



SPRAGUE-GRUNDY VALUES FOR 4-PILE PRINCESS AND THE ROSES

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Abstract

We study the game of The Princess and the Roses, an impartial combinatorial game consisting of heaps. On each player's turn they may either remove one element from a single heap or remove one element from each of two heaps. We find the Sprague-Grundy values for all positions in this game with four heaps or less.

1. Introduction

1.1. Basics of Combinatorial Games

Combinatorial games as they appear in this paper consist of two players who take turns changing the game state by making moves. In order to be considered a combinatorial game, the game must include no hidden information and no randomness. Examples of combinatorial games include Chess, Checkers, Go, and Nim.

When we refer to a “game” in this paper, we assume it is a combinatorial game that is impartial, finite, follows normal play, and always ends with a winner and loser (so no ties). An *impartial* game is one in which both players have the same set of moves from any given game state. Chess is an example of a partial game since one player can only move white pieces and the other can only move black pieces. A *finite* game is one which has no possible infinite sequence of moves, meaning the game will always come to an end. *Normal play* means that a player who cannot move loses the game.

Nim is one of the most famous examples of an impartial finite combinatorial game. In Nim, the game state consists of objects sorted into distinct piles, and a move is made by removing any positive number of objects from one pile.

1.2. Terminology and Notation

In the study of combinatorial games, a specific game state is referred to as a *position*. For example, in Chess a position is identified by where each piece is on the board, and in Nim it is identified by the multiset containing the number of objects in each pile.

Some games, like Nim and The Princess and the Roses, consist of objects sorted into piles. We can denote a position in one of these games with a tuple of positive integers representing the number of objects in each pile. For example, in Nim we may write $(1, 3)$ to denote a position which has a pile with 1 object and a pile with 3 objects.

If one can make a move that changes position G into position G' , then G' is called an *option* of G . For reasons that will become apparent later, we denote this as $G' \in G$. A position with no options is a *terminal position*. The word “game” is often used to refer not only to the set of rules, but also to a specific position.

1.3. $\mathcal{N}$ -Positions and $\mathcal{P}$ -Positions

One aspect of these combinatorial games that makes them interesting to study is that they can be solved. That is, from any position exactly one of the players has an optimal strategy they can use to guarantee that they win. If the next player has an optimal strategy then the position is called an $\mathcal{N}$ -*position*, and if the previous player has an optimal strategy then the position is called a $\mathcal{P}$ -*position*. For example, in the game of Nim, any position consisting of exactly one pile is an $\mathcal{N}$ -position since the next player can remove the entire pile to win. A position which is empty is a $\mathcal{P}$ -position since the previous player has already won by creating that position. Some games, like Nim, are solved already, meaning the set of $\mathcal{N}$ - and $\mathcal{P}$ -positions is known, while some are yet unsolved. The classification of a position as either an $\mathcal{N}$ -position or $\mathcal{P}$ -position is referred to as its *outcome class*.

1.4. Conway’s Method of Induction

In his book, *On Numbers and Games*, John Conway outlined a method of structural induction which he used for combinatorial games, and which we will use throughout this paper [3]. In this method, a position can be identified with the set of options of that position. So in a game of Nim, the position $(1, 3)$ can be identified with the set $\{(0, 3), (1, 2), (1, 1), (1, 0)\}$. So any terminal position can be identified with the empty set, and other positions are identified recursively as sets of their options. This method makes it very easy to do induction, as follows.

Theorem 1 (Conway Induction). *Suppose there exists some property P such that a game has property P if all of its options have property P . Then every finite game has property P .*

This is true because any terminal position has the property trivially, and then every other position builds off of them somehow. A more rigorous proof of the above theorem can be found in Schleicher and Stoll's article [4]. Note that this relies on the games being finite, since this guarantees that every option of a position will be strictly closer to a terminal position.

Sometimes we will use a different version of the induction argument, which we will call *strong Conway induction*. Instead of assuming that all options of a game have property P , we assume that all games that can be reached in some positive number of moves have property P . If this implies that P holds for the original game, then P holds for all games for the same reason. One important application of Conway induction is to recursively determine all of the $\mathcal{N}$ -positions and $\mathcal{P}$ -positions of a game.

Theorem 2. *When using normal play, a position is a $\mathcal{P}$ -position if and only if all of its options are $\mathcal{N}$ -positions.*

1.5. Sprague-Grundy Theorem, Sprague-Grundy Values, and Required Concepts

If G and H are two games, one can construct their *sum*, denoted $G + H$, by playing the two games side by side. To make a move on $G + H$, a player chooses either G or H and then makes a move on whichever game they chose. This means that $G + H$ is terminal if and only if both G and H are terminal, and the set of options of $G + H$ is $\{G' + H : G' \text{ is an option of } G\} \cup \{G + H' : H' \text{ is an option of } H\}$. We say that two games G and G' are *equivalent* if $G + H$ and $G' + H$ have the same outcome class for all games H . We denote this equivalence by $G \cong G'$. The Sprague-Grundy Theorem is a theorem for impartial combinatorial games that was proved independently by Sprague [5] and Grundy [6].

Theorem 3 (Sprague-Grundy Theorem). *If G is a combinatorial game, then there exists some nonnegative integer n such that $G \cong *n$, where $*n$ denotes a game of Nim consisting of one pile with n objects in it (or the empty game if $n = 0$).*

We call the number n from the above theorem the *Sprague-Grundy value* (*SG value*) of G , and denote it by $SG(G)$. If S is a set of nonnegative integers, then the *minimum excluded value* from S , denoted $\text{mex}S$, is the lowest nonnegative integer which is not an element of S . For example, $\text{mex}\{0, 1, 2, 5\} = 3$, and $\text{mex}\{1, 3, 5, 6\} = 0$.

Corollary 4. *If G is a combinatorial game, then*

$$SG(G) = \text{mex}\{SG(G') : G' \in G\}.$$

That is, the SG value of a game is the lowest nonnegative integer which is not an SG value of any of its options.

The above three results are proven by Sprague and Grundy in their respective works. Notice that the above corollary acts as a way to recursively determine the SG values of positions in a game. The base case for this recursion is that any terminal position will have an SG value of 0 since the set of options will be empty and $\text{mex}\emptyset = 0$. Since $\cong$ is an equivalence relation, we note that $G \cong H$ if and only if $\text{SG}(G) = \text{SG}(H)$. From this, the following results are easily derived.

Lemma 5. *If G and H are combinatorial games, then $G \cong H$ if and only if $G + H$ is a $\mathcal{P}$ -position.*

Theorem 6. *If G is a position in a combinatorial game, then G is a $\mathcal{P}$ -position if and only if $\text{SG}(G) = 0$.*

Remark 1. If G is a game, then $\text{SG}(G) \leq |G|$. That is, the maximum value of the SG value of a game is the number of options that game has. This is clear since the SG value is the minimum excluded value of all the game's options. If $\text{SG}(G) = n$, then G must have at least n options since there must be one option with each SG value from 0 to $n - 1$.

2. Background

2.1. The Game of The Princess and the Roses

The Princess and the Roses (P&R) was written about by Berlekamp, Conway, and Guy in *Winning Ways for your Mathematical Plays* [2]. The author was introduced to this game by his mentor, Zachary Wolske, who was himself introduced to it by the late Alfonso Gracia-Saz.

The story behind the game, as told by Gracia-Saz [7], is as follows. Princess Alice has two suitors, Laura and Renata, who take turns bringing her roses from the same garden. The garden consists of multiple bushes, each of which has roses of a unique color. On each suitor's turn she may either bring one rose, or she may bring two roses which are different colors. For example, if the garden consists of one bush with 3 red roses and one with 2 white roses, a suitor may either bring one red rose, one white rose, or one red rose and one white rose. The suitor who brings the final rose will win Alice's heart.

We will simulate this experience using objects that are sorted into piles. On each player's turn they may either remove one object or they may remove two objects, each from a different pile. As usual, the player who cannot make a move loses.

2.2. Notation

When writing about P&R, we will express positions as tuples, with each coordinate representing the number of roses in one of the bushes. For example, the position

where there is one bush with 3 roses and one with 2 will be expressed as $(3,2)$. We will always ensure that the piles are expressed in non-increasing order. So any 4-pile position will be expressed as (a,b,c,d) , with $a \geq b \geq c \geq d$.

Since parity comes up in many cases, we will use special notation for the parity of piles. An even pile will be denoted with an E , and an odd pile with an O . We will use the same tuple notation with the non-increasing restriction. For example, (E,E,O) can represent a position with 3 piles where the largest and second-largest piles have an even number of objects, and the smallest has an odd number of objects. We will also let (E,E,O) denote the set of such positions, and so we write $(6,2,1) \in (E,E,O)$ [1].

If $n < m$, we will consider the set of n -pile positions to be a subset of the set of m -pile positions. For example, we may use $(3,2)$ or $(3,2,0)$ depending on whether we want to use the perspective of 2 piles or 3 piles. We will express moves using tuples once again, except the tuple will represent the change in the values of piles instead of the piles themselves. For example, in a game with 4 piles, the move that takes one object from the largest pile and one from the third-largest pile will be written as $(-1,0,-1,0)$.

We say a move is *illegal* if it would either decrease a pile that has a size of zero, or if it would stop the piles from being in non-increasing order. Note that the first of these restrictions is already part of the game's rules, so it does not change anything. The second restriction does not change anything either, since if we would make a move that would remove the non-increasing condition, there always exists another move that reaches the same option but preserves the piles being non-increasing. For example, the move $(-1,0,-1,0)$ on the position $(4,2,2,2)$ would be illegal, but we can assume that the move $(-1,0,0,-1)$ is made instead since both moves would lead to the outcome $(3,2,2,1)$.

Sometimes we do not have enough information about a position to determine if a move is legal. For example, the move $(0,-1)$ might not be legal from (O,E) or (E,E) if the second pile has a size of zero. As another example, the move $(-1,0)$ might not be legal from (O,O) or (E,E) if the two piles have the same size.

2.3. Established Results

The $\mathcal{N}$ - and $\mathcal{P}$ -positions for up to 5 piles are known already, whereas for 6 piles it is an open problem.

Theorem 7. *If G is a position in $\mathcal{P}\mathcal{E}\mathcal{R}$ with 5 piles (or less), then G is a $\mathcal{P}$ -position if and only if $G \in (E,E,E,E,E) \cup (O,O,O,E,E) \cup (E,O,O,O,O) \cup (O,E,E,O,O)$.*

A proof of the above theorem can be found in Kenjoe Lim's article [1].

The Sprague-Grundy values for up to 3 piles are also known. Here is a sketch of the proof for 2 piles, and although the 3-pile case can be proven similarly, we will produce a novel proof of it using tools that are crucial for the 4-pile case.

Theorem 8. *If G is a position in $P\mathcal{E}R$ with 2 piles, then*

$$SG(G) = \begin{cases} 0 & \text{if } G \in (E, E) \\ 1 & \text{if } G \in (O, E) \\ 2 & \text{if } G \in (O, O) \\ 3 & \text{if } G \in (E, O). \end{cases}$$

Sketch of proof. We will prove this using Conway induction. Suppose that the result holds for all options of G . Notice that there are at most 3 possible moves: $(-1, 0)$, $(0, -1)$, and $(-1, -1)$. Since every move changes the parity of at least one pile, G cannot have an option whose piles are all the same parities as G . Note that this proof relies on the fact that all moves mentioned are always legal.

If $G \in (E, E)$, then none of the options of G are in (E, E) , so none of the options have a SG-value of 0. Thus, $SG(G) = 0$.

If $G \in (O, E)$, then none of the options of G have a SG value of 1. Additionally, the move $(-1, 0)$ will always create a position in (E, E) , so G always has an option with an SG value of 0. Thus, 1 is the minimum excluded value of the SG values of the options of G , and so $SG(G) = 1$.

The cases of $G \in (O, O)$ and $G \in (E, O)$ can be proven similarly, noting that all moves that lead to parity combinations associated with lower SG values are always legal. $\square$

3. Sprague-Grundy Values for 4 Piles

3.1. Reduction of the Problem

Unlike for 3 piles, the Sprague-Grundy values for 4 piles of P&R do not follow an immediately obvious pattern. As we will see, that is far from the case. One helpful step in the right direction will be to reduce this problem from one involving 4 degrees of freedom to one with only 3 degrees of freedom.

Lemma 9. *In 4-pile $P\mathcal{E}R$, if $a \geq b \geq c \geq d \geq 0$, and $n \geq 0$, then $SG(a, b, c, d) = SG(a + n, b + n, c + n, d)$.*

Proof. We will use induction on n . If $n = 0$ then the games are the same and so the result is obvious. Suppose $n > 0$, and assume that $SG(a, b, c, d) = SG(a + n - 1, b + n - 1, c + n - 1, d)$ for all (a, b, c, d) .

Now we will use Conway induction. let (a, b, c, d) be fixed and assume the result holds for all options of (a, b, c, d) . Recall that two games have the same SG value if and only if their sum is a $\mathcal{P}$ -position. So we will play the games (a, b, c, d) and $(a + n, b + n, c + n, d)$ side-by-side and show that there is a winning strategy for the previous player.

