player and being opposable differ. That is, for which non-opposable graphs does the second player win? Kakihara conjectured in [11] that a graph is opposable if and only if the second player has a winning strategy. In Section 3, we show that this conjecture is false, and do so by giving infinite families of counterexamples.

Arroyo [1] also defined a parallel concept to opposability, trying to begin to classify graphs for which the first player wins SNORT. In Section 4, we generalize this definition to what we call *almost opposable graphs*, and explore how this property behaves with respect to various graph products.

In Section 5, we demonstrate an application of our results to a natural family of games involving placing chess pieces onto a chessboard such that no two pieces can attack each other. These types of chess problems have been studied for centuries, particularly for placing queens. For a list of results, a list of real-world applications, and a summary of the history, we direct the reader to the excellent survey by Bell and Stevens [3]. The sequence-obsessed reader may be aware that the nth number in the sequence A250000 in the OEIS is the maximum value m such that m white queens and m black queens can be placed on an $n \times n$ chessboard such that no two queens of opposite color can attack each other. See [16, 20] for some relevant results on sequence A250000. We consider a gamified, adversarial version of the sequence A250000, which we call the Peaceable Queens Game, where two players are each assigned a color and take turns placing a queen of their assigned color on a square of an $n \times m$ chessboard. This turns out to be a special case of SNORT. We determine the outcome of the Peaceable Queens Game for chessboards with at least one odd side. We also determine the outcomes of analogous peaceable games played with other chess pieces.

Finally, in Section 6, we end with some open questions and directions for future work. For graph theorists unfamiliar with combinatorial game theory, we suggest Siegel's textbook [15].

2. Opposable Graphs

For completeness, we begin with Kakihara's proof [11] that the second player wins SNORT on opposable graphs—note that Kakihara in fact cites a preprint by Stacey Stokes and Mark D. Schlatter, but that work does not seem to have appeared anywhere, and the authors could not be reached. The technique used in this proof will be utilized throughout the paper.

Lemma 1 ([11, Proposition 3.6 on p. 24])**.** *If G is a graph that admits an opposition f, then playing* SNORT *on G is second-player win.*

Proof. We will show by induction that a winning strategy for the second player is to always respond with $f(v)$ when the first player plays on v. On the first move, since f is an opposition, the second player responding by coloring $f(v)$ is a valid move.

Suppose that play continues in this way, but that at some point, when the first player plays to some u, the second player cannot respond to $f(u)$. The only way that the second player would be unable to play $f(u)$ is if either $f(u)$ is already colored, or $f(u)$ is adjacent to some vertex w that the first player already claimed.

Suppose first that $f(u)$ is already colored. If $f(u)$ was colored by the second player, then, since f is an automorphism, it must be because the first player played u at some earlier point in the game. However, this contradicts the fact that u was just played by the first player. Otherwise, if $f(u)$ was colored by the first player, then the second player would have responded at the time on $f(f(u)) = u$. This contradicts the fact that the first player just played on u.

It must therefore be the case that $f(u)$ is adjacent to some vertex w that the first player already claimed. When the first player played on w, the second player would have responded to $f(w)$. Since f is an automorphism and $\{f(u), w\}$ is an edge, $\{f(f(u)), f(w)\}$ is an edge. Since f has order two, we obtain that $\{u, f(w)\}$ is an edge. This contradicts the fact that the first player just played on u since the first player's move would be adjacent to the earlier move $f(w)$ by the second player.

Thus, whenever the first player makes a move, the second player will be able to respond. So the player to run out of moves first will be the first player. Therefore, G is second-player win. □

Next, we establish a way of showing that graphs that partition into opposable graphs are themselves opposable. To do this, we introduce a definition involving matchings. We will see in Theorem 1 that this new definition gives a structural characterization of graphs that admit an opposition.

Definition 1. A perfect matching M is an *opposition matching* if for any two edges $v_1v_2 \neq v_3v_4$ in M, the graph H induced by $\{v_1, v_2, v_3, v_4\}$ is isomorphic to either K_4, C_4, or just the two edges of M.

Theorem 1. *A graph G has an opposition if and only if the complement $\overline{G}$ has an opposition matching.*

Proof. Suppose first that f is an opposition of G. Then in $\overline{G}$ we construct a matching M by matching v to $f(v)$. Since f has order two and no fixed points, this is well-defined. Since oppositions match vertices to non-adjacent vertices, they must be adjacent in the complement, and thus are a valid edge of a matching. Since f is an automorphism, we get that it is a bijection, and thus M is a perfect matching. Let v_1v_2 and v_3v_4 be two edges of M.

If v_1v_3 is an edge in G, then so too is v_2v_4 because f is an automorphism. By the contrapositive, if v_2v_4 is an edge in $\overline{G}$, then v_1v_3 is an edge in $\overline{G}$. Because f is an order two automorphism, this is an if and only if argument. The edge pair v_1v_3 and v_2v_4 either both exist, or both do not exist. Similarly, the edge pair v_1v_4 and v_2v_3 either both exist, or both do not exist. If no edge pairs exist in $\overline{G}$, then the subgraph H induced by $\{v_1, v_2, v_3, v_4\}$ is just $v_1v_2 + v_3v_4$. If exactly one edge pair exists in $\overline{G}$, then $H \cong C_4$, and if both edge pairs exist in $\overline{G}$, then $H \cong K_4$.

Conversely, suppose that $\overline{G}$ has an opposition matching M. Define $f\colon V(G) \to V(G)$ by mapping $v \in V(G)$ to the matched vertex in M. We claim f is an opposition in G. Let v_1v_2 be an edge of G, and let $f(v_1)f(v_2) = v_3v_4$. Then v_1v_3 and v_2v_4 are edges in M. Since M is an opposition matching, the subgraph H induced in $\overline{G}$ is isomorphic to either K_4, C_4, or just the two edges.

- If $H \cong K_4$, then $v_1v_2 \notin E(G)$, and this contradicts $v_1v_2 \in E(G)$ assumed.
- If $H \cong C_4$, then $v_1v_2 \in E(G)$ implies that the two edges not in H are v_1v_2 and v_3v_4. Thus, v_3v_4 is an edge of G.
- If H is just $v_1v_3 + v_2v_4$, then v_3v_4 is not an edge of H, and so $v_3v_4 \in E(G)$.

It follows that f is an edge-preserving function. Furthermore, f is a bijection because it was constructed through a perfect matching. Thus, f is an automorphism. Because M is a perfect matching, the function f has order two and has no fixed points. Since f is based on a perfect matching, the edge $vf(v)$ always exists in $\overline{G}$, and thus does not exist in G. Thus, the automorphism does not map vertices to adjacent vertices. So, f is an opposition of G. □

Now we can reason about graphs that nicely partition. We use $2K_2$ to denote the graph on 4 vertices made up of two disjoint edges and $4K_1$ to be the empty graph on 4 vertices.

Theorem 2. *If G can be vertex-partitioned into sets $V_1, \ldots, V_k$ such that the subgraph induced by V_i is isomorphic to an opposable graph H_i, and for any pair of paired vertices, $\{u_i, v_i\} \subseteq V_i$ and $\{u_j, v_j\} \subseteq V_j$, the set $\{u_i, v_i, u_j, v_j\}$ induces either a C_4, a $2K_2$, or a $4K_1$, then G is opposable.*

Proof. Let G be partitioned into V_i and let $f_i\colon V_i \to V(H_i)$ be an isomorphism between the subgraph induced by V_i and H_i. Let $\varphi_i\colon V(H_i) \to V(H_i)$ be an opposition on H_i. We create a function $\psi\colon V(G) \to V(G)$ by mapping $v \in V_i$ as $\psi(v) = f_i^{-1}(\varphi_i(f_i(v)))$ for each $i \in \{1, \ldots, k\}$ and claim it is an opposition. Note that ψ has no fixed points since φ_i has no fixed points and f_i is an isomorphism. It

Name	Symbol	Corresponding $\varphi(g_1, g_2, h_1, h_2)$
Cartesian	$G \square H$	$(g_1 = g_2 \wedge h_1h_2 \in E(H)) \vee$ $(g_1g_2 \in E(G) \wedge h_1 = h_2)$
Strong	$G \boxtimes H$	$(g_1 = g_2 \wedge h_1h_2 \in E(H)) \vee$ $(g_1g_2 \in E(G) \wedge h_1 = h_2) \vee$ $(g_1g_2 \in E(G) \wedge h_1h_2 \in E(H))$
Tensor	$G \times H$	$g_1g_2 \in E(G) \wedge h_1h_2 \in E(H)$
Lexicographic	$G \bullet H$	$g_1g_2 \in E(G) \vee (g_1 = g_2 \wedge h_1h_2 \in E(H))$
Co-normal	$G * H$	$g_1g_2 \in E(G) \vee h_1h_2 \in E(H)$
Homomorphic	$G \ltimes H$	$g_1 = g_2 \vee$ $(g_1g_2 \in E(G) \wedge \neg (h_1h_2 \in E(H)))$
Cihpromomoh	$G \rtimes H$	$h_1 = h_2 \vee$ $(h_1h_2 \in E(H) \wedge \neg (g_1g_2 \in E(G)))$

Table 1: Common graph products written in the notation of Definition 2.

also follows from the construction that ψ has order two, and does not map vertices to adjacent vertices, otherwise φ_i would not be an opposition on H_i.