Suppose the next player makes any move on the first game (a, b, c, d) which turns it into (a', b', c', d') (where either $a' = a$ or $a' = a - 1$ depending on which move was made, and likewise for the other letters). Whichever piles were taken from by, their counterparts in the second game have at least as many objects in them, and so they can be taken from by the previous player. So the previous player can make a move on the same piles of the second game, which will turn it into $(a' + n, b' + n, c' + n, d')$. Since (a', b', c', d') is an option of (a, b, c, d) , by the hypothesis for Conway induction, $\text{SG}(a', b', c', d') = \text{SG}(a' + n, b' + n, c' + n, d')$, and so the new position is a $\mathcal{P}$ -position, meaning the previous player now have a winning strategy.

Suppose instead that the next player makes any move on the second game which involves taking from the fourth pile, turning it into $(a' + n, b' + n, c' + n, d - 1)$. Then it must be the case that $d > 0$, and so the previous player can make the same move on the first game since all piles have a nonzero number of objects in them. This will turn the first game into $(a', b', c', d - 1)$. Since this is an option of (a, b, c, d) , by the hypothesis of Conway induction, $\text{SG}(a', b', c', d - 1) = \text{SG}(a' + n, b' + n, c' + n, d - 1)$, and so and so the new position is a $\mathcal{P}$ -position.

Finally, suppose that the next player makes any move on the second game which does not involve taking from the fourth pile. Then the previous player can make a move on the second game where the set of piles taken from is the complement of the set of piles the next player took from within the set of the first three piles. So after both of those moves, the game will be $(a, b, c, d) + (a + n - 1, b + n - 1, c + n - 1, d)$, which is a $\mathcal{P}$ -position by the inductive hypothesis for the induction on n .

Thus, regardless of what the next player does, the previous player can create a $\mathcal{P}$ -position where they have a winning strategy, and so they have a winning strategy from the original position. Thus, $(a, b, c, d) \cong (a + n, b + n, c + n, d)$. $\square$

Corollary 10. *In 3 pile P&R, the SG values are given by*

$$\text{SG}(G) = \begin{cases} 0 & \text{if } G \in (E, E, E) \cup (O, O, O) \\ 1 & \text{if } G \in (O, E, E) \cup (E, O, O) \\ 2 & \text{if } G \in (O, O, E) \cup (E, E, O) \\ 3 & \text{if } G \in (E, O, E) \cup (O, E, O). \end{cases}$$

Proof. Let $G = (a, b, c)$ with $a \geq b \geq c$. By the previous theorem, $(a, b, c) \cong (a - c, b - c, 0) = (a - c, b - c)$. Then we can simply use the SG values for 2 piles since the parities of $a - c$ and $b - c$ are given by those of a , b , and c . $\square$

As a particular application of the previous theorem, for any $a \geq b \geq c \geq d \geq 0$, we can see that

$$\text{SG}(a - c + d, b - c + d, d, d) = \text{SG}(a, b, c, d).$$

Thus, the SG value of a position relies only on the values $a - c + d$, $b - c + d$, and d , and so we only require three degrees of freedom to determine the SG value. We

will now work to construct a game that takes advantage of these three important values.

Let us define a new game called *simplified P&R*, which is the same as P&R except one may not take from the third-largest pile unless they also take from the fourth-largest pile. That is, the moves $(0, 0, -1, 0)$, $(-1, 0, -1, 0)$, and $(0, -1, -1, 0)$ are not allowed. We will always denote positions in this game with square brackets. For example, the position $[3, 3, 3, 0]$ is a terminal position in this game since there are no legal moves. Every pile is the third-largest and there is no fourth-largest.

Lemma 11. *Let $a \geq b \geq c \geq d \geq 0$. Then for all $n \geq 0$, we have $SG(a, b, c, d) = SG[a + n, b + n, c + n, d]$.*

Proof. We will use strong Conway induction. Suppose that this holds for all positions that can be reached from (a, b, c, d) with some positive number of moves. If we can show that it holds for (a, b, c, d) , then it holds for all positions.

We will also use the fact that if G and H are games, then $SG(G) = SG(H)$ if and only if $G + H$ is a $\mathcal{P}$ -position. So we will put the two games side-by-side and show that there is a winning strategy for the previous player.

Suppose that the next player makes any move on one game that can be legally replicated on the other. Then the previous player can replicate that move, and the position will be $(a', b', c', d') + [a' + n, b' + n, c' + n, d']$, which is a $\mathcal{P}$ -position by the hypothesis for Conway induction since (a', b', c', d') is an option of (a, b, c, d) .

Suppose instead that the next player's move cannot be replicated. This is either because they made one of the moves on the first game that is no longer available in simplified P&R, or because they took from a pile that was full in one game but empty in the other. However, we can actually notice that the second situation is impossible. Such a pile would have to be one of the first three piles in the second game, since any other pile has a value that is less than or equal to its counterpart in the other game. However, since they did not take from the fourth pile in the second game, they also cannot have taken from the third pile, meaning the value of whichever pile(s) they took from is strictly greater than the value of the third pile. Since the differences in size of the first, second, and third piles are the same for both games, this means that the corresponding piles in the first game are nonempty, and so the move can be replicated. Thus, they must have made the move $(0, 0, -1, 0)$, $(-1, 0, -1, 0)$, or $(0, -1, -1, 0)$ on the first game. In this case, the previous player can make the move $(-1, -1, 0, 0)$, $(0, -1, 0, 0)$, or $(-1, 0, 0, 0)$, respectively, on the first game. After that, the new position will be $(a - 1, b - 1, c - 1, d) + [a + n, b + n, c + n, d]$. By the inductive hypothesis for strong Conway induction, this is a $\mathcal{P}$ -position since $(a - 1, b - 1, c - 1, d)$ is reachable from (a, b, c, d) with a finite sequence of moves. $\square$

We will now define another new game called *P&R Helper* (because we are defining it to help us). The positions in this game are the ordered triples $[x, y, z]$ (any triple

with square brackets denotes a position in this game) satisfying $0 \leq x \leq y \leq z$, and the moves are $(-1, -1, -1)$, $(-1, 0, 0)$, $(-1, -2, -1)$, $(-1, -1, -2)$, $(0, -1, 0)$, $(0, -1, -1)$, and $(0, 0, -1)$. Any move that would cause the values to no longer satisfy $0 \leq x \leq y \leq z$ is illegal, and the player who cannot move loses. Note that unlike Simplified P&R, the only terminal move in this game is $[0, 0, 0]$ since otherwise one of the three moves that decreases only one number can be made on the smallest nonzero number.

For any position G and any move M , let $G + M$ denote the option of G attained by performing the move M . For example, if $G = [a, b, c]$, then $G + (-1, 0, 0) = [a - 1, b, c]$.

The reasoning behind this game is that we want a game whose state depends only on the three crucial values that affect the SG value of a 4-pile P&R game. Our x , y , and z in P&R Helper represent the values d , $b - c + d$, and $a - c + d$, respectively. Indeed, it must always be the case that $d \leq b - c + d \leq a - c + d$. The seven moves allowed for P&R Helper simply show how the values of d , $b - c + d$, and $a - c + d$ change when we make one of the seven allowed moves in simplified P&R.

Theorem 12. *For all $a \geq b \geq c \geq d \geq 0$, we have $SG(a, b, c, d) = SG[d, b - c + d, a - c + d]$.*

Proof. Since we already know that $SG(a, b, c, d) = SG[a, b, c, d]$, it suffices to prove that $SG[a, b, c, d] = SG[d, b - c + d, a - c + d]$. We will prove this using Conway induction. Assume that the result holds for all options of $[a, b, c, d]$. We will play the two games $[a, b, c, d]$ and $[d, b - c + d, a - c + d]$ side-by-side and show that there is a winning strategy for the previous player.

We define a correspondence between the moves available in the first game, Simplified P&R, and in the second game, P&R Helper:

$$\begin{aligned} (0, 0, 0, -1) &\sim (-1, -1, -1) \\ (0, 0, -1, -1) &\sim (-1, 0, 0) \\ (0, -1, 0, -1) &\sim (-1, -2, -1) \\ (-1, 0, 0, -1) &\sim (-1, -1, -2) \\ (0, -1, 0, 0) &\sim (0, -1, 0) \\ (-1, -1, 0, 0) &\sim (0, -1, -1) \\ (-1, 0, 0, 0) &\sim (0, 0, -1). \end{aligned}$$

The winning strategy for the previous player is to always make the move that corresponds to whichever move the next player makes. Then after their move, the game will look like $[a', b', c', d'] + [d', b' - c' + d', a' - c' + d']$, which is a $\mathcal{P}$ -position by the inductive hypothesis. In addition, we can see that a move is illegal if and only if its corresponding move is. For example, we have that $a = b$ if and only if $a - c + d = b - c + d$, in which case the fourth and seventh pairs of moves are illegal. Thus, the original game is indeed a $\mathcal{P}$ -position. $\square$

Now that we have shown that $\text{SG}(a, b, c, d) = \text{SG}[d, b - c + d, a - c + d]$, we need only find the SG values of every position in P&R Helper in order to find all SG values for 4-pile P&R.

3.2. Sprague-Grundy Values for P&R Helper

In order to find the Sprague-Grundy values for P&R Helper, a computer program was written to recursively generate the Sprague-Grundy values for the positions satisfying $0 \leq x \leq y \leq z \leq 50$. These positions have been posted to the author's GitHub¹. Then, much time was spent looking at these positions trying to spot the patterns, and a set of rules was found that was consistent with said positions. In this paper we will detail these rules and the way that they were able to be found, and verify that they are true.

Let $[x, y, z]$ with $0 \leq x \leq y \leq z$ be a position in P&R Helper. Since each position in P&R Helper has at most 7 options, the SG values only go up to 7. Like with fewer piles, patterns emerge that describe the SG values. However, it is not governed by only one pattern. Sometimes the emergent pattern is based on the parities of x , y , and z , and sometimes on the value of $y + z - x$ or $y + 2z - 2x$ modulo 4 or 8. Either way, these patterns often do not emerge immediately. There are plenty of “exceptions”, that is, positions with specific values or specific constraints on the values of x , y and z that are not part of a greater pattern.

Which of these patterns applies depends greatly on the relative values of x , y , and z . The positions are split into four different “regions”, each with their own rules governing the SG values. We will call these regions Up, Right, Left, and Down (abbreviated U , R , L , and D , respectively) due to their positioning when the positions were into a text file with y increasing going to the right and z increasing going down. This text file can be seen on the aforementioned GitHub page. The regions are defined as follows, assuming $0 \leq x \leq y \leq z$:

$$\begin{aligned} U &= \{[x, y, z] : z < x + y, y \leq 2x + 1\} \\ R &= \{[x, y, z] : z < x + y, y > 2x + 1\} \\ L &= \{[x, y, z] : z \geq x + y, y \leq 2x\} \\ D &= \{[x, y, z] : z \geq x + y, y > 2x\}. \end{aligned}$$

Notice that these regions come from comparing the values of $a - b$, $b - c$, and d where (a, b, c, d) is a game of P&R. Indeed, when $[x, y, z] = [d, b - c + d, a - c + d]$, we can see that

$$\begin{aligned} z < x + y &\quad \text{if and only if} \quad a - b < d, \\ y \leq 2x &\quad \text{if and only if} \quad b - c \leq d. \end{aligned}$$

¹<https://github.com/Zev1/Princess-and-the-Roses-4-pile>

So these three values not only uniquely determine the SG values for 4-pile P&R positions, but the pattern of SG values is centered around them.

All of the aforementioned “exceptions” exist in region U , since this is the region where x , y , and z are closest to each other. There is still a pattern that governs most of the region, but it can be broken when x is small, or when y or z are close to x , or when $x + y$ is close to z (that is, when a position is close to being in L instead of U). Exceptions tend to happen due to certain moves being illegal at certain positions, and those differences ripple out and affect the positions that lead to them, and the ones that lead to those ones, and so on until it stabilizes into a pattern.

Now we will list the rules determining the SG value for a position $[x, y, z]$ of P&R Helper. Similarly to before, we will let E denote an even number and let O denote an odd number, so for instance $[E, E, O]$ is the set of positions where x and y are even and z is odd.

Theorem 13. *Let $G = [x, y, z]$ with $0 \leq x \leq y \leq z$. Then we have the following:*

- $\text{SG}(G) = 0$ if and only if $G \in [E, E, E]$
- $\text{SG}(G) = 1$ in the following cases:
 - If $G \in U \cup R$, then $\text{SG}(G) = 1$ if and only if one of the following holds:
 - * $G = [1, 1, 1]$,
 - * $G = [x, x, x]$ with $x \equiv 3 \pmod{4}$,
 - * $G = [x, x, x+1]$ with $x \equiv 1 \pmod{4}$ and $x \neq 1$,
 - * $G = [x, x+1, x+1]$ with $x \equiv 0 \pmod{4}$ and $x \neq 0$, or
 - * $y + z - x \equiv 1 \pmod{4}$ and $z > x + 1$.
 - If $G \in L \cup D$, then $\text{SG}(G) = 1$ if and only if $G \in [E, E, O]$.
- $\text{SG}(G) = 2$ in the following cases:
 - If $G \in U$, then $\text{SG}(G) = 2$ if and only if one of the following holds:
 - * $G = [x, x, x]$ with $x \equiv 1 \pmod{4}$ and $x \neq 1$,
 - * $G = [x, x, x+1]$ with $x \equiv 3 \pmod{4}$,
 - * $G = [x, x+1, x+1]$ with $x \equiv 2 \pmod{4}$ and $x \neq 2$,
 - * $G = [x, x, x+2]$ with $x \equiv 1 \pmod{4}$ and $x \neq 1$,
 - * $G = [x, x+1, x+2]$ with $x \equiv 0 \pmod{4}$ and $x \neq 0$,
 - * $G = [x, x+2, x+2]$ with $x \equiv 3 \pmod{4}$,
 - * $y + z - x \equiv 7 \pmod{8}$ and either:
 - $z > x + 4$ and $y = x$,
 - $z > x + 5$ and $y = x + 1$, or
 - $z > x + 2$ and $z = x + y - 1$; or
 - * $y + z - x \equiv 4 \pmod{8}$, $x = y$, x is odd, and $z > x + 4$,
 - * $y + z - x \equiv 2 \pmod{4}$, $G \notin [E, E, E]$, $z > x + 2$, and neither of the following apply:

- $y + z - x \equiv 6 \pmod{8}$, $z > x + 4$, and $y \leq x + 1$,
 - $y + z - x \equiv 6 \pmod{8}$ and $z = x + y - 2$.
 - If $G \in R$, then $\text{SG}(G) = 2$ if and only if $G \in [E, O, O]$.
 - If $G \in L$, then $\text{SG}(G) = 2$ if and only if $y + 2z - 2x \equiv 3 \pmod{4}$.
 - If $G \in D$, then $\text{SG}(G) = 2$ if and only if $G \in [E, O, O]$.
- $\text{SG}(G) = 3$ in the following cases:
 - If $G \in U$, then $\text{SG}(G) = 3$ if and only if one of the following holds:
 - * $G \in \{[1, 2, 2], [2, 2, 3], [5, 6, 6], [6, 6, 7], [3, 4, 5], [5, 5, 8]\}$,
 - * $G = [x, x, x + 1]$ with $x \equiv 0 \pmod{4}$ and $x \neq 0, 4$,
 - * $G = [x, x + 1, x + 1]$ with $x \equiv 3 \pmod{4}$ and $x \neq 3, 7$,
 - * $G = [x, x + 1, x + 2]$ with $x \equiv 1 \pmod{4}$ and $x \neq 1, 5$,
 - * $y + z - x \equiv 0 \pmod{8}$, $x = y$, x is odd, and $z > x + 5$,
 - * $y + z - x \equiv 6 \pmod{8}$, $G \notin [E, E, E]$, and either:
 - $z > x + 4$ and $y \leq x + 1$, or
 - $z > x + 2$ and $z = x + y - 2$; or
 - * $y + z - x \equiv 3 \pmod{4}$, $z > x + 2$, and neither of the following apply:
 - $y + z - x \equiv 7 \pmod{8}$, $z > x + 4$, and $y \leq x + 2$,
 - $y + z - x \equiv 7 \pmod{8}$, $y \leq 2x$, and $x + y - z \leq 3$.
 - If $G \in R$, then $\text{SG}(G) = 3$ if and only if $y + z - x \equiv 3 \pmod{4}$.
 - If $G \in L$, then $\text{SG}(G) = 3$ if and only if $y + 2z - 2x \equiv 1 \pmod{4}$.
 - If $G \in D$, then $\text{SG}(G) = 3$ if and only if $G \in [E, O, E]$.
- $\text{SG}(G) = 4$ in the following cases:
 - If $G \in U$, then $\text{SG}(G) = 4$ if and only if one of the following holds:
 - * $G \in \{[2, 3, 3], [5, 6, 7], [4, 4, 5], [7, 8, 8]\}$,
 - * $G = [x, x, x + 1]$ with $x \equiv 2 \pmod{4}$ and $x \neq 2, 6$,
 - * $G = [x, x + 1, x + 1]$ with $x \equiv 1 \pmod{4}$ and $x \neq 1, 5$,
 - * $G = [x, x + 1, x + 2]$ with $x \equiv 3 \pmod{4}$ and $x \neq 3, 7$,
 - * $G = [x, x + 1, x + 5]$ with $x \equiv 1 \pmod{8}$, or
 - * $y + z - x \equiv 0 \pmod{4}$, $G \notin [E, E, E]$, $z > x + 2$, and none of the following apply:
 - $G = [5, 5, 8]$,
 - $z > x + 4$, $y + z - x \equiv 4 \pmod{8}$, and $x = y$,
 - $z > x + 5$, $y + z - x \equiv 0 \pmod{8}$, and $x = y$,
 - $G = [x, x + 1, x + 3]$ with $x \equiv 0 \pmod{4}$ and $x \neq 0, 4, 8$,
 - $G = [x, x + 2, x + 3]$ with $x \equiv 3 \pmod{4}$ and $x \neq 3, 7$,
 - $G = [x, x + 1, x + 5]$ with $x \equiv 2 \pmod{8}$,
 - $G = [x, x + 2, x + 5]$ with $x \equiv 1 \pmod{8}$,
 - $G = [x, x + 1, x + 6]$ with $x \equiv 1 \pmod{8}$.
 - If $G \in R$, then $\text{SG}(G) = 4$ if and only if $y + z - x \equiv 0 \pmod{4}$ with x odd

	$z - (x + y)$	$y - 2x$	$y + z - x$	$y + 2z - 2x$
$(-1, -1, -1)$	+1	+1	-1	-1
$(-1, 0, 0)$	+1	+2	+1	+2
$(-1, -2, -1)$	+2	0	-2	-2
$(-1, -1, -2)$	0	+1	-2	-3
$(0, -1, 0)$	+1	-1	-1	-1
$(0, -1, -1)$	0	-1	-2	-3
$(0, 0, -1)$	-1	0	-1	-2

Table 1: How each move changes some important values.

- If $G \in L \cup D$, then $\text{SG}(G) = 4$ if and only if $G \in [O, E, O]$
- $\text{SG}(G) = 5$ in the following cases:
 - If $G \in U$, then $\text{SG}(G) = 5$ if and only if one of the following holds:
 - * $G \in \{[3, 4, 4], [7, 8, 9]\}$,
 - * $G = [x, x + 1, x + 3]$ with $x \equiv 0 \pmod{4}$ and $x \neq 0, 4, 8$,
 - * $G = [x, x + 2, x + 3]$ with $x \equiv 3 \pmod{4}$ and $x \neq 3, 7$,
 - * $G = [x, x + 1, x + 5]$ with $x \equiv 2 \pmod{8}$,
 - * $G = [x, x + 1, x + 6]$ with $x \equiv 1 \pmod{8}$,
 - * $y + z - x \equiv 7 \pmod{8}$, $z > x + 4$, $x + 2 = y$, and $x + y - z > 2$, or
 - * $y + z - x \equiv 7 \pmod{8}$, $z = x + y - 3$, $y > x + 2$, and $y \leq 2x$.
 - If $G \in R$, then $\text{SG}(G) = 5$ if and only if $y + z - x \equiv 2 \pmod{4}$ with x odd.
 - If $G \in L \cup D$, then $\text{SG}(G) = 5$ if and only if $G \in [O, E, E]$.
- $\text{SG}(G) = 6$ if and only if one of the following holds:
 - $G \in U$ and $G = [x, x + 2, x + 5]$ with $x \equiv 1 \pmod{8}$, or
 - $G \in D$ and $G \in [O, O, E]$.
- $\text{SG}(G) = 7$ if and only if $G \in D$ and $G \in [O, O, O]$

Proof. The proof of this uses a large but finite amount of case analysis, where each of the bullet points above is one of the cases that needs to be considered. We will be sketching it here and leaving further details to the reader.

The proof uses Conway induction once again, so let $G = [x, y, z]$ be a position in P&R Helper and assume the result holds for all options of G . Let n be the SG value that we want to show G to have, as indicated by the theorem.

There are many important things to note that will make this process easier. Firstly, we note that the seven possible moves in this game, along with $(0, 0, 0)$, cover all possible equivalence classes modulo 2 of x , y , and z .

In Table 1 we list the change each move has on some important values. We use the table to see which moves are capable of moving between different regions. Any move that increases the value of $z - (x + y)$ can move from U into L or from R into D , and vice versa. Any move that increases the value of $y - 2x$ can move from U into R or from L into D , and vice versa.

For a move to be able to go from U to D , it needs to increase $z - (x + y)$ without decreasing $y - 2x$. The moves that satisfy this are $(-1, -1, -1)$, $(-1, 0, 0)$, and $(-1, -2, -1)$. The move $(0, 0, -1)$ can likewise go from D to U .

For a move to be able to go from L to R , it needs to decrease $z - (x + y)$ and increase $y - 2x$ by at least 2, and vice versa. Neither of these are possible, so no move can go directly between L and R .

The other two values in the table, $y + z - x$ and $y + 2z - 2x$, are important modulo 4 or 8 for certain cases. The following lemmas all put together complete the proof. $\square$

Lemma 14. *Consider the cases in Theorem 13 that are defined by a finite set of positions. Suppose that G is one of these positions, or that some option of G is one of these positions. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. We can easily verify these positions using a computer, and in fact all such positions can be found in the GitHub repository². $\square$

Since we can account for all these finite-size cases, and any positions with one of them as an option, we can assume for the remainder of the paper that any G we are working with does not have any of those cases as an option.

Lemma 15. *Let $G \in D$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. If $G \in D$, then n is given by:

- 0 if $G \in [E, E, E]$
- 1 if $G \in [E, E, O]$
- 2 if $G \in [E, O, O]$
- 3 if $G \in [E, O, E]$
- 4 if $G \in [O, E, O]$
- 5 if $G \in [O, E, E]$
- 6 if $G \in [O, O, E]$
- 7 if $G \in [O, O, O]$.

²<https://github.com/Zev1/Princess-and-the-Roses-4-pile>

If $G \in D$ then no move can be illegal unless $x = 0$, in which case we know the result is true already since it is equivalent to a 3-pile game of P&R. If all options of G are in D then we easily verify the result since every move goes to a different set of parities.

Our boundary cases occur when $z = x + y$ or $y = 2x + 1$. If $z = x + y$ and $y > 2x + 1$ then every option of G is in D except for $G' = [x, y, z - 1] \in R$. Since $x + y + z$ is even, n is 0, 2, 4, or 6. If this were not a boundary case, then the SG value of $[x, y, z - 1]$ would be 1, 3, 5, or 7, respectively. Thus, the options of G that are still in D account for all SG values less than n . Since $x + y + (z - 1)$ is odd, looking at the rules for R we see that $\text{SG}(G')$ is 1 or 3, and thus $\text{SG}(G') \neq n$, so $\text{SG}(G) = n$.

If $y = 2x + 1$ and $z > x + y$, then every option of G is in D except for $[x, y - 1, z], [x, y - 1, z - 1] \in L$. Since $y = 2x + 1$, we know $y - 1$ is even and so, looking at the rules for L we see that the SG values for these options are the same as they would have been in the non-boundary case, so the result still holds.

If $z = x + y$ and $y = 2x + 1$, then the moves that go into L still do not affect G for the reasons mentioned in the previous paragraph. Just like in the $z = x + y$, $y > 2x + 1$ case, the moves that go into L and D account for all SG values less than n . It remains to show that $G' = [x, y, z - 1]$ does not have an SG value equal to n . Then we have

$$y + (z - 1) - x = y + y + x - 1 - x = 2y - 1 = 2(2x + 1) - 1 = 4x + 1 \equiv 1 \pmod{4}.$$

If $z - 1 > x + 1$, this means that $\text{SG}(G') = 1$. Otherwise, we have $z \leq x + 2$. Since $x \neq 0$ we have $z \geq y = 2x + 1$, and this means that $x + 2 \geq 2x + 1$ and so $x \leq 1$. We can see that $z > x + 2$ in this case if $x = 1$, and we already covered the case when $x = 0$. Thus, $\text{SG}(G') = 1$, and since n must be even we are done. $\square$

Lemma 16. *Let $G \in R$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. If $G \in R$, then n is given by:

- 0 if $G \in [E, E, E]$
- 1 if $y + z - x \equiv 1 \pmod{4}$
- 2 if $G \in [E, O, O]$
- 3 if $y + z - x \equiv 3 \pmod{4}$
- 4 if x is odd and $y + z - x \equiv 0 \pmod{4}$
- 5 if x is odd and $y + z - x \equiv 2 \pmod{4}$.