It remains to show that ψ preserves edges so that it is a proper automorphism. Since φ_i is an automorphism, we find that edges within V_i are preserved. If two vertices $v_i \in V_i$ and $v_j \in V_j$ with $V_i \neq V_j$ are not adjacent (and $v_i\psi(v_j) \notin E(G)$), then $\{v_i, v_j, \psi(v_i), \psi(v_j)\}$ induces a $4K_1$. Suppose instead that v_iv_j is an edge between V_i and V_j, with $i \neq j$. Let u_i, u_j be the corresponding paired vertices. Then, by the assumptions made in the statement of the theorem, $\{u_i, v_i, u_j, v_j\}$ induces either a C_4 or a $2K_2$ (a $4K_1$ is not possible as the edge v_iv_j is assumed to exist), and in either case there must be an edge u_iu_j. □

Definition 2. Let $\varphi(g_1, g_2, h_1, h_2)$ be any logical formula consisting of atomics $g_1g_2 \in E(G)$, $h_1h_2 \in E(H)$, $g_1 = g_2$, $h_1 = h_2$, and connectives $\wedge$, $\vee$, $\neg$, and $\Rightarrow$. We abuse graph product notation and define $G\varphi H$ to be the graph on vertices $V(G) \times V(H)$ where $(g_1, h_1)(g_2, h_2) \in E(G\varphi H)$ if and only if $\varphi(g_1, g_2, h_1, h_2)$.

We provide some definitions of common graph products in terms of this notation in Table 1.

For a given graph product φ and vertex $v \in V(H)$, the graph $G.v \subseteq G\varphi H$ is defined as the subgraph induced by the set of vertices $\{(u, v) \mid u \in V(G)\}$. The subgraph $u.H$ is defined similarly. For more on graph products we direct the reader to [10].

We can use Theorem 2 to prove that graph products preserve opposability. However, it is cleaner to show this by working directly from the definition of a graph product. To denote that two vertices x and y are adjacent (respectively, not adjacent), we write $x \sim y$ (respectively, $x \not\sim y$). When we want to specify the graph G in which the vertices x and y are adjacent, we write $x \sim_G y$.

Theorem 3. *If φ satisfies*

$$((h_1 = h_2) \wedge \neg(g_1g_2 \in E(G))) \Rightarrow \neg\varphi(g_1, g_2, h_1, h_2)$$

and G is opposable, then $G\varphi H$ is opposable.

Proof. Let f be an opposition of G. Define $\hat{f}\colon V(G\varphi H) \to V(G\varphi H)$ by $\hat{f}((g, h)) = (f(g), h)$. Since f is an order two bijection with no fixed points, $\hat{f}$ is also an order two bijection with no fixed points. Let $(g, h) \in V(G\varphi H)$. Then $\hat{f}((g, h)) = (f(g), h)$. Since $gf(g) \notin E(G)$ and $h = h$, we obtain that $\neg\varphi(g, f(g), h, h)$, and so $(g, h) \not\sim_{G\varphi H} \hat{f}((g, h))$. Finally, let $(g_1, h_1)(g_2, h_2)$ be an edge in $G\varphi H$. Note that h_1, h_2 are unchanged by $\hat{f}$, and $\hat{f}$ changes g_1 and g_2 to $f(g_1)$ and $f(g_2)$, respectively. Since f is an automorphism, $g_1 = g_2$ if and only if $f(g_1) = f(g_2)$, and $g_1g_2 \in E(G)$ if and only if $f(g_1)f(g_2) \in E(G)$. Thus, $\varphi(g_1, g_2, h_1, h_2)$ is satisfied if and only if $\varphi(f(g_1), f(g_2), h_1, h_2)$ is satisfied. Since by assumption $(g_1, h_1)(g_2, h_2) \in E(G\varphi H)$, we obtain that $\hat{f}$ is a homomorphism. Thus, $\hat{f}$ is an opposition, and $G\varphi H$ is opposable. □

Corollary 1. *If G is opposable and H is any graph, then the Cartesian product $G \square H$, the strong product $G \boxtimes H$, the tensor product $G \times H$, the lexicographic product $G \bullet H$, the co-normal product $G * H$, and the homomorphic product $G \ltimes H$ are all opposable.*

Proof. The proof is by checking the definitions of each of the graph products against the condition for φ in order to apply Theorem 3. □

To see that the conditions in Theorem 3 are necessary, we examine the graphs of Figure 1. Note that $C_4 \ltimes P_3$ meets the conditions of the theorem, and is thus opposable. However, $P_3 \ltimes C_4$ does not meet the conditions of the theorem, and is not opposable since the four vertices of degree five form a clique (thus disallowing the existence of an opposition). This example can be interpreted as showing that some condition on φ is necessary, as this graph could instead be defined by swapping G and H (and the graph product accordingly) to obtain the graph product $C_4 \rtimes P_3$ which fails the φ condition, or by interpreting it directly and noting that it is necessary that G is opposable.

3. Non-opposable Second-Player Win Graphs

Kakihara [11] conjectured that the second player wins SNORT on a graph if and only if that graph is opposable. We show that this conjecture is false by giving a number of counterexamples and constructions. First, we show the example with the least number of vertices, as determined by a simple search.

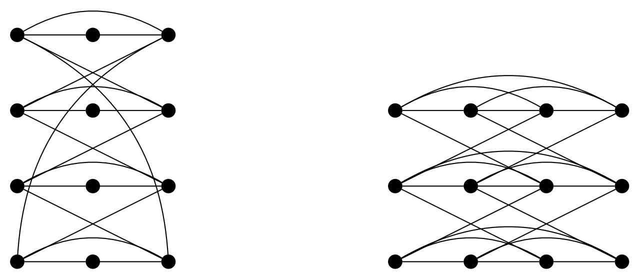

Figure 1: Opposable $C_4 \ltimes P_3$ and non-opposable $P_3 \ltimes C_4 \cong C_4 \rtimes P_3$.

Consider the graph $P_3 \cup C_3$ illustrated in Figure 2. We begin by showing that $P_3 \cup C_3$ is not opposable. Since there are only two pendants in $P_3 \cup C_3$, every automorphism of $P_3 \cup C_3$ either maps the two pendants to themselves or to each other. Since automorphisms preserve adjacency, every automorphism of $P_3 \cup C_3$ has the vertex of degree two in the P_3 map to itself. Thus, an opposition of $P_3 \cup C_3$ cannot exist since an automorphism of $P_3 \cup C_3$ would be forced to have a fixed point.

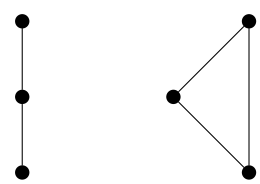

Figure 2: The smallest non-opposable graph that is second-player win.

Next we show that $P_3 \cup C_3$ is second-player win. Left has two options for her first move. Either Left can play on the P_3 or the C_3. Suppose that Left plays on the C_3 for her first move. This reserves two vertices for Left that Right cannot play on. In response, Right can play on the vertex of degree two in the P_3. This reserves two vertices for Right that Left cannot play on. Therefore, both players have exactly two moves remaining with Left going first. Thus, Right wins. If instead Left played her first move on the P_3, she can either play on a leaf or on the vertex of degree two. Regardless of which vertex she plays on first, Right can play on any vertex in the C_3, which preserves two vertices for himself that Left cannot play on. Therefore, both players have two moves remaining with Left going first and so Right wins.

Next, we consider connectedness as a natural constraint on our graphs. Figure 3 illustrates the five connected graphs on seven vertices that are non-opposable and second-player win. By a computer search, among all connected, second-player win, non-opposable graphs, the five graphs in Figure 3 are all of the graphs with the least vertices. Since all of the graphs in Figure 3 have an odd number of vertices, they are not opposable. In the following example, we show that one of the graphs is second-player win. A similar proof can be done for the other four graphs.

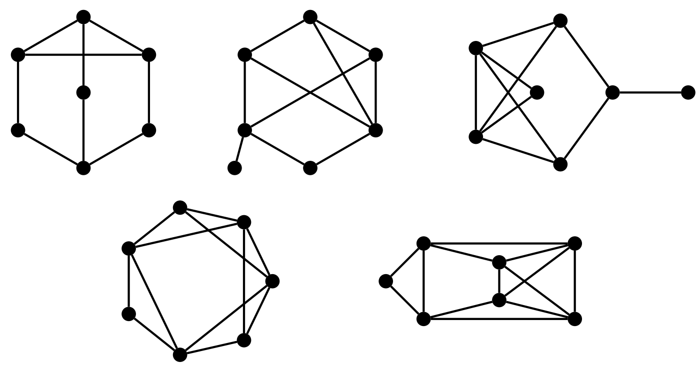

Figure 3: The five non-opposable, connected graphs of order seven that are second-player win.

Let G be the graph in Figure 4. Up to the symmetry of the graph, Left has five options for her first move: v_1, v_2, v_3, v_4, and v_7.

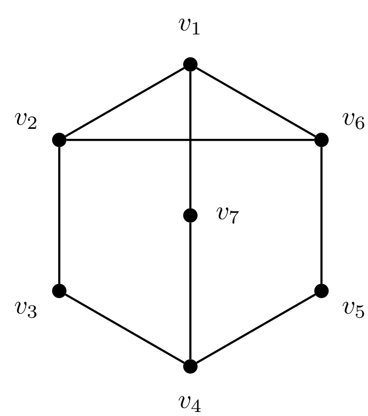

Figure 4: The graph with the fewest edges among all second-player win, connected graphs with seven vertices.