If every move is legal and all options of G are in R , since every move goes to a different residue class mod 2, every other SG value has a move to 0 or 2, and 0

and 2 cannot move to themselves. Since every value of $y + z - x$ goes to something different mod 4 and all three other residue classes are reached, every other SG value has a move to 1 or 3, and none of 1, 3, 4, or 5 can move to themselves. The move $(0, -1, -1)$ will move from 5 to 4.

Suppose that some option of G is in D . Then $z = x + y - 1$ and so $x + y + z$ is odd, and so $n = 1$ or 3. Since the SG values 0 and 2 follow the exact same rules in R and D , we know that those options will be accounted for. We can see that the moves $(-1, -1, -1)$, $(-1, 0, 0)$, $(0, -1, 0)$, and $(-1, -2, -1)$ will move into D . The first three will move to a position in D with an even sum, and so they will move to a position with an SG value of 0, 2, 4, or 6. The latter one will move to a position in D with an odd sum, and so it will move to a position with an SG value of 1, 3, 5, or 7. If $n = 1$ then $z = x + y - 1$ gives us $2y = 2 \pmod{4}$ and so y is odd, meaning y will still be odd after the move and so $\text{SG}(G') \neq 1$. A similar argument shows that $\text{SG}(G') \neq 3$ if $n = 3$. The move $(0, -1, -1)$ will still go from 3 to 1.

Suppose that some option of G is in U . Then $y = 2x + 2$ and the moves $(0, -1, 0)$ and $(0, -1, -1)$ will move into U , and they will move to positions where $y = 2x + 1$. Any of the rules concerning specific values of $x + y - z$ will not apply in this case. For each of the SG values, we can compare their rules in U and R and discover that the rules are either the same, or that the differences between them do not apply in this case.

Suppose that some move is illegal. Then $y = z$, meaning the moves $(-1, -1, -2)$ and $(0, 0, -1)$ will be illegal. If $y = z$ and $x + y + z$ is even then x is even, meaning n is not 4 or 5. Of course, any move can still go to 0. The move $(-1, -1, -2)$ decreases $y + z - x$ by 2, and the move $(0, 0, -1)$ decreases $y + z - x$ by 1. If $G \in [E, O, O]$, then either $(0, -1, 0)$ or $(-1, 0, 0)$ will move to some position with an SG value of 1 if $y + z - x$ is 1 or 3, respectively. If $y + z - x \equiv 3 \pmod{4}$, then $x + y + z$ is odd and so x must be odd. Then the move $(0, -1, -1)$ will still move to 1 and the move $(-1, 0, 0)$ or $(-1, -1, -1)$ will move to 2 if the starting position is $[O, O, O]$ or $[O, E, E]$, respectively. $\square$

Lemma 17. *Let $G \in L$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. If $G \in L$, then n is given by:

- 0 if $G \in [E, E, E]$
- 1 if $G \in [E, E, O]$
- 2 if $y + 2z - 2x \equiv 3 \pmod{4}$
- 3 if $y + 2z - 2x \equiv 1 \pmod{4}$
- 4 if $G \in [O, E, O]$
- 5 if $G \in [O, E, E]$.

Similarly to the previous cases, we can easily verify the result if all moves are legal and all options of G are in L .

Suppose that some option of G is in U . Then $z = x + y$ and so $x + y + z$ is even, and so n is 0, 2, 3, or 4. The move $(0, 0, -1)$ is the only move that will move into U , and it would otherwise move to 1, 3, 2, or 5, respectively. Thus, all moves that go to lower SG values still occur, except the move to an SG value of 2 when $y + 2z - 2x \equiv 1 \pmod{4}$. In this case, there are still other moves to an SG value of 2. The move $(0, 0, -1)$ will go to a position in U with $z = x + y - 1$. Since this position has an odd sum of x, y, z , its SG value cannot be 0 or 4. If the starting position was 2 or 3, then this position will have an odd sum of x, y, z with odd y , and so $2y \equiv 2 \pmod{4}$. Along with the fact that $z = x + y - 1$, this implies $y + z - x \equiv 1 \pmod{4}$, and so its SG value is 1.

Suppose that some option of G is in D and that $y = 2x$. Then y is even, meaning n is 0, 1, 4, or 5. The moves $(-1, -1, -1)$, $(-1, 0, 0)$, and $(-1, -1, -2)$ will move into D . The moves $(-1, -1, -1)$ and $(-1, -1, -2)$ move to positions with odd y and so would have moved to 2 and 3, but there is still a move that goes to 2 and one that goes to 3. Those two moves will now go to positions in D with odd y , so the SG values of those options will be 2, 3, 6, or 7. The move $(-1, 0, 0)$ will go to a position in D where y is even, and when y is even the SG values in L and D follow the exact same rules.

Suppose that some option of G is in D and that $y = 2x - 1$. Then y is odd, meaning n is 2 or 3. The move $(-1, 0, 0)$ will move into D , and if this option were still in L then its SG value would have been 3 or 2, respectively. Now, it will move to a position $G' \in D$ with odd y , meaning $\text{SG}(G')$ is 2, 3, 6, or 7. If $n = 2$, then $3 \equiv y + 2z - 2x = 2x - 1 + 2z - 2x = 2z - 1 \pmod{4}$, and so $2z \equiv 4 \pmod{4}$ and z is even, meaning the move $(-1, 0, 0)$ cannot move to 2. A similar argument applies if $n = 3$.

Suppose that some move is illegal. Then $y = x$, meaning the moves $(-1, -2, -1)$, $(0, -1, 0)$, and $(0, -1, -1)$ are illegal. If x and y are odd, then n is 2 or 3, and the latter two moves would have gone to 4 and 5. The first move would have gone to 3 or 2, respectively, but there are still other moves that do that. If the starting x and y are even, then n is 0 or 1. The move $(0, 0, -1)$ still goes from 1 to 0 in this case. $\square$

Lemma 18. *Consider the cases in Theorem 13 that cover a family of positions uniquely determined by x . Suppose that G is in one of these cases. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. For the cases which are defined only in terms of x , we can verify them by looking at all their options and seeing which cases they fall under. We also note that for each of these cases, there are only finitely many possible values for x for which any option will lie outside of U , and any such position is one we have checked

Move	Result	SG value of result
$(-1, -1, -1)$	$[x, x+1, x+1], x \equiv 2 \pmod{4}, x \neq 2, 6$	2
$(-1, 0, 0)$	$[x, x+2, x+2], x \equiv 2 \pmod{4}, x \neq 2, 6$	0
$(-1, -2, -1)$	$[x, x, x+1], x \equiv 2 \pmod{4}, x \neq 2, 6$	4
$(-1, -1, -2)$	Illegal	
$(0, -1, 0)$	$[x, x, x+1], x \equiv 3 \pmod{4}, x \neq 3, 7$	2
$(0, -1, -1)$	$[x, x, x], x \equiv 3 \pmod{4}, x \neq 3, 7$	1
$(0, 0, -1)$	Illegal	

Table 2: Options of G for Lemma 18.

manually.

For example, consider the case where $G = [x, x+1, x+1]$ with $x \equiv 3 \pmod{4}$ and $x \neq 3, 7$. In this case, $n = 3$. We can analyze the options of G for each move, as shown in Table 2. Thus, we can see in this case that $\text{SG}(G) = 3$, so the result of Theorem 13 holds for G . The other such cases are left as an exercise. $\square$

So we have done the cases which lie in R , L , and D , and those in U which are either one single position or a family of positions uniquely determined by x . The remaining cases are all defined by the value of $y+z-x$ modulo 4 or 8, along with some restrictions. Note that none of these cases can have an option in that same case, since every move changes the value of $y+z-x$ by 1, -1 , or -2 . So for each of these cases, we must show two things: one is that none of its options have an SG value of n , and one is that every SG value less than n is accounted for by one of its options.

We will very often have to deal with edge cases where some option lies outside U , so we will deal with those first. For the remaining theorems, we will use x, y, z to denote the entries of the original position G and x', y', z' for any of its options, which we will denote by G' .

Lemma 19. *Let $G \in U$ and suppose that G is in one of the cases not determined by finitely many positions or uniquely by x . Suppose that the result of Theorem 13 holds for all options of G . Then if G' is an option of G which does not lie in U , then $n \neq \text{SG}(G')$, where n is as described above.*

Proof. We will show this by considering each possibility of the region that G' is in, and the move used to get there. Note that $n \neq 7$ since $G \in U$, and that n cannot be 6 since $G \in U$ but is not in one of the cases determined uniquely by x . We also know by Theorems 2 and 6 that $n \neq \text{SG}(G')$ if either $n = 0$ or $\text{SG}(G') = 0$.

Case 1: $G' = G + (-1, -1, -1) \in D$. Then $z' = x' + y'$ and $y' = 2x' + k$ for some $k \in \{1, 2\}$. Then $G' = [x', 2x' + k, 3x' + k]$, so $G = [x' + 1, 2x' + k + 1, 3x' + k + 1]$.

Case 1a: $k = 1$ and x' is even. Then $\text{SG}(G') = 2$ and $y + z - x = 4x' + 3 \equiv$

$3 \pmod{8}$, so $n \neq 2$.

Case 1b: $k = 1$ and x' is odd. Then $\text{SG}(G') = 6$.

Case 1c: $k = 2$ and x' is even. Then $\text{SG}(G') = 0$.

Case 1d: $k = 2$ and x' is odd. Then $\text{SG}(G') = 4$ and $y + z - x = 4x' + 5 \equiv 1 \pmod{4}$, so $n \neq 4$.

Case 2: $G' = G + (-1, 0, 0) \in D$. Then $z' = x' + y'$ and $y' = 2x' + k$ for some $k \in \{1, 2, 3\}$. Then $G' = [x', 2x' + k, 3x' + k]$, so $G = [x' + 1, 2x' + k, 3x' + k]$ and so $y + z - x = 4x' + 2k - 1$.

Case 2a: x' is even and $k = 2$. Then $\text{SG}(G') = 0$.

Case 2b: x' is even and k is odd. Then $\text{SG}(G') = 2$. Then $y + z - x \equiv 2k - 1 \pmod{8}$, which is either 1 or 5 $\pmod{8}$, and thus $n \neq 2$.

Case 2c: x' is odd and $k = 2$. Then $\text{SG}(G') = 4$. Then $y + z - x = 4x' + 2k - 1 \equiv 7 \pmod{8}$, so $n \neq 4$.

Case 2d: x' is odd and k is odd. Then $\text{SG}(G') = 6$.

Case 3: $G' = G + (-1, -2, -1) \in D$. Then $y' = 2x' + 1$ and $z' = x' + y' + k$ for some $k \in \{0, 1\}$. Then $G' = [x', 2x' + 1, 3x' + k + 1]$, so $G = [x' + 1, 2x' + 3, 3x' + k + 2]$. Then $y + z - x = 4x' + 4 + k$.

Case 3a: x' is even and $k = 0$. Then $\text{SG}(G') = 2$ and $y + z - x \equiv 4 \pmod{8}$, so n can only equal 2 if $x = y$. But $(-1, -2, -1)$ is illegal if $x = y$, so this cannot be the case.

Case 3b: x' is even and $k = 1$. Then $\text{SG}(G') = 3$ and $y + z - x \equiv 5 \pmod{8}$, so $n \neq 3$.

Case 3c: x' is odd and $k = 0$. Then $\text{SG}(G') = 6$.

Case 3d: x' is odd and $k = 1$. Then $\text{SG}(G') = 7$.

Case 4: $G' = G + (-1, -1, -1) \in L$. Then $z' = x' + y'$. So $G' = [x', y', x' + y']$, so $G = [x' + 1, y' + 1, x' + y' + 1]$. Then $y + z - x = 2y' + 1$ and $y' + 2z' - 2x' = 3y'$.

Case 4a: x' and y' are even. Then $\text{SG}(G') = 0$.

Case 4b: x' is odd and y' is even. Then $\text{SG}(G') = 4$ and since $y + z - x$ is odd, $n \neq 4$.

Case 4c: $y' \equiv 1 \pmod{4}$. Then $\text{SG}(G') = 2$ and $y + z - x \equiv 3 \pmod{8}$, so $n \neq 2$.

Case 4d: $y' \equiv 3 \pmod{4}$. Then $\text{SG}(G') = 3$ and $y + z - x \equiv 7 \pmod{8}$. Since $x + y - z \leq 3$, we have $n \neq 3$.

Case 5: $G' = G + (-1, 0, 0) \in L$. Then $z' = x' + y'$. So $G' = [x', y', x' + y']$, so

$G = [x' + 1, y', x' + y']$. Then $y + z - x = 2y' - 1$ and $y' + 2z' - 2x' = 3y'$.

Case 5a: x' and y' are even. Then $\text{SG}(G') = 0$.

Case 5b: x' is odd and y' is even. Then $\text{SG}(G') = 4$ and since $y + z - x$ is odd, $n \neq 4$.

Case 5c: $y' \equiv 1 \pmod{4}$. Then $\text{SG}(G') = 2$ and $y + z - x \equiv 1 \pmod{8}$, so $n \neq 2$.

Case 5d: $y' \equiv 3 \pmod{4}$. Then $\text{SG}(G') = 3$ and $y + z - x \equiv 5 \pmod{8}$, meaning $n \neq 3$.

Case 6: $G' = G + (-1, -2, -1) \in L$. Then $z' = x' + y' + k$ for some $k \in \{0, 1\}$. So $G' = [x', y', x' + y' + k]$, so $G = [x' + 1, y' + 2, x' + y' + k + 1]$. Then $y + z - x = 2y' + k + 2$ and $y' + 2z' - 2x' = 3y' + 2k$.

Case 6a: y' is even, x' is even, and $k = 0$. Then $\text{SG}(G') = 0$.

Case 6b: y' is even, x' is even, and $k = 1$. Then $\text{SG}(G') = 1$. Then $y + z - x \equiv 3 \pmod{4}$ since y' is even, and thus $n \neq 1$.

Case 6c: y' is even, x' is odd, and $k = 0$. Then $\text{SG}(G') = 4$ and $n = 0$.

Case 6d: y' is even, x' is odd, and $k = 1$. Then $\text{SG}(G') = 5$ and $y + z - x \equiv 3 \pmod{4}$. But in this case $x + y - z = 1$, so $n \neq 5$.

Case 6e: $y' \equiv 1 \pmod{4}$. Then $\text{SG}(G') = 2$.

Case 6ei: $k = 0$. Then $y + z - x \equiv 4 \pmod{8}$. But $x \neq y$ since otherwise this move would be illegal, thus $n \neq 2$.

Case 6eii: $k = 1$. Then $\text{SG}(G') = 2$ and $y + z - x \equiv 5 \pmod{8}$, so $n \neq 2$.

Case 6f: $y' \equiv 3 \pmod{4}$. Then $\text{SG}(G') = 3$.

Case 6fi: $k = 0$. Then $y + z - x \equiv 0 \pmod{8}$. But $x \neq y$ since otherwise this move would be illegal, thus $n \neq 3$.

Case 6fii: $k = 1$. Then $y + z - x \equiv 1 \pmod{8}$, so $n \neq 3$.

Case 7: $G' = G + (0, -1, 0) \in L$. Then $z' = x' + y'$, so $G' = [x', y', x' + y']$ and $G = [x', y' + 1, x' + y']$. Then $y + z - x = 2y' + 1$ and $y' + 2z' - 2x' = 3y'$.

Case 7a: y' is even and x' is even. Then $\text{SG}(G') = 0$.

Case 7b: y' is even and x' is odd. Then $\text{SG}(G') = 4$ and $y + z - x \equiv 1 \pmod{4}$, so $n \neq 4$.

Case 7c: $y' \equiv 1 \pmod{4}$. Then $\text{SG}(G') = 2$ and $y + z - x \equiv 3 \pmod{8}$, so $n \neq 2$.

Case 7d: $y' \equiv 3 \pmod{4}$. Then $\text{SG}(G') = 3$ and $y + z - x \equiv 7 \pmod{8}$. Since $x + y - z = 1$, we have $n \neq 3$.

Case 8: $G' = G + (-1, -1, -1) \in R$. Then $y' = 2x' + 2$, so $G' = [x', 2x' + 2, z']$, so $G = [x' + 1, 2x' + 3, z' + 1]$. Then $y + z - x = x' + z' + 3$ and $y' + z' - x' = x' + z' + 2$.

Case 8a: x' is even and z' is even. Then $\text{SG}(G') = 0$.

Case 8b: $x' + z' \equiv 1 \pmod{4}$. Then $\text{SG}(G') = 3$ and $y + z - x \equiv 0 \pmod{4}$, so n can be 3 only if $x = y$. But we cannot have $x = y$ since $y = 2x + 1$, and so $n \neq 3$.

Case 8c: $x' + z' \equiv 3 \pmod{4}$. Then $\text{SG}(G') = 1$ and $y + z - x \equiv 2 \pmod{4}$, so $n \neq 1$.

Case 8d: x' is odd and $x' + z' \equiv 0 \pmod{4}$. Then $\text{SG}(G') = 5$ and $y + z - x \equiv 3 \pmod{4}$. Then n can be 5 only if $x + 2 = y$ or $y \leq 2x$. But since we must have $y = 2x + 1$, the latter cannot be the case. If $y = 2x + 1$ and $x + 2 = y$, then $x = 1$, meaning $x' = 0$, and so $\text{SG}(G') < 4$.

Case 8e: x' is odd and $x' + z' \equiv 2 \pmod{4}$. Then $\text{SG}(G') = 4$ and $y + z - x \equiv 1 \pmod{4}$, so $n \neq 4$.

Case 9: $G' = G + (-1, 0, 0) \in R$. Then $y' = 2x' + k$ for some $k \in \{2, 3\}$. Then $G' = [x', 2x' + k, z']$ and $G = [x' + 1, 2x' + k, z']$. Then $y + z - x = x' + z' + k - 1$ and $y' + z' - x' = x' + z' + k$.

Case 9a: x' is even, z' is even, and $k = 2$. Then $\text{SG}(G') = 0$.

Case 9b: x' is even, z' is odd, and $k = 3$. Then $\text{SG}(G') = 2$ and $y = 2x + 1$. Then $y + z - x$ is odd, so n can be 2 only if $y = x$, $y = x + 1$, or $z = x + y - 1$. In the former two cases, we cannot have $y = 2x + 1$ since this would force x to be -1 or 0 , respectively. If $z = x + y - 1$, then $G + (-1, 0, 0) \in D$. Thus, $n \neq 2$.

Case 9c: $x' + z' + k \equiv 1 \pmod{4}$. Then $\text{SG}(G') = 1$ and $y + z - x \equiv 2 \pmod{4}$, so $n \neq 1$.

Case 9d: $x' + z' + k \equiv 3 \pmod{4}$. Then $\text{SG}(G') = 3$ and $y + z - x \equiv 2 \pmod{4}$. Then n can be 3 only if $y \leq x + 1$ or $z = x + y - 2$. If $y = x$ then $2x' + k = y' = x' + 1$, so $x' = 1 - k$ which is not possible. If $y = x + 1$ then $2x' + k = y' = x' + 2$, so $x' = 2 - k$ which is not possible. If $z = x + y - 2$, then $z' = x' + y' - 1$, so $x' + z' + k = 4x' + 2k - 1 \equiv 3 \pmod{4}$, so $k = 2$, so $y = 2x$. Then $z = x + y - 2 = 3x - 2$, so $G = [x, 2x, 3x - 2]$. If x is odd then $y + z - x \equiv 2 \pmod{8}$, so $n \neq 3$ and if x is even then $n = 0$. Thus, $n \neq 3$.

Case 9e: x' is odd and $x' + z' + k \equiv 0 \pmod{4}$. Then $\text{SG}(G') = 4$ and $y + z - x \equiv 3 \pmod{4}$, so $n \neq 4$.

Case 9f: x' is odd and $x' + z' + k \equiv 2 \pmod{4}$. Then $\text{SG}(G') = 5$ and $y + z - x \equiv 1 \pmod{4}$, so $n \neq 5$.

Case 10: $G = G + (-1, -1, -2) \in R$. Then $y' = 2x' + 2$. Then $G' = [x', 2x' + 2, z']$ and $G = [x' + 1, 2x' + 3, z' + 2]$. Then $y + z - x = x' + z' + 4$ and $y' + z' - x' = x' + z' + 2$.

Move	$y' + z' - x' \pmod{4}$		Can be in L
$(-1, -1, -1)$	0	$z' > x' + 1$	Yes
$(-1, 0, 0)$	2	$z' > x' + 2$	Yes
$(-1, -2, -1)$	3	$z' > x' + 1$	Yes
$(-1, -1, -2)$	3	$z' > x'$	No
$(0, -1, 0)$	0	$z' > x' + 1$	Yes
$(0, -1, -1)$	3	$z' > x'$	No
$(0, 0, -1)$	0	$z' > x'$	No

Table 3: Options of G for Lemma 20.

Case 10a: x' is even and z' is even. Then $\text{SG}(G') = 0$.

Case 10b: $x' + z' \equiv 1 \pmod{4}$. Then $\text{SG}(G') = 3$ and $y + z - x \equiv 1 \pmod{4}$, so $n \neq 3$.

Case 10c: $x' + z' \equiv 3 \pmod{4}$. Then $\text{SG}(G') = 1$ and $y + z - x \equiv 3 \pmod{4}$, so $n \neq 1$.

Case 10e: x' is odd and $x' + z' \equiv 0 \pmod{4}$. Then $\text{SG}(G') = 5$ and $y + z - x \equiv 0 \pmod{4}$, so $n \neq 5$.

Case 10f: x' is odd and $x' + z' \equiv 2 \pmod{4}$. Then $\text{SG}(G') = 4$ and $y + z - x \equiv 2 \pmod{4}$, so $n \neq 4$. $\square$

Now, it suffices to show that for each case in U , there is always a move going to each SG value lower than n , and there are no options in U whose SG value is equal to n . Note that for all $G \notin [E, E, E]$, there is always a legal move into $[E, E, E]$, so there will always be an option with an SG value of 0.

Lemma 20. *Let $G \in U$, and suppose that $y + z - x \equiv 1 \pmod{4}$ and $z > x + 1$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. In this case, $n = 1$. The options of G are as shown in Table 3. It suffices to show that G has no option whose SG value is 1. Since $z' > x'$ for all options, none of them can be of the form $[x, x, x]$. The other cases in U with an SG value of 1 that we need to consider are $[x, x, x + 1]$ with $x \equiv 1 \pmod{4}$, and $[x, x + 1, x + 1]$ with $x \equiv 0 \pmod{4}$. These cases both have $y' + z' - x' \equiv 2$, but of the options of G that can have $z' = x' + 1$ either have $y' + z' - x' \equiv 0$ or $3 \pmod{4}$. $\square$

Lemma 21. *Let $G \in U$, and suppose that $y + z - x \equiv 7 \pmod{8}$, $z > x + 4$, and $y = x$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. In this case, $n = 2$. The options of G are as shown in Table 4. We determine

Move	$y' + z' - x' \pmod{8}$			SG value if in U
$(-1, -1, -1)$	6	$z' > x' + 4$	$y' = x'$	0 or 3
$(-1, 0, 0)$	0	$z' > x' + 5$	$y' = x' + 1$	4
$(-1, -2, -1)$	Illegal			
$(-1, -1, -2)$	5	$z' > x' + 3$	$y' = x'$	1
$(0, -1, 0)$	Illegal			
$(0, -1, -1)$	Illegal			
$(0, 0, -1)$	6	$z' > x' + 3$	$y' = x'$	0 or 3

Table 4: Options of G for Lemma 21.

Move	$y' + z' - x' \pmod{8}$			SG value if in U
$(-1, -1, -1)$	6	$z' > x' + 5$	$y' = x' + 1$	3
$(-1, 0, 0)$	0	$z' > x' + 6$	$y' = x' + 2$	0 or 4
$(-1, -2, -1)$	5	$z' > x' + 5$	$y' = x'$	1
$(-1, -1, -2)$	5	$z' > x' + 4$	$y' = x' + 1$	1
$(0, -1, 0)$	6	$z' > x' + 5$	$y' = x'$	0 or 3
$(0, -1, -1)$	5	$z' > x' + 4$	$y' = x'$	1
$(0, 0, -1)$	6	$z' > x' + 4$	$y' = x' + 1$	3

Table 5: Options of G for Lemma 22.

the last column of this table by examining which case each of these options falls under. For the options labeled “0 or 3”, one of their SG values will be 0 and the other will be 3.

Since $x = y$, any option of G satisfies $y' \leq x' + 1 \leq 2x' + 1$, so no option can be in R . Thus, in particular, $G + (-1, -1, -2)$ will be in U and so there is an option of G whose SG value is 1. We also note that $(-1, -1, -2)$ is legal since $z > y$.

Thus, G has options whose SG values are 0 and 1, and no option whose SG value is 2, and thus $\text{SG}(G) = 2$. $\square$

Lemma 22. *Let $G \in U$, and suppose that $y + z - x \equiv 7 \pmod{8}$, $z > x + 5$, and $y = x + 1$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. In this case, $n = 2$. The options of G are as shown in Table 5. Note that $(0, -1, -1)$ is legal and $G + (0, -1, -1)$ must be in U , so $\text{SG}(G + (0, -1, -1)) = 1$. We can see that none of the other SG values are 2 as long as they are in U .