Suppose that, on her first turn, Left colors v_1. In response, Right can color v_4. If Left colors v_2 on her second turn, then Right can color v_5 to win the game. If instead Left colors v_6 on her second turn, then Right can color v_3 to win the game. The analogous strategy works if Left begins with v_4.

Suppose that, on her first turn, Left colors v_2. In response, Right can color v_4. If Left colors v_1 on her second turn, then Right can color v_5 to win the game. If instead Left colors v_6 on her second turn, then Right can color v_7 to win the game.

Again, the analogous strategy works if Left begins with v_3.

Suppose that, on her first turn, Left colors v_7. In response, Right can color v_2. If Left colors v_4 on her second turn, then Right can color v_6 to win the game. If instead Left colors v_5 on her second turn, then Right can color v_3 to win the game.

Therefore, regardless of which vertex Left colors on her first turn, Right can still win the game. So, this graph is second-player win.

As a further natural constraint on our graphs, Figure 5 illustrates two trees. As determined by a computer search, out of all second-player win trees that have an even (respectively, odd) number of vertices, the tree with 12 (respectively, 15) vertices in Figure 5 is the smallest in terms of number of vertices. Note that no tree is opposable, as proven by Kakihara [11, Theorem 4.3 on p. 49].

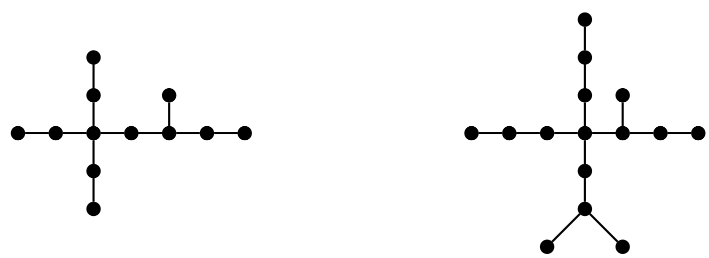

Figure 5: The unique smallest tree that is second-player win with an even number of vertices (left) and an odd number of vertices (right).

Next, we give examples of infinite families of graphs that are not opposable but are second-player win. First we construct the infinite family $\{G_{2n+1}\}_{n=2}^{\infty}$. For a fixed $n \geq 2$, begin with a K_{2n+1}. Take any two vertices u and v of the K_{2n+1} and subdivide the edge uv by adding a vertex x. Attach $2n-2$ pendants to x to complete the construction of G_{2n+1}. Figure 6 illustrates G_5, the graph in the family $\{G_{2n+1}\}_{n=2}^{\infty}$ that has the fewest number of vertices.

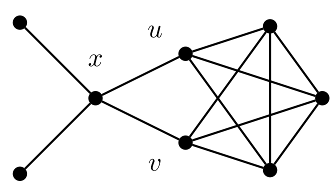

Figure 6: The graph G_5 on eight vertices; it is not opposable but second-player win.

To see that every graph in $\{G_{2n+1}\}_{n=2}^{\infty}$ is non-opposable, note that for any $n \geq 2$, G_{2n+1} contains exactly one vertex x that is adjacent to a pendant. Therefore, every automorphism of G_{2n+1} maps x to itself and so G_{2n+1} is not opposable. Next, we show that the second player always wins SNORT on any graph in $\{G_{2n+1}\}_{n=2}^{\infty}$.

Fix $n \geq 2$, with u, v and x as defined in the construction. Let S be the set of vertices in the K_{2n+1} that are not u, v, or x. There are four cases for Left's first move.

Suppose that, on her first move, Left colors a vertex in S. Right can respond by coloring x. This results in the $2n-2$ pendants being accessible to Right but not Left and $2n-2$ vertices in S accessible to Left but not Right. Since both players have $2n-2$ moves remaining with Left going first, Right wins.

Suppose that, on her first move, Left colors u or v. In response, Right can color v or u. Afterwards, the only vertices available to both players are the $2n-2$ pendants. Since there are an even number of moves left in the game with Left going first, Right wins.

Suppose that, on her first move, Left colors x. Right can respond by coloring any of the vertices in S. This results in a similar position to the first case, with each player having $2n-2$ moves remaining with Left going first. So Right wins.

For the final case, suppose that, on her first move, Left colors a pendant. Right can follow the same strategy as when Left began with coloring x, by coloring any of the vertices in S. From this position, Left has at most $2n-2$ moves remaining and Right has at least $2n-2$ moves remaining. Since Left moves first, Right wins.

Next, we build a tool called the *firework* of a graph, which will enable us to construct infinitely many second-player win graphs from a given second-player win graph. In particular, we use it to construct an infinite family of trees that are second-player win.

Definition 3. If G is a graph that is second-player win, then we call a pair of vertices $(u, v) \in V(G)$ a *response pair* if the game resulting from the first player playing on u, and the second player responding on v, is second-player win.

Note that, if (u, v) is a response pair, then so too is (v, u). This is because the game of SNORT on an (untinted) graph must be a symmetric form; i.e. it is isomorphic to its conjugate ($G \cong \overline{G}$).

We note that response pairs may provide a way to generalize Kakihara's conjecture that graphs are second-player win if and only if they have an opposition. In essence, if the second player has a strategy which only looks at the most recent move and not the full history, does this suffice to make the graph opposable?

Open Problem 1. If there is a set of response pairs (second-player limited history winning strategies) $R = \{(u_i, v_i)\}$ such that all $u \in V(G)$ have some response pair $(u, v_i) \in R$, is it necessary and sufficient that G is opposable? Are response pairs the correct definition to capture that the second player only looks at the previous move?

Observing the opposable graph $4K_1$ may enable a set of response pairs in a 4-cycle, we note that an opposition may not be immediately extracted, yet the graph remains opposable.

Definition 4. If G is a graph that is second-player win, and (u, v) is a response pair such that $\mathrm{d}(u, v) \geq 3$, then we construct the *n-firework* of G corresponding to the pair (u, v) by adding n leaves to u and n leaves to v.

Using response pairs, we may also construct an infinite family of trees where the second player wins. In Definition 4, one might like to refer to the vertices u and v as the *shells*, and the leaves attached to u and v as the *confetti* (a single leaf in particular would be a *confetto*).

Lemma 2. *If G is a graph such that the neighbors of u are a subset of the neighbors of v, then, when playing* SNORT *on G, it is always at least as good to play on v as it is to play on u.*

Proof. Suppose, without loss of generality, that at some moment in the game Left has the opportunity to play on either u or v. We must show that, if she has a winning strategy playing on u, then she also has a winning strategy playing on v.

Notice that playing on u removes u and tints its open neighborhood blue, and similarly for playing on v. However, every neighbor of u is also a neighbor of v, and so the move on v is at least as advantageous for Left as the move on u; the only possible difference between the graph obtained by moving on v and the graph obtained by moving on u is that the former may have more vertices tinted blue, which can only help Left. □

Theorem 4. *If the second player wins* SNORT *on a graph G with a response pair (u, v) of distance at least 3, then the second player wins* SNORT *on the n-firework of G for all $n \geq |V(G)|$.*

Proof. Consider playing the n-firework F of G constructed from the response pair (u, v). Since this is a symmetric form, we need only show that Right wins playing second.

If Left plays on u, then Right can win by playing on v since (u, v) is a response pair. Similarly, if Left plays on v, then Right can win by playing on u. So suppose now that Left plays on neither u nor v. Since $\mathrm{d}_G(u, v) \geq 3$, from the construction of a firework we have that $\mathrm{d}_F(u, v) \geq 3$. So, assume now that Left plays on neither u nor v, and assume without loss of generality that Right responds with v.

By Lemma 2, we may assume that Left responds either on u or on some other vertex of G; in particular, Left does not play on a confetto. If Left plays on u, then Right can resume the second player winning strategy, since the order of Left's moves does not matter. Otherwise, Right can play either on u or some confetto of u. In either case, what results is necessarily the disjunctive sum of a subposition G' of the original game G, a sum of copies of $*$ (from the confetti, if both players play adjacent to u), and the negative integer $-n$ (from the leaves of v).

Since $|V(G)| \leq n$, the subposition G' of G has value $G' \leq n - 1$ (some vertices of G have been removed). Thus, Right wins the resulting graph. □

In many cases, Theorem 4 could be improved by allowing n to in fact be significantly smaller than $|V(G)|$, but we have no need for such increased power here; we already have all we need to find our second-player win trees.

Corollary 2. *There exists an infinite number of trees on which the second player wins* SNORT.

Proof. Consider the tree in Figure 7. It is a straightforward calculation to show that it is second-player win and has a response pair of distance three. Thus, by Theorem 4, we have the result; notice that the firework construction only adds leaves, and so every firework of a tree is itself a tree. □

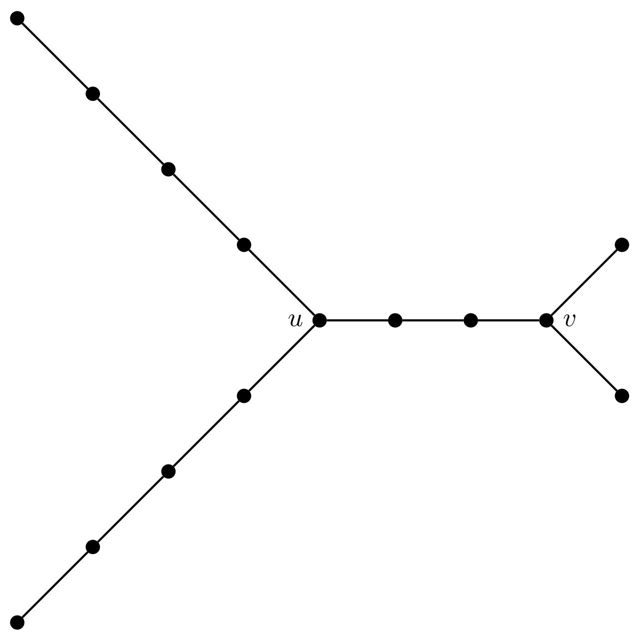

Figure 7: A second-player win tree with a response pair (u, v) of distance 3.