Thus, G has options whose SG values are 0 and 1, and no option whose SG value is 2, and thus $\text{SG}(G) = 2$. $\square$

Lemma 23. *Let $G \in U$, and suppose that $y + z - x \equiv 7 \pmod{8}$, $z > x + 2$, and*

Move	$y' + z' - x' \pmod{8}$		
$(-1, -1, -1)$	6	$z' > x' + 2$	$z' = x' + y'$
$(-1, 0, 0)$	0	$z' > x' + 3$	$z' = x' + y'$
$(-1, -2, -1)$	5	$z' > x' + 2$	$z' = x' + y' + 1$
$(-1, -1, -2)$	5	$z' > x' + 1$	$z' = x' + y' - 1$
$(0, -1, 0)$	6	$z' > x' + 2$	$z' = x' + y'$
$(0, -1, -1)$	5	$z' > x' + 1$	$z' = x' + y' - 1$
$(0, 0, -1)$	6	$z' > x' + 1$	$z' = x' + y' - 2$

Table 6: Options of G for Lemma 23.

$z = x + y - 1$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .

Proof. In this case, $n = 2$. The options of G are as shown in Table 6. The options $G + (-1, -1, -2)$ and $G + (0, -1, -1)$ satisfy $z' < x' + y'$, so they are in $U \cup R$, and they have $y' + z' - x' \equiv 1 \pmod{4}$ and $z' > x' + 1$, so those options both have an SG value of 1. Also, at least one of these moves is legal unless $x = y = z$, which is not possible. It remains to show that no option of G has an SG value of 2.

The option $G + (0, 0, -1)$ must be in U . If $z' > x' + 2$ then its SG value is 0 or 3 since it satisfies $y' + z' - x' \equiv 6 \pmod{8}$ and $z' = x' + y' - 2$. If $z' = x' + 2$, then in order to have an SG value of 2, we would need $y' + z' - x'$ to be odd (we can see this by checking the three cases in which $n = 2$ and $z = x + 2$). But $y' + z' - x'$ is even, so this cannot be the case.

The other options besides the three mentioned above are all in L or D , so by Lemma 19 we are done.

Thus, G has options whose SG values are 0 and 1, and no option whose SG value is 2, and thus $\text{SG}(G) = 2$. $\square$

Lemma 24. Let $G \in U$, and suppose that $y + z - x \equiv 4 \pmod{8}$, $z > x + 4$, $x = y$, and x is odd. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .

Proof. In this case, $n = 2$. The options of G are as shown in Table 7. The move $(-1, 0, 0)$ is legal unless $x = 0$, in which case we already know the result to be true. Since $x = y$, any option of G satisfies $y' \leq x' + 1 \leq 2x' + 1$, so no option can be in R .

The only options of G that can be in $L \cup D$ are $G + (-1, -1, -1)$ and $G + (-1, 0, 0)$. However, if either of these are in L then they must satisfy $z' = x' + y'$ since they both only increase $z - (x + y)$ by 1. However, this would mean $y' + z' - x'$ would

Move	$y' + z' - x' \pmod{8}$			x'	SG if in U
$(-1, -1, -1)$	3	$z' > x' + 4$	$y' = x'$	E	3
$(-1, 0, 0)$	5	$z' > x' + 5$	$y' = x' + 1$	E	1
$(-1, -2, -1)$	Illegal				
$(-1, -1, -2)$	2	$z' > x' + 3$	$y' = x'$	E	0
$(0, -1, 0)$	Illegal				
$(0, -1, -1)$	Illegal				
$(0, 0, -1)$	3	$z' > x' + 4$	$y' = x'$	O	3

Table 7: Options of G for Lemma 24.

Move	$y' + z' - x' \pmod{4}$	
$(-1, -1, -1)$	1	$z' > x' + 2$
$(-1, 0, 0)$	3	$z' > x' + 3$
$(-1, -2, -1)$	0	$z' > x' + 2$
$(-1, -1, -2)$	0	$z' > x' + 1$
$(0, -1, 0)$	1	$z' > x' + 2$
$(0, -1, -1)$	0	$z' > x' + 1$
$(0, 0, -1)$	1	$z' > x' + 2$

Table 8: Options of G for Lemma 25.

need to be even, which is not the case for either of these. Thus, all options of G are in U .

Thus, G has options whose SG values are 0 and 1, and no option whose SG value is 2, and thus $\text{SG}(G) = 2$. $\square$

Lemma 25. *Let $G \in U$, and suppose that $y + z - x \equiv 2 \pmod{4}$, $G \notin [E, E, E]$, $z > x + 2$, and neither of the following apply:*

- $y + z - x \equiv 6 \pmod{8}$, $z > x + 4$, and $y \leq x + 1$,
- $y + z - x \equiv 6 \pmod{8}$ and $z = x + y - 2$.

If Theorem 13 holds for all options of G , then Theorem 13 holds for G .

Proof. In this case, $n = 2$. The options of G are as shown in Table 8. Suppose $y' + z' - x' \equiv 1 \pmod{4}$. If $G' \in U$, then $\text{SG}(G') = 1$. If $z \neq y$, then $G + (0, 0, -1) \in U$ and so G has an option with an SG value of 1. If $z = y$ then $x \neq y$ and so $(0, -1, 0)$ is legal, and $G + (0, -1, 0) \in U$ as long as $x > 1$, and there are only finitely many positions in U with $x \leq 1$, so we have checked those. Thus, G has an option whose SG value is 1.

It remains to show that no option has an SG value of 2.

Suppose $y' + z' - x' \equiv 0 \pmod{4}$. Suppose for a contradiction that $G' \in U$ with $\text{SG}(G') = 2$. Noting that $z' > x' + 1$, the only $n = 2$ case which satisfies this is the case where $y' + z' - x' \equiv 4 \pmod{8}$, $x' = y'$, x' is odd, and $z' > x' + 4$. But regardless of which of those three moves was taken, this would imply that $y + z - x \equiv 6 \pmod{8}$, $z > x + 4$, and $y \leq x + 1$, which is excluded in the assumption of this lemma.

Consider the option $G' = G + (-1, 0, 0)$. If $G' \in U$ then $\text{SG}(G') = 3$ unless $y' + z' - x' \equiv 7 \pmod{8}$ and either $z' > x' + 4$ and $y' \leq x' + 2$, or $y' \leq 2x'$ and $x' + y' - z' \leq 3$.

Case 1: $y' + z' - x' \equiv 7 \pmod{8}$, $z' > x' + 4$, and $y' \leq x' + 2$. This would mean that $y + z - x \equiv 6 \pmod{8}$, $z > x + 3$, and $y \leq x + 1$. This cannot be the case by the assumption of this lemma unless $z = x + 4$. If this is true, then we have $z' = x' + 5$ and $y' = x' + k$ for some $k \in \{1, 2\}$ (k cannot be 0 since otherwise the original position would have $x > y$). Then $7 \equiv y' + z' - x' = x' + k + x' + 5 - x' = x' + 5 + k \pmod{8}$, so $x' \equiv 2 - k \pmod{8}$. If $k = 1$ then $G' = [x', x' + 1, x' + 5]$ with $x' \equiv 1 \pmod{8}$ in which case $\text{SG}(G') = 4$. If $k = 2$ then $x' + 2 = y'$ and $x' + y' - z' = x' + (x' + 2) - (x' + 5) = x' - 3$, which is greater than 2 since $x' \equiv 0 \pmod{8}$ and so $x' \geq 8$. In this case $\text{SG}(G') = 5$.

Case 2: $y' + z' - x' \equiv 7 \pmod{8}$, $y' \leq 2x'$, and $x' + y' - z' \leq 3$. Suppose $x' + y' - z' = 3$. Then if $y' > x' + 1$ we will have $\text{SG}(G') = 5$. If $y' = x' + 1$ then that means $y = x$ and $z > x + 4$, which cannot be the case. Suppose instead that $x' + y' - z' \leq 2$. Then $x' + y' - z' = 1$ since the sum of all three numbers is odd. But then $y + z - x \equiv 6 \pmod{8}$ and $x + y - z = 2$, which is a contradiction.

Thus, G has options whose SG values are 0 and 1, and no option whose SG value is 2, and thus $\text{SG}(G) = 2$. $\square$

Lemma 26. *Let $G \in U$, and suppose that $y + z - x \equiv 0 \pmod{8}$, $z > x + 5$, $x = y$, and x is odd. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. In this case, $n = 3$. The options of G are as shown in Table 9.

The move $(-1, 0, 0)$ is always legal if $x \neq 0$. Even if $G' = G + (-1, 0, 0)$ is in L or D , its SG value will still be 1 since $G' \in [E, E, O]$. Also, $(0, 0, -1)$ is legal since $z \neq y$ and $G + (0, 0, -1)$ must be in U , so its SG value is 2.

Thus, G has options whose SG values are 0, 1, and 2, and no option whose SG value is 3, and thus $\text{SG}(G) = 3$. $\square$

Lemma 27. *Let $G \in U$, and suppose that $y + z - x \equiv 6 \pmod{8}$, $G \notin [E, E, E]$, $z > x + 4$, and $y = x$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. In this case, $n = 3$. The options of G are as shown in Table 10.

Move	$y' + z' - x' \pmod{8}$			x'	SG in U
$(-1, -1, -1)$	7	$z' > x' + 5$	$y' = x'$	E	2
$(-1, 0, 0)$	1	$z' > x' + 6$	$y' = x' + 1$	E	1
$(-1, -2, -1)$	Illegal				
$(-1, -1, -2)$	6	$z' > x' + 4$	$y' = x'$	E	0
$(0, -1, 0)$	Illegal				
$(0, -1, -1)$	Illegal				
$(0, 0, -1)$	7	$z' > x' + 4$	$y' = x'$	O	2

Table 9: Options of G for Lemma 26.

Move	$y' + z' - x' \pmod{8}$			SG if in U
$(-1, -1, -1)$	5	$z' > x' + 4$	$y' = x'$	1
$(-1, 0, 0)$	7	$z' > x' + 5$	$y' = x' + 1$	2
$(-1, -2, -1)$	Illegal			
$(-1, -1, -2)$	4	$z' > x' + 3$	$y' = x'$	0
$(0, -1, 0)$	Illegal			
$(0, -1, -1)$	Illegal			
$(0, 0, -1)$	5	$z' > x' + 3$	$y' = x'$	1

Table 10: Options of G for Lemma 27.

Since there is always an option whose SG value is 0 and such an option must have $y' + z' - x'$ even, we know that $\text{SG}(G + (-1, -1, -2)) = 0$. Also, $(0, 0, -1)$ is legal since $z \neq y$ and $G + (0, 0, -1)$ must be in U , so its SG value is 1.

The move $(-1, 0, 0)$ is legal since $x \neq 0$. If $G' = G + (-1, 0, 0) \in L$, then $z' = x' + y'$, so $G' = [x', x' + 1, 2x' + 1]$, meaning $y' + z' - x' = 2x' + 2$ is even, which is a contradiction. Thus, $G' \in U$ (a position with $y' = x' + 1$ cannot be in R or D unless $x = 0$) and so $\text{SG}(G') = 2$.

Thus, G has options whose SG values are 0, 1, and 2, and no option whose SG value is 3, and thus $\text{SG}(G) = 3$. $\square$

Lemma 28. *Let $G \in U$, and suppose that $y + z - x \equiv 6 \pmod{8}$, $G \notin [E, E, E]$, $z > x + 4$, and $y = x + 1$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. In this case, $n = 3$. The options of G are as shown in Table 11.

All options with $y' + z' - x' \equiv 5 \pmod{8}$ have an SG value of 1 if they are in U . The move $(0, 0, -1)$ is legal since $y \neq z$, and $G + (0, 0, -1)$ must be in U , so it has an SG value of 1.

Move	$y' + z' - x' \pmod{8}$		
$(-1, -1, -1)$	5	$z' > x' + 4$	$y' = x' + 1$
$(-1, 0, 0)$	7	$z' > x' + 5$	$y' = x' + 2$
$(-1, -2, -1)$	4	$z' > x' + 4$	$y' = x'$
$(-1, -1, -2)$	4	$z' > x' + 3$	$y' = x' + 1$
$(0, -1, 0)$	5	$z' > x' + 4$	$y' = x'$
$(0, -1, -1)$	4	$z' > x' + 3$	$y' = x'$
$(0, 0, -1)$	5	$z' > x' + 3$	$y' = x' + 1$

 Table 11: Options of G for Lemma 28.

Suppose x is even. Then $(-1, -2, -1)$ is legal since $x \neq y$, and $G' = G + (-1, -2, -1)$ has an SG value of 2 if it is in U . If $G' \in L$, then $z' = x' + y' + k$ for some $k \in \{0, 1\}$. Then $G' = [x', x', 2x' + k]$. But since x' is odd, $y' + z' - x' = 2x' + k$ cannot be congruent to 4 (mod 8), so $G' \notin L$. Thus, in this case $\text{SG}(G') = 2$.

Suppose x is odd. Then $(0, -1, -1)$ is legal since $x \neq y$ and $G' = G + (0, -1, -1)$ has an SG value of 2 as long as $z' > x' + 4$. We know that $z' > x' + 3$, so suppose $z' = x' + 4$. Then $y' + z' - x' = x' + 4$ cannot be congruent to 4 (mod 8), which is a contradiction. Thus, $\text{SG}(G') = 2$ in this case. So there is always an option whose SG value is 2.