We can use a similar trick to create untinted, second-player win graphs from tinted, second-player win graphs. The trick is to add a new vertex and make it adjacent to all the blue-tinted vertices, and add another vertex and make it adjacent to all the red-tinted vertices. Then, as in the firework construction, add many leaves to each of these two new vertices. If one adds a sufficiently large number of leaves, then the untinted resulting graph must be second-player win.

4. Almost Opposable Graphs

In this section, we define a graph property analogous to opposability where the player moving first is able to use a "mirroring" strategy to win.

Definition 5. Let G be a graph. We say that a set $S \subseteq V(G)$ is *admissible* if $u \in S \subseteq N[u]$ for some $u \in V(G)$. If there exists an admissible set S such that $G - S$ is opposable then we say that G is *almost opposable.*

Definition 5 is a generalization of one of the automorphisms studied by Arroyo [1]. In his thesis, Arroyo considered automorphisms of order two on graphs with an odd number of vertices such that these automorphisms had exactly one fixed point and, besides the fixed point, the automorphism mapped vertices outside of their closed neighborhoods. Graphs with such automorphisms satisfy our definition of being almost opposable because deleting the vertex that is the fixed point of the automorphism yields a graph that has an opposition.

Now we show that the first player wins on almost opposable graphs.

Lemma 3. *If G is almost opposable, then the first player wins* SNORT *on G.*

Proof. Let v be a vertex in G such that $G - S$ is opposable for some $v \in S \subseteq N[v]$. Fix S to be any such subset of $N[v]$. Let f be an opposition of $G - S$. The first player can win by implementing the following strategy. On her first move, the first player colors v. By doing this, every vertex that the second player colors must be in $G - N[v]$. Consequently, every vertex the second player colors is in the subgraph $G - S$. Since $G - S$ is opposable, whenever the second player colors u in $G - S$, the first player can respond by coloring $f(u)$. By the same inductive argument as in the proof of Lemma 1, the first player can continue this strategy until the second player runs out of moves. □

The converse of Lemma 3 does not hold. By considering brooms, we can obtain infinitely many trees where the first player wins, but the graphs are not almost opposable.

A broom graph $B(n, m)$ is a path on n vertices with m leaves attached to one end. In particular, the longest path in $B(n, m)$ has $n + 1$ vertices. We claim that brooms are not almost opposable when $n \geq 5$ and $m \geq 3$, and also when $n \geq 8$ and $m = 2$.

If $n \geq 5$ and $m \geq 3$, then the vertex v adjacent to the m leaves has degree at least four. Any vertex deletions that do not include v can remove at most one vertex adjacent to v. So the resulting graph would have a single vertex of degree at least three and therefore no opposition. Deleting an admissible set that includes v must leave a path on at least two vertices, which is not opposable since any automorphism will have either a fixed point or adjacent vertices mapped to each other at the center of the path.

If $n = 5$, 6, or 7 and $m = 2$, then it is possible to remove one, two, or three path vertices, respectively, to yield a $P_3 \cup P_3$, which allows an opposition. If $n \geq 8$ and $m = 2$, then the broom is not almost opposable since the deletion of any admissible set will leave either a single vertex of degree greater than two, the disjoint union of two paths of different lengths, a single path, or a path plus two isolated vertices. Observing that none of these is opposable, we conclude $B(n, 2)$ is not almost opposable for $n \geq 8$.

To show that the first player wins on infinitely many brooms, we can use the *temperature* of SNORT played on a path. For the definition of the temperature of a combinatorial game we direct the reader to [9]. Suppose that the first player makes their initial move on the end of the path adjacent to the m leaves. This leaves a path on $n - 1$ vertices to play on, with one end only available to the first player. Since the temperature of paths in SNORT is at most 7.5 [9], with the second player having the next move they can reserve no more than 8 vertices in optimal play. Therefore, if $m \geq 9$ and $n \in \mathbb{Z}^+$, then the first player wins on $B(n, m)$.

Unlike opposability, graph products do not behave quite as well with almost opposability. A stronger condition can instill better behavior.

Definition 6. Let G be an almost opposable graph. If there exists an automorphism of order two $\alpha\colon V(G) \to V(G)$, an admissible set S, and an opposition f of $G - S$ such that $\alpha(x) = f(x)$ for all $x \in V(G - S)$, then we say that f is a *compatible* opposition. If $G - S$ has a compatible opposition for some admissible set S then we say that G is *compatibly* almost opposable.

Theorem 5. *If G and H are compatibly almost opposable graphs then $G \boxtimes H$ is compatibly almost opposable.*

Proof. Since G and H are compatibly almost opposable, let g and h be compatible oppositions of G and H, respectively. There must exist a vertex $u \in V(G)$ and a set $S_G \subseteq N_G[u]$ such that g is a compatible opposition of $G - S_G$. Similarly, there must exist a vertex $v \in V(H)$ and a set $S_H \subseteq N_H[v]$ such that h is a compatible opposition of $H - S_H$.

Let $g'\colon V(G) \to V(G)$ and $h'\colon V(H) \to V(H)$ be automorphisms of order two such that $g'(x) = g(x)$ and $h'(y) = h(y)$ for all $x \in V(G) \backslash S_G$ and $y \in V(H) \backslash S_H$. Let $S = \{(x, y) \in V(G \boxtimes H) \mid x \in S_G,\ y \in S_H\}$. Note that $(u, v) \in S \subseteq N_{G \boxtimes H}[(u, v)]$. Define $f\colon V(G \boxtimes H - S) \to V(G \boxtimes H - S)$ by $f((x, y)) = (g'(x), h'(y))$. We claim that f is an opposition of $G \boxtimes H - S$.

Let $(x_1, y_1), (x_2, y_2) \in V(G \boxtimes H - S)$ be distinct. Since g' and h' are bijections, f is also a bijection. Suppose that $(x_1, y_1) \sim_{G \boxtimes H - S} (x_2, y_2)$. If $x_1 = x_2$ then $g'(x_1) = g'(x_2)$. If $x_1 \sim_G x_2$ then $g'(x_1) \sim_G g'(x_2)$. Similar statements can be made for y_1, y_2, and h'. Since g' never maps a vertex outside of S_G to a vertex in S_G and h' never maps a vertex outside of S_H to a vertex in S_H, $f((x_1, y_1)) \sim_{G \boxtimes H - S} f((x_2, y_2))$. Therefore, f is an automorphism of $G \boxtimes H - S$.

Let $(x,y) \in V(G \boxtimes H - S)$. Since g' and h' are automorphisms of order two, $f(f((x,y))) = (x,y)$. It remains to show that $(x,y) \nsim_{G\boxtimes H-S} f((x,y))$. By the way S is defined, either $x \notin S_G$ or $y \notin S_H$. If $x \notin S_G$ then since g is an opposition, $g(x) = g'(x) \nsim_G x$. Thus, $(x,y) \nsim_{G\boxtimes H-S} (g'(x), h'(y)) = f((x,y))$. If instead $x \in S_G$, then $y \notin S_H$. By a similar argument, $h'(y) \nsim_H y$ and so $(x,y) \nsim_{G\boxtimes H-S} (g'(x), h'(y)) = f((x,y))$. Therefore, f is an opposition.

Let $f' \colon G \boxtimes H \to G \boxtimes H$ be defined by $f'((x,y)) = (g'(x), h'(y))$. Using a similar argument as with f, we have that f' is an automorphism of order two. Furthermore, for any $(x,y) \in V(G \boxtimes H - S)$, we have $f'((x,y)) = (g'(x), h'(y)) = f((x,y))$. This proves the theorem. □

Theorem 5 tells us that strong products of compatibly almost opposable graphs are themselves compatibly almost opposable (in particular, they are almost opposable). As a consequence of this, given any finite number of graphs $G_1, \ldots, G_k$ that are compatibly almost opposable, the strong product $\boxtimes_{i=1}^{k} G_i$ also compatibly almost opposable. One example of this is the k-dimensional strong grid. Since strong grids will be relevant to us again in Section 5, we prove here that all such strong grids are almost opposable.

Corollary 3. *If $k, n_i \in \mathbb{Z}^+$ for each $1 \le i \le k$, then $\boxtimes_{i=1}^{k} P_{n_i}$ is almost opposable.*

Proof. Let $n \in \mathbb{Z}^+$ and let $v_0, \ldots, v_{n-1}$ be the vertices of P_n. Consider the function $f \colon V(P_n) \to V(P_n)$ defined by $f(v_i) = v_{n-i-1}$. Note that f is an automorphism of order two. If n is odd then f, when restricted to the domain $V(P_n - v_{\frac{n-1}{2}})$, is an opposition of $P_n - v_{\frac{n-1}{2}}$. If n is even then f, when restricted to the domain $V(P_n - \{v_{\frac{n}{2}-1}, v_{\frac{n}{2}}\})$, is an opposition of $P_n - \{v_{\frac{n}{2}-1}, v_{\frac{n}{2}}\}$.

Therefore, P_n is compatibly almost opposable for any $n \in \mathbb{Z}^+$. Thus, by Theorem 5, the product $\boxtimes_{i=1}^{k} P_{n_i}$ is almost opposable, where $k, n_1, \ldots, n_k \in \mathbb{Z}^+$. □

If one desires to show that a strong product is almost opposable, it may be tempting to weaken the hypothesis of Theorem 5 to require that G and H only be almost opposable rather than compatibly almost opposable. But this does not work, as the following counterexample demonstrates.

Let G be the graph in Figure 8, with vertices labeled in the same way. Note that G is almost opposable since $G - \{c,d,g\} \cong P_2 \cup P_2$ is opposable and $\{c,d,g\} \subseteq N[d]$. There are two oppositions of $G - \{c,d,g\}$: the automorphism of order two that maps a to f and b to e, and the automorphism of order two that maps a to e and b to f. We claim that $\{c,d,g\}$ is the only admissible set of vertices such that deleting them from G yields an opposable graph. Since G has an odd number of vertices, the number of vertices we delete must be odd for an opposition to be possible. Deleting any single vertex from G will not yield an opposable graph. The only admissible sets of vertices of size three are $\{a,b,c\}$, $\{b,c,d\}$, $\{c,d,e\}$, $\{c,d,g\}$, $\{d,e,g\}$ and $\{d,e,f\}$. Note that $G - \{a,b,c\} \cong P_4$, $G - \{b,c,d\} \cong 2K_1 \cup P_2$, $G - \{c,d,e\} \cong 2K_1 \cup P_2$,

$G - \{d, e, g\} \cong P_3 \cup K_1$, and $G - \{d, e, f\} \cong P_3 \cup K_1$, none of which are opposable. There are no admissible sets of size five or larger and so $\{c, d, g\}$ is the only admissible set S such that $G - S$ is opposable.

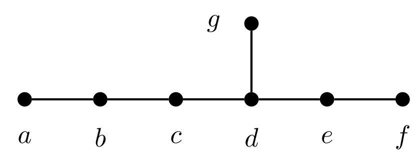

Figure 8: An almost opposable graph that is not compatibly almost opposable.

We now show that G does not have a non-trivial automorphism. Let α be an automorphism of G. The three leaves, a, f, and g, would be forced to be mapped to each other by α. Since d is the unique vertex of degree three, $\alpha(d) = d$. However, α must satisfy $d(x, d) = d(\alpha(x), d)$ for each $x \in \{a, f, g\}$. The only way this is possible is if each leaf is a fixed point. It then follows that α is the identity map. Therefore, G does not have an automorphism that maps a to f and b to e, nor an automorphism that maps a to e and b to f. Thus, G is not compatibly almost opposable.

Consider the graph $G \boxtimes P_4$. We claim that $G \boxtimes P_4$ is not almost opposable, despite both G and P_4 being almost opposable. Let v_1, v_2, v_3, v_4 be the vertices of P_4 labeled in the natural way. Suppose, for a contradiction, that $G \boxtimes P_4$ is almost opposable. Let S be an admissible set of vertices such that $G \boxtimes P_4 - S$ is opposable. Since G has no non-trivial automorphisms and since S cannot contain vertices from both $G.v_1$ and $G.v_4$, an opposition of $G \boxtimes P_4 - S$ must map every vertex $(x, v_1) \in V(G.v_1)$ to $(x, v_4) \in V(G.v_4)$. Consequently, every vertex $(x, v_2) \in V(G.v_2)$ must map to $(x, v_3) \in V(G.v_3)$. However, $(x, v_2) \sim (x, v_3)$. Since S cannot contain all vertices in $G.v_2$ and $G.v_3$, an opposition of $G \boxtimes P_4 - S$ cannot exist. This justifies our claim that $G \boxtimes P_4$ is not almost opposable, and shows why we cannot drop the word 'compatibly' in Theorem 5.

Similarly, it is possible to have a strong product that is almost opposable where the factor graphs are not necessarily compatibly almost opposable. We demonstrate this now.

Let v_0, v_1, and v_2 be the vertices of P_3 (labeled in the natural way). Let G be the graph in Figure 8. Recall that G does not have any non-trivial automorphisms, and hence cannot be compatibly almost opposable. We claim that the graph $G \boxtimes P_3$, illustrated in Figure 9, is almost opposable. Let $S = \{(u, v_x) \mid u \in \{c, d, g\}, x \in \{0, 1, 2\}\}$. Note that $(d, v_1) \in S \subseteq N_{G \boxtimes P_3}[(d, v_1)]$. The graph $G \boxtimes P_3 - S$ is isomorphic to $(P_2 \boxtimes P_3) \cup (P_2 \boxtimes P_3)$, which is opposable. Therefore, $G \boxtimes P_3$ is almost opposable, even though G is not compatibly almost opposable.

The property of P_3 that allows this strong product to be almost opposable is that it has a vertex adjacent to every other vertex: we call such a vertex *universal*. We

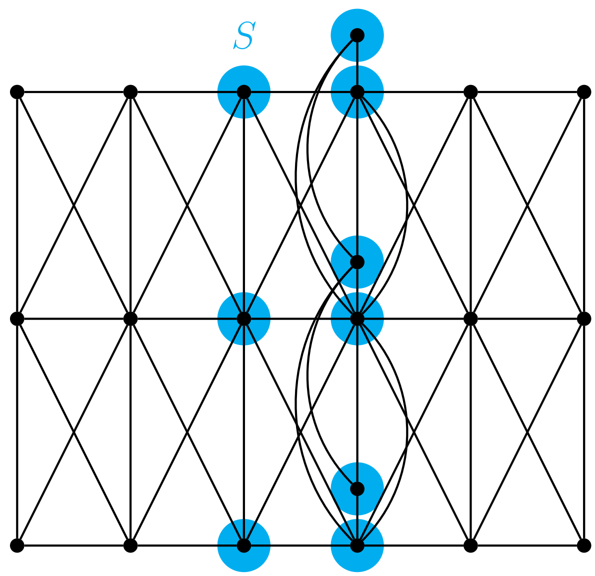

Figure 9: An example of an almost opposable strong product with a factor that is not compatibly almost opposable.

can generalize this result to any almost opposable graph G and any graph with a universal vertex H.

Theorem 6. *If G is almost opposable and H contains a universal vertex, then $G \boxtimes H$ is almost opposable.*

Proof. Let v be a universal vertex of H. Let $u \in V(G)$ and $u \in S_G \subseteq N_G[u]$ such that $G - S_G$ is opposable. Let g be an opposition of $G - S_G$. Let $S = \{(x, y) \in V(G \boxtimes H) \mid x \in S_G, y \in V(H)\}$. Note that $(u, v) \in S \subseteq N_{G\boxtimes H}[(u, v)]$. Define $f \colon V(G \boxtimes H - S) \to V(G \boxtimes H - S)$ by $f((x, y)) = (g(x), y)$. We claim that f is an opposition of $G \boxtimes H - S$.

Let $(x_1, y_1), (x_2, y_2) \in V(G \boxtimes H - S)$ be distinct. Since g is a bijection, f is also a bijection. Suppose that $(x_1, y_1) \sim_{G\boxtimes H - S_G} (x_2, y_2)$. If $x_1 = x_2$ then $g(x_1) = g(x_2)$. If $x_1 \sim_{G-S_G} x_2$ then $g(x_1) \sim_{G-S_G} g(x_2)$. Since g is an automorphism of $G - S_G$, g never maps a vertex outside of S_G to a vertex in S_G. So $f((x_1, y_1)) = (g(x_1), y_1) \sim_{G\boxtimes H-S} (g(x_2), y_2) = f((x_2, y_2))$ and f is an automorphism of $G \boxtimes H - S$.

Let $(x, y) \in V(G \boxtimes H - S)$. Note that $f(f((x, y))) = (g(g(x)), y) = (x, y)$ and so f is of order two. Since g is an opposition of $G - S_G$, we have $g(x) \nsim_G x$. Thus, $(x, y) \nsim_{G\boxtimes H-S} (g(x), y) = f((x, y))$. Therefore, f is an opposition of $G \boxtimes H - S$ and so $G \boxtimes H$ is almost opposable. □

Using similar criteria for two graphs G and H as in Theorem 5, we can construct larger almost opposable graphs using the Cartesian product.

Theorem 7. *If G and H are compatibly almost opposable, and there exists a $u \in V(G)$ such that $G - u$ has a compatible opposition, then $G \Box H$ is compatibly almost opposable.*

Proof. Let g be a compatible opposition of $G - u$. Let $v \in V(H)$ and $v \in S_H \subseteq N_H[v]$ such that $H - S_H$ has a compatible opposition h. Let $g' \colon V(G) \to V(G)$ and $h' \colon V(H) \to V(H)$ be automorphisms of order two such that $g'(x) = g(x)$ and $h'(y) = h(y)$ for all $x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$. Let $S = \{(x, y) \in V(G \Box H) \mid x = u, y \in S_H\}$. Define $f \colon V(G \Box H - S) \to V(G \Box H - S)$ by $f((x, y)) = (g'(x), h'(y))$. We claim that f is an opposition of $G \Box H - S$.