None of the options which satisfy $y' + z' - x' \equiv 4 \pmod{8}$ can have an SG value of 3. We can also see that $G' = G + (-1, 0, 0)$ cannot have an SG value of 3 since $z' > x' + 4$ and $y' \leq x' + 2$.

Thus, G has options whose SG values are 0, 1, and 2, and no option whose SG value is 3, and thus $\text{SG}(G) = 3$. $\square$

Lemma 29. *Let $G \in U$, and suppose that $y + z - x \equiv 6 \pmod{8}$, $G \notin [E, E, E]$, $z > x + 2$, and $z = x + y - 2$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. In this case, $n = 3$. The options of G are as shown in Table 12. All options with $y' + z' - x' \equiv 5 \pmod{8}$ have an SG value of 1 if they are in U . If $y \neq z$ then $G + (0, 0, -1)$ must be in U , so it has an SG value of 1. If $y = z$ then $x \neq y$ meaning $(0, -1, 0)$ is legal, and $G + (0, -1, 0) \in U$ since $z = x + y - 2$. Thus, G has an option whose SG value is 1.

The move $(-1, 0, 0)$ is legal since $x \neq 0$. Let $G' = G + (-1, 0, 0)$. If $G' \in U$, then $\text{SG}(G') = 2$ since $y' + z' - x' \equiv 7 \pmod{8}$, $z' > x' + 2$, and $z' = x' + y' - 1$. Suppose $G' \notin U$. Then $G' \in R$ since $z' = x' + y' - 1$. So $y' = 2x' + k$ for some $k \in \{2, 3\}$. Then $G' = [x', 2x' + k, 3x' + k - 1]$. Then $y' + z' - x' = 4x' + 2k - 1 \equiv 7 \pmod{8}$ implies that $k = 2$ and that x' is odd. But then this would imply that $G \in [E, E, E]$, which is not the case. Thus, $G' \in U$ and so $\text{SG}(G') = 2$.

Move	$y' + z' - x' \pmod{8}$		
$(-1, -1, -1)$	5	$z' > x' + 2$	$z' = x' + y' - 1$
$(-1, 0, 0)$	7	$z' > x' + 3$	$z' = x' + y' - 1$
$(-1, -2, -1)$	4	$z' > x' + 2$	$z' = x' + y'$
$(-1, -1, -2)$	4	$z' > x' + 1$	$z' = x' + y' - 2$
$(0, -1, 0)$	5	$z' > x' + 2$	$z' = x' + y' - 1$
$(0, -1, -1)$	4	$z' > x' + 1$	$z' = x' + y' - 1$
$(0, 0, -1)$	5	$z' > x' + 1$	$z' = x' + y' - 3$

Table 12: Options of G for Lemma 29.

Move	$y' + z' - x' \pmod{4}$	
$(-1, -1, -1)$	2	$z' > x' + 2$
$(-1, 0, 0)$	0	$z' > x' + 3$
$(-1, -2, -1)$	1	$z' > x' + 2$
$(-1, -1, -2)$	1	$z' > x' + 1$
$(0, -1, 0)$	2	$z' > x' + 2$
$(0, -1, -1)$	1	$z' > x' + 1$
$(0, 0, -1)$	2	$z' > x' + 1$

Table 13: Options of G for Lemma 30.

None of the options with $y' + z' - x' \equiv 4 \pmod{8}$ can have an SG value of 3 if they are in U unless they satisfy $G' = [x', x' + 1, x' + 2]$ with $x' \equiv 1 \pmod{4}$. Both these options either satisfy $z' = x' + y' - 1$ or $z' = x' + y' - 2$, in which case x' would be 3 or 2, respectively, which would contradict $x' \equiv 1 \pmod{4}$.

Thus, G has options whose SG values are 0, 1, and 2, and no option whose SG value is 3, and thus $\text{SG}(G) = 3$. $\square$

Lemma 30. *Let $G \in U$, and suppose that $y + z - x \equiv 3 \pmod{4}$, $z > x + 2$, and neither of the following apply:*

- $y + z - x \equiv 7 \pmod{8}$, $z > x + 4$, and $y \leq x + 2$,
- $y + z - x \equiv 7 \pmod{8}$, $y \leq 2x$, and $x + y - z \leq 3$.

If Theorem 13 holds for all options of G , then Theorem 13 holds for G .

Proof. In this case, $n = 3$. The options of G are as shown in Table 13. Every option satisfying $y' + z' - x' \equiv 1 \pmod{4}$ has an SG value of 1 if it is in U . Note that $G + (0, -1, -1)$ must be in U unless it is illegal, in which case $x = y$ and so $G + (-1, -1, -2) \in U$. So there is always an option whose SG value is 1.

Now we will show that there is an option whose SG value is 2. Suppose $y \leq 2x$. Let G' be one of the options $G + (-1, -1, -1)$ or $G + (0, -1, 0)$. If $x = y$ and $G \in [O, O, O]$ (in which case $z > x + 3$), then instead let $G' = G + (0, 0, -1)$. Then G' satisfies $y' + z' - x' \equiv 2 \pmod{4}$ and $z' > x' + 2$, and at least one of these options satisfies $G' \notin [E, E, E]$.

Suppose $G' \in L$. Then we have $z' = x' + y'$. Then $y' + z' - x' = 2y' \equiv 2 \pmod{4}$ implies that y' is odd. If y' is odd for $G' \in L$, then $\text{SG}(G')$ is either 2 or 3, but by Lemma 19 it cannot be 3. Suppose for a contradiction that $y' + z' - x' \equiv 6 \pmod{8}$, $z' > x' + 4$, and $y' \leq x' + 1$. Then regardless of which option G' is, this would imply that $y + z - x \equiv 7 \pmod{8}$, $z > x + 4$, and $y' \leq x' + 2$, which is a contradiction. Now, suppose for a contradiction that $y' + z' - x' \equiv 6 \pmod{8}$ and $z' = x' + y' - 2$. Then regardless of which option G' is, this would imply that $y + z - x \equiv 7 \pmod{8}$ and $x + y - z \leq 3$, which is a contradiction since we are assuming $y \leq 2x$. Thus, $\text{SG}(G') = 2$.

Suppose instead that $y = 2x + 1$. Since y is odd and $x + y + z$ is odd, G is either in $[O, O, O]$ or $[E, O, E]$.

Case 1: $G \in [O, O, O]$. Then let $G' = G + (-1, 0, 0)$, which is legal since $x \neq 0$. Then $G' \in (R \cup D) \cap [E, O, O]$, so $\text{SG}(G') = 2$.

Case 2: $G \in [E, O, E]$. Then let $G' = G + (0, 0, -1)$, which is legal since $y \neq z$. Then $G' \in U$ satisfying $y' + z' - x' \equiv 2 \pmod{4}$. Suppose for a contradiction that $y' + z' - x' \equiv 6 \pmod{8}$, $z' > x' + 4$, and $y' \leq x' + 1$. Then this implies $y + z - x \equiv 7 \pmod{8}$, $z > x + 5$, and $y \leq x + 1$, which is a contradiction. Now, suppose for a contradiction that $y' + z' - x' \equiv 6 \pmod{8}$ and $z' = x' + y' - 2$. Then this implies that $y + z - x \equiv 7 \pmod{8}$ and $z = x + y - 1$. Then $y + z - x = 4x + 1$ which cannot be congruent to 7 (mod 8), so this is not possible. Thus, $\text{SG}(G') = 2$.

Now it remains to show that no option in U has $\text{SG}(G') = 3$. This is clearly true for the options satisfying $y' + z' - x' \equiv 1 \pmod{4}$.

Let $G' = G + (-1, 0, 0)$ and suppose $G' \in U$ and $\text{SG}(G') = 3$. Then we must be in the case where $y' + z' - x' \equiv 0 \pmod{8}$, $x' = y'$, x' is odd, and $z' > x' + 5$. But the move $(-1, 0, 0)$ cannot result in an option where $x' = y'$.

Let $G' = G + (-1, -1, -1)$. If $G \in [O, O, O]$ then $\text{SG}(G') = 0$, and otherwise we have already shown that $\text{SG}(G') = 2$ if $G' \in U$.

Let $G' = G + (0, -1, 0)$. If $G \in [E, O, E]$ then $\text{SG}(G') = 0$, and otherwise we have already shown that $\text{SG}(G') = 2$ if $G' \in U$.

Let $G' = G + (0, 0, -1)$. If $\text{SG}(G') = 3$, then G' must satisfy $y' + z' - x' \equiv 6 \pmod{8}$ and $G' \notin [E, E, E]$. This implies that $y + z - x \equiv 7 \pmod{8}$. Suppose for a contradiction that $z' > x' + 4$, and $y' \leq x' + 1$. Then this implies $y + z - x \equiv 7 \pmod{8}$, $z > x + 5$, and $y \leq x + 1$, which is a contradiction. Now, Suppose for a contradiction that $z' > x' + 2$ and $z' = x' + y' - 2$. Then we will have $z > x + 3$ and $z = x + y - 1$. If $y \leq 2x$ then this is a contradiction, and if $y = 2x + 1$ then we

Move	$y' + z' - x' \pmod{4}$	
$(-1, -1, -1)$	3	$z' > x' + 2$
$(-1, 0, 0)$	1	$z' > x' + 3$
$(-1, -2, -1)$	2	$z' > x' + 2$
$(-1, -1, -2)$	2	$z' > x' + 1$
$(0, -1, 0)$	3	$z' > x' + 2$
$(0, -1, -1)$	2	$z' > x' + 1$
$(0, 0, -1)$	3	$z' > x' + 1$

Table 14: Options of G for Lemma 31.

have shown that this is impossible.

Thus, G has options whose SG values are 0, 1, and 2, and no option whose SG value is 3, and thus $\text{SG}(G) = 3$. $\square$

Lemma 31. *Let $G \in U$, and suppose that $y + z - x \equiv 0 \pmod{4}$, $G \notin [E, E, E]$, $z > x + 2$, and none of the following apply:*

- $G = [5, 5, 8]$,
- $z > x + 4$, $y + z - x \equiv 4 \pmod{8}$, and $x = y$,
- $z > x + 5$, $y + z - x \equiv 0 \pmod{8}$, and $x = y$,
- $G = [x, x + 1, x + 3]$ with $x \equiv 0 \pmod{4}$ and $x \neq 0, 4, 8$,
- $G = [x, x + 2, x + 3]$ with $x \equiv 3 \pmod{4}$ and $x \neq 3, 7$,
- $G = [x, x + 1, x + 5]$ and $x \equiv 2 \pmod{8}$,
- $G = [x, x + 2, x + 5]$ and $x \equiv 1 \pmod{8}$,
- $G = [x, x + 1, x + 6]$ and $x \equiv 1 \pmod{8}$.

If Theorem 13 holds for all options of G , then Theorem 13 holds for G .

Proof. In this case, $n = 4$. The options of G are as shown in Table 14. Since $x \neq 0$, the move $(-1, 0, 0)$ is legal. Let $G' = G + (-1, 0, 0)$. If $G' \in U \cup R$ then $\text{SG}(G') = 1$. Otherwise, $z' = x' + y'$, which is not possible since $x' + y' + z'$ is odd. Thus, G has an option whose SG value is 1.

We will show that there are options of G whose SG values are 2 and 3.

Case 1: $z = x + 3$. Then $y + z - x = y + 3 \equiv 0 \pmod{4}$, so $y \equiv 1 \pmod{4}$.

Case 1a: $x = y$. Then $G = [x, x, x + 3]$ with $x \equiv 1 \pmod{4}$. Consider $G' = G + (0, 0, -1)$, then $G' = [x, x, x + 2]$ with $x \equiv 1 \pmod{4}$, so $\text{SG}(G') = 2$ since we may assume $x > 5$. Next, consider $G' = G + (-1, -1, -1)$. Then $y' + z' - x' \equiv 3 \pmod{4}$ and $z' > x' + 2$. However, $z' \leq x' + 4$ and $x' + y' - z' = x' - 3$ which we can assume is greater than 3, so neither of the exceptional cases apply, and thus $\text{SG}(G') = 3$.

Case 1b: Suppose that $x + 1 = y$. Then $G = [x, x + 1, x + 3]$ with $x \equiv 0 \pmod{4}$, so to avoid a contradiction x must equal 0, 4, or 8, and we have checked that these positions have an SG value of 4.

Case 1c: Suppose that $x + 2 = y$. Then $G = [x, x + 2, x + 3]$ with $x \equiv 3 \pmod{4}$, so to avoid a contradiction x must equal 3 or 7, and we have checked that these positions have an SG value of 4.

Case 1d: Suppose that $x + 3 = y$. Then $G = [x, x + 3, x + 3]$ with $x \equiv 2 \pmod{4}$. Consider $G' = G + (0, -1, -1)$. Then $G' = [x, x + 2, x + 2]$ with $x \equiv 2 \pmod{4}$, so $\text{SG}(G') = 2$. Next, consider $G' = G + (-1, -1, -1)$; then $G' = [x', x' + 3, x' + 3]$ with $x' \equiv 1 \pmod{4}$. Then $y' + z' - x' \equiv 3 \pmod{4}$ and $z' > x' + 2$. However, $z' \leq x' + 4$ and $x' + y' - z' = x'$, which we can assume is greater than 3. Thus, neither of the exceptional cases apply, so $\text{SG}(G') = 3$.