By using the same argument as in the proof of Theorem 5, we get that f is well-defined, injective, and surjective. Let $(x_1, y_1), (x_2, y_2) \in V(G \Box H - S)$. If $(x_1, y_1) \sim_{G \Box H - S} (x_2, y_2)$, then either $x_1 = x_2$ and $y_1 y_2 \in E(H)$, or $x_1 x_2 \in E(G)$ and $y_1 = y_2$. If $x_1 = x_2$, then since g' and h' are automorphisms, we have

$$\begin{aligned} f((x_1, y_1)) &= (g'(x_1), h'(y_1)) \\ &= (x_2, h'(y_1)) \\ &\sim_{G \Box H - S} (x_2, h'(y_2)) \\ &= (g'(x_2), h'(y_2)) \\ &= f((x_2, y_2)). \end{aligned}$$

Similarly, if $y_1 = y_2$, then $f((x_1, y_1)) \sim_{G \Box H - S} f((x_2, y_2))$. Thus, f is an automorphism of order two. If $(x_1, y_1) \nsim_{G \Box H - S} (x_2, y_2)$, then either $x_2 \notin N_G[x_1]$ or $y_2 \notin N_H[y_1]$. Without loss of generality, suppose that $x_2 \notin N_G[x_1]$. If $x_1 \neq u$, then $g'(x_2) \notin N_G[g'(x_1)]$ and so $f((x_1, y_1)) \nsim_{G \Box H - S} f((x_2, y_2))$. If $x_1 = u$, then $y_1 \notin S_H$. Thus, $h'(y_1) \nsim_{H - S_H} h'(y_2)$ and so $f((x_1, y_1)) \nsim_{G \Box H - S} f((x_2, y_2))$. Therefore, f is an opposition of $G \Box H - S$.

Define $f' \colon V(G \Box H) \to V(G \Box H)$ by $f'((x, y)) = (g'(x), h'(y))$. Using a similar argument as with f, we have that f' is an automorphism of order two. Furthermore, for any $(x, y) \in V(G \Box H - S)$, we have $f'((x, y)) = f((x, y))$. This proves the theorem. $\Box$

Independently of Kakihara and Arroyo, Uiterwijk [17] characterized the outcomes of SNORT on $P_n \Box P_m$ by using the opposability of $P_n \Box P_m$ when both n and m are even and the almost opposability of $P_n \Box P_m$ when either n or m is odd. By using Corollary 1 and Theorem 7, we can generalize Uiterwijk's result to k-dimensional Cartesian grids.

Corollary 4. *If $k \in \mathbb{Z}^+$ and $n_1, \ldots, n_k \in \mathbb{Z}^+$, then the graph $\Box_{i=1}^{k} P_{n_i}$ is almost opposable if and only if there is at most one n_i that is even; otherwise, $\Box_{i=1}^{k} P_{n_i}$ is opposable.*

Proof. First, suppose that there are at least two even values of n_i. Without loss of generality, suppose that n_1 and n_2 are even. Note that $\square_{i=1}^{k} P_{n_i}$ is isomorphic to $(P_{n_1} \square P_{n_2}) \square (\square_{\ell=3}^{k} P_{n_\ell})$.

We claim that $P_{n_1} \square P_{n_2}$ is opposable. Let $x_1, \ldots, x_{n_1}$ be the vertices of P_{n_1} such that $x_i \sim x_{i+1}$ for each $1 \leq i \leq n_1 - 1$. Let $y_1, \ldots, y_{n_2}$ similarly denote the vertices of P_{n_2}. The function $f\colon V(P_{n_1}) \to V(P_{n_2})$ defined by $f((x_i, y_j)) = (x_{n_1-i+1}, y_{n_2-j+1})$ is an opposition of $P_{n_1} \square P_{n_2}$, and so $P_{n_1} \square P_{n_2}$ is opposable. Since $P_{n_1} \square P_{n_2}$ is opposable, $\square_{i=1}^{k} P_{n_i}$ is opposable by Corollary 1.

Suppose instead that each n_i is odd. Note that any path on an odd number of vertices satisfies the conditions of G in the statement of Theorem 7. Thus, the Cartesian product of any two odd paths is compatibly almost opposable. By induction, $\square_{i=1}^{k} P_{n_i}$ is almost opposable.

Finally, suppose that there is exactly one n_i that is even. Without loss of generality, assume n_k is even. The graph $\square_{i=1}^{k} P_{n_i}$ is isomorphic to $(\square_{i=1}^{k-1} P_{n_i}) \square (P_k)$. By the argument made in the case where n_i is odd for all i, we know that $\square_{i=1}^{k-1} P_{n_i}$ satisfies the conditions for G in the statement of Theorem 7. Furthermore, P_{n_k} is compatibly almost opposable by the argument given in the proof of Corollary 3. Therefore, by applying Theorem 7, $(\square_{i=1}^{k-1} P_{n_i}) \square (P_k)$ is almost opposable. $\square$

If one were to weaken the hypothesis of Theorem 7 such that H is only required to be almost opposable instead of compatibly almost opposable, then not only are we unable to conclude that the Cartesian product is compatibly almost opposable, but in fact we sometimes cannot even conclude that the product is almost opposable. To see this, let H be the graph in Figure 8 with the vertices labeled in the same way. Recall that we showed H is almost opposable but does not have any non-trivial automorphisms (in particular, it is not compatibly almost opposable). We claim that $P_3 \square H$ is not almost opposable; but note that P_3 is compatibly almost opposable, and the graph resulting from removing the middle vertex admits a compatible opposition (thus, it satisfies the requirements of G in Theorem 7).

Let v_1, v_2, v_3 be the vertices of P_3 labeled in the natural way. Suppose, for a contradiction, that $P_3 \square H$ is almost opposable. Let S be an admissible set such that $P_3 \square H - S$ is opposable. If $x \in V(H)$ and $(v_1, x), (v_3, x) \notin S$, then any non-trivial automorphism of $P_3 \square H - S$ is forced to map (v_1, x) to (v_3, x). For any given $x \in V(H)$, since the only vertex adjacent to both (v_1, x) and (v_3, x) is (v_2, x), if $(v_1, x), (v_3, x) \notin S$ then any automorphism of $P_3 \square H - S$ is forced to map (v_2, x) to itself. It is then not too difficult to see that there exists no admissible set S such that $G - S$ is opposable. Thus, $P_3 \square H$ is not almost opposable.

5. Peaceable Queens Game

Consider a game played on a chessboard where two players take turns placing a queen on an unoccupied space of the board. The player to go first always places white queens while the player that moves second always places black queens. Neither player is allowed to place a queen that is within attacking range of any of the queens previously placed by their opponent. If one of the players is unable to place a queen on the chessboard, then that player loses the game. We call this the *Peaceable Queens Game* and in this section we explore the outcomes of the game for $n \times m$ chessboards.

Denote $\text{PQ}(n, m)$ for the Peaceable Queens Game on an $n \times m$ grid. We can also define similar games using other chess pieces: we write $\text{PR}(n, m)$, $\text{PN}(n, m)$, $\text{PB}(n, m)$, and $\text{PK}(n, m)$ for the Peaceable Rooks Game, the Peaceable Knights Game, the Peaceable Bishops Game, and the Peaceable Kings Game, respectively.

Each of these peaceable chess piece games is isomorphic to particular SNORT positions. Consider an $n \times m$ array of vertices labeled (x, y) where $0 \le x \le n-1$ and $0 \le y \le m-1$. The vertex (x, y) corresponds to the square (x, y) on the chessboard. Two vertices (x_1, y_1) and (x_2, y_2) will be adjacent in the array if the corresponding squares on the chessboard are within attacking range of each other based on the chess piece being considered. For example, the vertices $(0, 0)$ and $(4, 4)$ are adjacent to each other when considering Queens but not when we are considering Knights. We refer to the graph generated by considering Queens as the *Queen's grid*. We similarly define the *Bishop's grid*, the *Knight's grid*, the *Rook's grid*, and the *King's grid*. The Peaceable Queens game is isomorphic to SNORT played on the Queen's grid, and similar statements can be made for the other grids. To determine the outcomes of each of the peaceable games, we will show that the relevant grid graphs are either opposable or almost opposable. For each of the proofs we will label the vertices in the same way as described above.

We begin with the Peaceable Kings game as its outcomes follow directly from Corollary 3.

Theorem 8. *If $n, m \in \mathbb{Z}^+$, then the first player wins $\text{PK}(n, m)$.*

Proof. The $n \times m$ King's grid is isomorphic to $P_n \boxtimes P_m$. By Corollary 3, all k-dimensional strong grids are almost opposable, and so every King's grid is almost opposable. By Lemma 3, the first player wins SNORT on every King's grid. □

For the Peaceable Knights, Bishops, and Rooks games, we completely determine the outcomes by directly showing that the relevant grids are opposable or almost opposable.

Theorem 9. *Let $n, m \in \mathbb{Z}^+$. The second player wins $\text{PN}(n, m)$ if and only if either n or m is even.*

Proof. In the Knight's grid G_N, two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either

- $x_2 = x_1 \pm 1$ and $y_2 = y_1 \pm 2$ or
- $x_2 = x_1 \pm 2$ and $y_2 = y_1 \pm 1$.

Suppose, without loss of generality, that n is even. Define $f\colon V(G_N) \to V(G_N)$ by $f((x, y)) = (n - x - 1, y)$. Note that f is an automorphism of order two. Since n is even, f has no fixed points. Furthermore, since $y \neq y \pm 1$ and $y \neq y \pm 2$, it follows that $f((x, y))$ and (x, y) are not adjacent. Therefore, f is an opposition of G_N. Thus, by Lemma 1, the second player wins SNORT on G_N, and so the second player wins PN(n, m).

Now suppose instead that both n and m are odd. Let $g\colon V(G_N - (\frac{n-1}{2}, \frac{m-1}{2})) \to V(G_N - (\frac{n-1}{2}, \frac{m-1}{2}))$ be defined by $g((x, y)) = (n - x - 1, m - y - 1)$. We claim that g is an opposition of $G_N - (\frac{n-1}{2}, \frac{m-1}{2})$. Note that g is an automorphism of order two with no fixed points. Suppose, for a contradiction, that (x, y) and $g((x, y))$ were adjacent. Then either $n - x - 1 = x \pm 1$ or $n - x - 1 = x \pm 2$. Thus, either $n - 2x - 1 = \pm 1$ or $n - 2x - 1 \pm 2$. Since $2x + 1$ is odd and n is odd, $n - (2x + 1) = n - 2x - 1$ is even. So $n - 2x - 1 \neq \pm 1$ and consequently $n - 2x - 1 = \pm 2$. This forces $m - y - 1 = y \pm 1$ which is a contradiction since $m - 2y - 1$ is even. Therefore, $g((x, y))$ and (x, y) are not adjacent for any $0 \leq x \leq n - 1$ and $0 \leq y \leq m - 1$. Since g is an opposition of $G_N - (\frac{n-1}{2}, \frac{m-1}{2})$, G_N is almost opposable. By Lemma 3, the first player wins SNORT on G_N, and so PN(n, m) is first-player win when both n and m are odd. □

Theorem 10. *Let $n, m \in \mathbb{Z}^+$. The second player wins* PB(n, m) *if and only if either n or m is even.*

Proof. In the Bishop's grid G_B, two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $|x_1 - x_2| = |y_1 - y_2|$.

Without loss of generality, suppose that n is even. We define $f\colon V(G_B) \to V(G_B)$ by $f((x, y)) = (n - x - 1, y)$. Note that f is a automorphism of order two. Since n is even, $x \neq n - x - 1$ and so f has no fixed points. Suppose, for a contradiction, that (x, y) and $(n - x - 1, y)$ are adjacent to each other for some $0 \leq x \leq n - 1$ and $0 \leq y \leq m - 1$. Then $|x - (n - x - 1)| = |y - y|$ and so $n = 2x + 1$, which contradicts n being even. Therefore, f is an opposition. Thus, by Lemma 1, the second player wins on G_B.

Now suppose that both n and m are odd. We claim that G_B is almost opposable. Consider the subgraph $H = G_B - N[(\frac{n-1}{2}, \frac{m-1}{2})]$. We define $f\colon V(H) \to V(H)$ by $f((x, y)) = (n - x - 1, m - y - 1)$ and claim that f is an opposition of H. Note that f is an automorphism of order two with no fixed points. Let $(x, y) \in V(H)$ and suppose, for a contradiction, that (x, y) is adjacent to $f((x, y))$. Then

$|x-(n-x-1)| = |y-(m-y-1)|$ and so $|2x+1-n| = |2y+1-m|$. Thus, either

$$2x+1-n = 2y+1-m \tag{1}$$

or

$$2x+1-n = m-(2y+1). \tag{2}$$

If Equation (1) holds, then rearranging yields $x-\frac{n-1}{2} = y-\frac{m-1}{2}$. Thus, $|x-\frac{n-1}{2}| = |y-\frac{m-1}{2}|$ and so $(x,y) \in N[(\frac{n-1}{2},\frac{m-1}{2})]$ which is a contradiction. If Equation (2) holds, then rearranging yields $x-\frac{n-1}{2} = -\left(y-\frac{m-1}{2}\right)$. Thus, $|x-\frac{n-1}{2}| = |y-\frac{m-1}{2}|$ and again $(x,y) \in N[(\frac{n-1}{2},\frac{m-1}{2})]$. Therefore, for all $v \in V(H)$, $f(v)$ is not adjacent to v, and so f is an opposition of H. So G_B is almost opposable and the first player wins SNORT on G_B by Lemma 3. □

Theorem 11. *Let $n,m \in \mathbb{Z}^+$. The second player wins* PR(n,m) *if and only if both n and m are even.*

Proof. In the Rook's grid G_R, two vertices (x_1,y_1) and (x_2,y_2) are adjacent if and only if either $x_1 = x_2$ or $y_1 = y_2$.

Suppose that both n and m are even. Define $f\colon V(G_R) \to V(G_R)$ by $f((x,y)) = (n-x-1, m-y-1)$ and note that f is an automorphism of order two with no fixed points. Since $x = n-x-1$ and $y = m-y-1$ if and only if n and m are odd, $f((x,y))$ is not adjacent to (x,y) for all $(x,y) \in V(G_R)$. Therefore, f is an opposition and so by Lemma 1, the second player wins SNORT on G_R.

Suppose instead that at least one of n or m is odd. We consider two cases, the first being exactly one of n and m is odd and the second being both n and m are odd. Without loss of generality, assume n is odd and m is even. Let $H = G_R - \{(\frac{n-1}{2},y) \mid 0 \le y \le m-1\}$. Note that $\{(\frac{n-1}{2},y) \mid 0 \le y \le m-1\} \subseteq N_{G_R}[(\frac{n-1}{2},y)]$ for any $0 \le y \le m-1$. Define $f\colon V(H) \to V(H)$ by $f((x,y)) = (n-x-1,y)$. Note that f is an automorphism of order two with no fixed points. For any $0 \le x \le n-1$, $x = n-x-1$ if and only if $x = \frac{n-1}{2}$. Therefore, for all $(x,y) \in V(H)$, (x,y) and $f((x,y)$ are not adjacent, and so f is an opposition. Thus, by Lemma 3, the first player wins SNORT on G_R.

Now suppose that both n and m are odd. Consider the automorphism g on $G_R - N[(\frac{n-1}{2},\frac{m-1}{2})]$ defined by $g((x,y)) = (n-x-1,m-y-1)$. Note that g is an automorphism of order two with no fixed points. Since (x,y) and $(n-x-1,m-x-1)$ are adjacent if and only if either $x = \frac{n-1}{2}$ or $y = \frac{m-1}{2}$, g is an opposition of $G_R - N[(\frac{n-1}{2},\frac{m-1}{2})]$. Therefore, G_R is almost opposable and so the first player wins SNORT on G_R by Lemma 3. □

Next, we determine the outcome of the Peaceable Queens game played on a board with either an odd number of rows or an odd number of columns.

Theorem 12. *Let $n,m \in \mathbb{Z}^+$. If either n or m is odd, then the first player wins* PQ(n,m).

Proof. In the Queen's grid G_Q, two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2$, $y_1 = y_2$, or $|x_1 - x_2| = |y_1 - y_2|$. Note that these conditions for adjacency are the same as testing for adjacency in the Rook's grid and the Bishop's grid simultaneously.

To show that the first player wins on G_Q when either n or m is odd, we will show that G_Q is almost opposable when either n or m is odd. Without loss of generality, suppose that n is odd and m is even. Let $S = \{(\frac{n-1}{2}, y) \mid 0 \leq y \leq m\}$ and let $f : V(G_Q - S) \to V(G_Q - S)$ be defined by $f((x, y)) = (n - x - 1, m - y - 1)$. By the work done in the proofs of Theorems 10 and 11, (x, y) and $(n - x - 1, m - y - 1)$ are not adjacent in both the Bishop's grid and the Rook's grid for all $0 \leq x \leq n - 1$ and $0 \leq y \leq m - 1$. Therefore, (x, y) and $f((x, y))$ are not adjacent in the Queen's grid and so f is an opposition of $G_Q - S$.

Now suppose that both n and m are odd. Let $H = G_Q - N[(\frac{n-1}{2}, \frac{m-1}{2})]$ and let $g : V(H) \to V(H)$ be defined by $g((x, y)) = (n - x - 1, m - x - 1)$. By the work done in the proofs of Theorem 10 and Theorem 11, (x, y) and $g((x, y))$ are not adjacent for any $0 \leq x \leq n - 1$ and $0 \leq y \leq m - 1$. So g is an opposition of H.

Therefore, when at least one of n or m is odd, the Queen's grid is almost opposable and thus the first player wins PQ(n, m) by Lemma 3. □

We believe that determining who wins the Peaceable Queens Game on $2n \times 2m$ chessboards cannot be done with the techniques used in this paper. Without loss of generality, assume $n \leq m$. To see that the $2n \times 2m$ Queen's grid is not opposable, suppose, for a contradiction, that α is an opposition of the Queen's grid. The Queen's grid has four vertices of minimum degree: $(0, 0)$, $(2n-1, 0)$, $(0, 2m-1)$, and $(2n - 1, 2m - 1)$. Thus, α maps these four vertices to each other. In the case where $n = m$, these four vertices are all adjacent to each other and so we already have a contradiction. From here we assume that $n < m$. Since $(0, 0)$ is adjacent to $(2n-1, 0)$ and $(0, 2m - 1)$, $\alpha((0, 0)) = (2n - 1, 2m - 1)$ and thus $\alpha((2n - 1, 0)) = (0, 2m - 1)$. To preserve adjacency, for each $1 \leq x \leq 2n - 1$ and $1 \leq y \leq 2m - 1$, α is forced to map $(x, 0)$ to $(2n - x - 1, 2m - 1)$ and $(0, y)$ to $(2n - 1, 2m - y - 1)$. Consequently, α is forced to map (x, y) to $(2n - x - 1, 2m - y - 1)$ for all $0 \leq x \leq 2n - 1$ and $0 \leq y \leq 2m - 1$.

Let $A = \{(x, m - n + x) \mid 0 \leq x \leq 2n - 1\}$ and $B = \{(x, m + n - x - 1) \mid 0 \leq x \leq 2n - 1\}$. Informally, A and B are each sets of $2n$ vertices and the vertices of $A \cup B$ form the diagonals containing the four center vertices of the grid. Note that all of the vertices in A are adjacent to each other and all of the vertices in B are adjacent to each other. However, by the above argument, every vertex in A is mapped by α to another vertex in A. Similarly, α maps vertices in B to other vertices in B. This is a contradiction and so the $2n \times 2m$ Queen's grid is not opposable.

While it is easy to show that $2n \times 2m$ Queen's grids are not opposable, proving that they are not almost opposable is more challenging. It is easy to observe that the $2 \times 2m$ Queen's grid is always almost opposable for $m \geq 2$: picking $v = (0, m)$ and admissible set $S = \{(0, m-1), (1, m-1), (1, m)\} \subseteq N[v]$ results in a graph with an opposition given by the pairs $(0, i) \leftrightarrow (1, 2m-1-i)$ and $(1, i) \leftrightarrow (0, 2m-1-i)$ for all $i \leq m-2$. There also exist non-obvious instances of almost opposability, as demonstrated in Figure 10 for the 4×4 case. Computationally, however, the 4×6 and 4×8 boards appear to not be almost opposable. We leave the full solution as Open Problem 2.

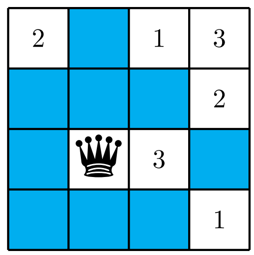

Figure 10: An illustration of the 4×4 Queen's grid being almost opposable, with the admissible set colored cyan (as in Figure 9), and the opposition pairs $(v, f(v))$ labeled by like numbers.

Open Problem 2. When is the $2n \times 2m$ Queen's grid almost opposable for $n, m \in \mathbb{Z}^+$?

6. Further Directions

Instead of having black and white queens like in the Peaceable Queens Game, we could consider an impartial game where all queens are the same color and the two players are not allowed to place a queen that is within attacking range of any queen that is already on the chessboard. Noon and Van Brummelen [12, 13] introduced and studied this game on $n \times n$ chessboards and Brown and Ladha [5] explored variations. Noon and Van Brummelen [12, 13] showed that on all $n \times n$ boards for $1 \leq n \leq 9$, the first player wins the impartial game. However, the second player wins on the 10×10 board. This leads naturally to the question of whether the second player can ever win the Peaceable Queens Game on a $2n \times 2m$ board.

Open Problem 3. Is it possible for the second player to win $\text{PQ}(2n, 2m)$?

In [2], it is shown that the function that counts the number of automorphisms of a graph is $\exp(O(\sqrt{n \log n}))$-enumerable.

Open Problem 4. What can be said about the enumerability of counting the number of oppositions of a graph? What about the number of ways a graph can be almost opposable?

We note that counting perfect matchings (in possible service of counting opposition matchings) is #P-complete [19].

One can see in multiple ways that the problem of deciding whether a given graph is opposable is in NP. Perhaps easiest is to non-deterministically guess an opposition matching on the complement, and check that every pair of edges in the matching induces a valid subgraph. Another way is to construct an auxiliary graph G' where edges in G correspond to vertices in G', and vertices in G' are adjacent if and only if the corresponding edges in G induce a valid subgraph within an opposition matching. Then opposition matchings correspond to $|V(G)|/2$-cliques in G', and CLIQUE is an NP-complete problem. Similarly, almost-opposability is also in NP, as one can non-deterministically guess a partial neighborhood and opposition matching on the resulting graph, then verify the validity of these guesses.

The fact that (almost) opposability is in NP contrasts with the problem of identifying whether a given SNORT position is winning being PSPACE-complete [14]. To correct this dissonance, we observe that a SNORT strategy based around (almost) opposability is one where each move relies on only (respectively, at most) the previous move. However, when reducing from QBF to show PSPACE-hardness, strategies may observe the entire history of plays. As a final note on complexity, it is known that deciding if a graph has a perfect matching is solvable in P time by contracting odd cycles to a point and using the Ford–Fulkerson algorithm to solve a max-flow problem across the resulting bipartite graph [8]. However, these natural techniques do not guarantee that the resulting matchings satisfy the conditions of opposition matchings. Further complicating the issue, graphs such as $P_2 \Box P_{2n}$ yield both opposition matchings (lift opposition matching of P_2) and non-opposition perfect matchings (lift perfect non-opposition matching of P_{2n}). We leave open the issue of the complexity of identifying if a graph is (almost) opposable.

Next, we mention two combinatorial games played on graphs where determining the structural properties of first-player winning graphs and second-player winning graphs remains open.

The IMPARTIAL SNORT ruleset, also written as ISNORT, is played nearly identically to SNORT, with the only difference being that both players have access to both colors—instead of just one color each. Despite the similarities of SNORT and ISNORT, not all of our results hold for ISNORT. Uiterwijk [18] showed that the second player wins ISNORT on $P_n \Box P_m$ if and only if either n or m is even. In the case where at most one of n and m are even, we know the first player wins SNORT on $P_n \Box P_m$ by Corollary 4.

Open Problem 5. For which graphs does the second (respectively, first) player win SNORT and the first (respectively, second) player win ISNORT?

A combinatorial game for digraphs called DIGRAPH PLACEMENT was recently introduced in [6]. In DIGRAPH PLACEMENT, the vertices of a given digraph are colored either red or blue. The two players, Left and Right, take turns deleting vertices along with their out-neighborhoods. Left is only allowed to delete blue vertices and Right is only allowed to delete red vertices. Just as we did with SNORT in this paper, we can ask questions regarding the structural properties of digraphs where a player wins DIGRAPH PLACEMENT by using a "mirroring" strategy.

Open Problem 6. What structural properties for digraphs yield simple winning strategies in DIGRAPH PLACEMENT for either the first or second player?

References

[1] Edward Arroyo, *Dawson's Chess, Snort on Graphs and Graph Involutions*, Ph.D. thesis, City University of New York, 1998.

[2] Robert Beals, Richard Chang, William Gasarch, and Jacobo Torán, On finding the number of graph automorphisms. *Chicago J. Theoret. Comput. Sci.* (1999), Article 1, 28 pp.

[3] Jordan Bell and Brett Stevens, A survey of known results and research areas for n-queens, *Discrete Math.* **309** (1) (2009), 1–31.

[4] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, *Winning Ways for Your Mathematical Plays,* Vol. 1, A K Peters, Ltd., Natick, MA, 2001.

[5] Tricia Muldoon Brown and Abrahim Ladha, Exploring mod 2 n-queens games, *Recreat. Math. Mag.* **6** (11) (2019), 15–25.

[6] Alexander Clow and Neil A McKay, Digraph placement games, preprint, `arXiv:2407.12219`.

[7] J. H. Conway, *On Numbers and Games*, A K Peters, Ltd., Natick, MA, 2001.

[8] Jack Edmonds, Paths, trees, and flowers, *Canadian J. Math.* **17** (1965), 449–467.

[9] Svenja Huntemann, Richard J. Nowakowski, and Carlos Pereira dos Santos, Bounding game temperature using confusion intervals, *Theoret. Comput. Sci.* **855** (2021), 43–60.

[10] Wilfried Imrich and Sandi Klavžar, *Product Graphs*, Wiley-Interscience, New York, 2000.

[11] Keiko Kakihara, *Snort: a combinatorial game*, Master's thesis, California State University, San Bernardino, 2010.

[12] Hassan Noon, *Surreal numbers and the n-queens game*, Master's thesis, Bennington College, 2002.

[13] Hassan Noon and Glen Van Brummelen, The non-attacking queens game, *College Math. J.* **37** (2006), 223–227.

[14] Thomas J. Schaefer, On the complexity of some two-person perfect-information games, *J. Comput. System Sci.* **16** (2) (1978), 185–225.

[15] A.N. Siegel, *Combinatorial Game Theory*, American Mathematical Society, Providence, RI, 2013.

[16] OEIS Foundation Inc, The On-Line Encyclopedia of Integer Sequences, `https://oeis.org`.

[17] Jos W. H. M. Uiterwijk, Solving bicoloring-graph games on rectangular boards—Part 1: partisan Col and Snort, in *Advances in Computer Games*, *Lecture Notes in Comput. Sci.* **13262**, Springer, Cham, 2022, 96–106.

[18] Jos W. H. M. Uiterwijk, Solving bicoloring-graph games on rectangular boards—Part 2: impartial Col and Snort, in *Advances in Computer Games*, *Lecture Notes in Comput. Sci.* **13262**, Springer, Cham, 2022, 107–117.

[19] L.G. Valiant, The complexity of computing the permanent, *Theoret. Comput. Sci.* **8** (2) (1979), 189–201.

[20] Yukun Yao and Doron Zeilberger, Numerical and symbolic studies of the peaceable queens problem, *Exp. Math.* **31** (1) (2022), 269–279.