So from now on we will assume $z > x + 3$. Then all options of G satisfy $z' > x' + 2$.

Case 2: Suppose that $x = y$ and $z = x + 4$. Then $G = [x, x, x + 4]$ and $y + z - x = x + 4$, so $x \equiv 0 \pmod{4}$. This is a contradiction since then we would have $G \in [E, E, E]$.

Case 3: Suppose that $x = y$ and $z = x + 5$, so we must have $y + z - x \equiv 0 \pmod{8}$. Then $G = [x, x, x + 5]$ and $y + z - x = x + 5$, so $x \equiv 3 \pmod{8}$. Consider $G' = G + (-1, -1, -1)$. Then $y' + z' - x' \equiv 7 \pmod{8}$, $y' = x'$, and $z' > x' + 4$, so $\text{SG}(G') = 2$. Next, consider $G' = G + (0, 0, -1)$. Then $y' + z' - x' \equiv 7 \pmod{8}$, $z' = x' + 4$, and $x' + y' - z' = x' - 5$ which we can assume is greater than 3. Thus, neither of the exceptional cases apply, so $\text{SG}(G') = 3$.

All other cases where $x = y$ are excluded, so from now on we will assume $x < y$. We will look at the options of each move.

Consider the moves which result in $y' + z' - x' \equiv 3 \pmod{4}$. We have the following:

- If $G' \in R$ then $\text{SG}(G') = 3$. Otherwise, assume $G' \in U$.
- If $y' + z' - x' \equiv 7 \pmod{8}$, $z' > x' + 4$, and $y' = x'$, then $\text{SG}(G') = 2$.
- If $G' = [x', x' + 1, x' + 5]$ with $x \equiv 1 \pmod{8}$, then $\text{SG}(G') = 4$.
- If $y' + z' - x' \equiv 7 \pmod{8}$, $z' > x' + 5$, and $y' = x' + 1$, then $\text{SG}(G') = 2$.
- If $y' + z' - x' \equiv 7 \pmod{8}$, $z' > x' + 4$, and $z' = x' + y' - 1$, then $\text{SG}(G') = 2$.
- If $y' + z' - x' \equiv 7 \pmod{8}$, $z' > x' + 4$, $x' + 2 = y$, and $x' + y' - z' > 2$, then $\text{SG}(G') = 5$.
- If $y' + z' - x' \equiv 7 \pmod{8}$, $z' = x' + y' - 3$, $y' > x' + 2$, and $y' \leq 2x'$, then $\text{SG}(G') = 5$.
- Otherwise, $\text{SG}(G') = 3$. Note that we are ignoring options where there is an upper bound on both the size of $z - x$ and $x + y - z$, since for any such restrictions there are only finitely many positions that satisfy them, all of which have been checked.

Then we have the following moves to consider (for $G' = G + (-1, -1, -1)$, note that $G' \in U \cup R$ since $x + y - z$ is odd):

- If $y = 2x + 1$, then $\text{SG}(G') = 3$. Otherwise, we will have $G' \in U$.
- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 4$, and $y = x$, then $\text{SG}(G') = 2$ (exclude since $y > x$).
- If $G = [x, x + 1, x + 5]$ with $x \equiv 2 \pmod{8}$, then $\text{SG}(G') = 4$ (excluded in statement of lemma).
- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 5$, and $y = x + 1$, then $\text{SG}(G') = 2$.
- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 4$, and $z = x + y - 2$, then $\text{SG}(G') = 2$.
- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 4$, $x + 2 = y$, and $x + y - z > 3$, then $\text{SG}(G') = 5$.
- If $y + z - x \equiv 0 \pmod{8}$, $z = x + y - 4$, $y > x + 2$, and $y \leq 2x - 1$, then $\text{SG}(G') = 5$.
- Otherwise, $\text{SG}(G') = 3$.

For $G' = G + (0, -1, 0)$, note that $G' \in U$ since $x + y - z$ is odd. We have

- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 4$, and $y = x + 1$, then $\text{SG}(G') = 2$.
- If $G = [x, x + 2, x + 5]$ with $x \equiv 1 \pmod{8}$, then $\text{SG}(G') = 4$ (excluded in statement of lemma).
- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 5$, and $y = x + 2$, then $\text{SG}(G') = 2$.
- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 4$, and $z = x + y - 2$, then $\text{SG}(G') = 2$.
- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 4$, $x + 3 = y$, and $x + y - z > 3$, then $\text{SG}(G') = 5$.
- If $y + z - x \equiv 0 \pmod{8}$, $z = x + y - 4$, $y > x + 3$, and $y \leq 2x + 1$, then $\text{SG}(G') = 5$.
- Otherwise, $\text{SG}(G') = 3$.

For $G' = G + (0, 0, -1)$, note that $G' \in U$. We have

- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 5$, and $y = x$, then $\text{SG}(G') = 2$ (exclude since $y > x$).
- If $G = [x, x + 1, x + 6]$ with $x \equiv 1 \pmod{8}$, then $\text{SG}(G') = 4$ (excluded in statement of lemma).
- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 6$, and $y = x + 1$, then $\text{SG}(G') = 2$.
- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 5$, and $z = x + y$, then $\text{SG}(G') = 2$ (exclude since $G \in U$).
- If $y + z - x \equiv 0 \pmod{8}$, $z > x + 5$, $x + 2 = y$, and $x + y - z > 1$, then $\text{SG}(G') = 5$.
- If $y + z - x \equiv 0 \pmod{8}$, $z = x + y - 2$, $y > x + 2$, and $y \leq 2x$, then $\text{SG}(G') = 5$.
- Otherwise, $\text{SG}(G') = 3$.

Consider the moves which result in $y' + z' - x' \equiv 2 \pmod{4}$. Then we have the following when $G' \in U$:

- If $G' \in [E, E, E]$, then $\text{SG}(G') = 0$.
- If $y' + z' - x' \equiv 6 \pmod{8}$, $G' \notin [E, E, E]$, $z' > x' + 4$, and $y' \leq x' + 1$, then $\text{SG}(G') = 3$.
- If $y' + z' - x' \equiv 6 \pmod{8}$, $G' \notin [E, E, E]$, and $z' = x' + y' - 2$, then $\text{SG}(G') = 3$.
- Otherwise, $\text{SG}(G') = 2$.

Then we have the following moves to consider. For $G' = G + (-1, -2, -1)$, we have

- If $z = x + y - 2$ then $G' \in L \cup D$. In this case $y + z - x = 2y - 2 \equiv 0 \pmod{4}$ and thus y is odd. Then y' is odd. Otherwise, we will have $G' \in U$.
 - If $G' \in D$ then $y = 2x + 1$. Then $G = [x, 2x + 1, 3x - 1]$, so $G' = [x - 1, 2x - 1, 3x - 2]$. Then if x is odd, $\text{SG}(G') = 2$ and if x is even then $\text{SG}(G') = 6$.
 - If $G' \in L$ then since y' is odd, $\text{SG}(G')$ is 2 or 3. Then $y + 2z - 2x = 3y - 4 \equiv 3y \pmod{4}$, so if $y' \equiv 1 \pmod{4}$ then $\text{SG}(G') = 2$ and if $y' \equiv 3 \pmod{4}$ then $\text{SG}(G') = 3$.
- If $G \in [O, E, O]$, then $\text{SG}(G') = 0$.
- If $y + z - x \equiv 0 \pmod{8}$, $G \notin [O, E, O]$, $z > x + 4$, and $y \leq x + 2$, then $\text{SG}(G') = 3$.
- If $y + z - x \equiv 0 \pmod{8}$, $G \notin [O, E, O]$, and $z = x + y - 4$, then $\text{SG}(G') = 3$.
- Otherwise, $\text{SG}(G') = 2$.

For $G' = G + (-1, -1, -2)$, we have

- If $y = 2x + 1$, then $G' \in R$. Otherwise, we have $G' \in U$.
 - If x is even, then $\text{SG}(G') = 5$.
 - If $G \in [O, O, E]$ then $\text{SG}(G') = 0$.
 - If $G \in [O, E, O]$ then $\text{SG}(G') = 2$.
- If $G \in [O, O, E]$, then $\text{SG}(G') = 0$.
- If $y + z - x \equiv 0 \pmod{8}$, $G \notin [O, O, E]$, $z > x + 5$, and $y \leq x + 1$, then $\text{SG}(G') = 3$.
- If $y + z - x \equiv 0 \pmod{8}$, $G \notin [O, O, E]$, and $z = x + y - 2$, then $\text{SG}(G') = 3$.
- Otherwise, $\text{SG}(G') = 2$.

For $G' = G + (0, -1, -1)$, note that $G' \in U$. We have

- If $G \in [E, O, O]$, then $\text{SG}(G') = 0$.
- If $y + z - x \equiv 0 \pmod{8}$, $G \notin [E, O, O]$, $z > x + 5$, and $y \leq x + 2$, then $\text{SG}(G') = 3$.
- If $y + z - x \equiv 0 \pmod{8}$, $G \notin [E, O, O]$, and $z = x + y - 2$, then $\text{SG}(G') = 3$.
- Otherwise, $\text{SG}(G') = 2$.

So now we have characterized the SG values of all these options based on G . We can see that no option of G that is in U can have an SG value of 4. It remains to

show that there is at least one option with an SG value of 2, and at least one option with an SG value of 3.

Suppose $\text{SG}(G + (0, -1, -1)) \neq 2$. Then we find ourselves in one of the following cases (which are not necessarily mutually exclusive with cases 1-3 from earlier in the proof).

Case 4: $y + z - x \equiv 0 \pmod{8}$, $G \notin [E, O, O]$, $z > x + 5$, and $y \leq x + 2$.

Case 4a: $y = x + 1$. Then $\text{SG}(G + (-1, -1, -1)) = 2$.

Case 4b: $y = x + 2$. Then $\text{SG}(G + (0, -1, 0)) = 2$.

Case 5: $y + z - x \equiv 0 \pmod{8}$, $G \notin [E, O, O]$, and $z = x + y - 2$. Then $\text{SG}(G + (-1, -1, -1)) = 2$.

Case 6: $G \in [E, O, O]$. Suppose $\text{SG}(G + (-1, -1, -2)) \neq 2$. Then we find ourselves in one of the following cases.

Case 6a: $y + z - x \equiv 0 \pmod{8}$, $z > x + 5$, and $y \leq x + 1$. Then $y = x + 1$ and $\text{SG}(G + (-1, -1, -1)) = 2$.

Case 6b: $y + z - x \equiv 0 \pmod{8}$ and $z = x + y - 2$. Then $\text{SG}(G + (-1, -1, -1)) = 2$.

Case 6c: $y = 2x + 1$. Suppose $\text{SG}(G + (-1, -2, -1)) \neq 2$. Then one of the following cases must hold.

Case 6ci: $y + z - x \equiv 0 \pmod{8}$, $z > x + 4$, and $y \leq x + 2$. Then $2x + 1 \leq x + 2$, so $x \leq 1$. There are only finitely many such cases where $G \in U$, all of which have been checked already.

Case 6cii: $y + z - x \equiv 0 \pmod{8}$ and $z = x + y - 4$. Then $y + z - x = (2x + 1) + (x + (2x + 1) - 4) - x = 4x - 2 \equiv 2 \pmod{4}$, which is a contradiction, so this is impossible.

Case 6ciii: $z = x + y - 2$. Then $y + z - x = (2x + 1) + (x + (2x + 1) - 2) - x = 4x \equiv 0 \pmod{8}$ since x is even. If $z \leq x + 4$ then $3x - 1 \leq x + 4$ so $x < 3$. There are only finitely many such cases where $G \in U$, all of which have been checked already. If instead $z > x + 4$, then $\text{SG}(G + (-1, -1, -1)) = 2$.

Thus, we have shown that there is an option whose SG value is 2.

Now, to show there is an option whose SG value is 3. Suppose $\text{SG}(G + (0, 0, -1)) \neq 3$. Then we find ourselves in one of the following cases (which are not necessarily mutually exclusive with cases 1-6 from earlier in the proof).

Case 7: $y + z - x \equiv 0 \pmod{8}$, $z > x + 6$, and $y = x + 1$.

Case 7a: $G \notin [E, O, O]$. Then $\text{SG}(G + (0, -1, -1)) = 3$.

Case 7b: $G \notin [O, O, E]$. Consider $G' = G + (-1, -1, -2)$. Then $G' \in R$ if and only if $y = 2x + 1$, so $x = 0$. There are no such cases where $G \in U$, so we have

Move	$y' + z' - x' \pmod{8}$			
$(-1, -1, -1)$	6	$z' > x' + 4$	$y' = x' + 2$	$x' + y' - z' > 1$
$(-1, 0, 0)$	0	$z' > x' + 5$	$y' = x' + 3$	$x' + y' - z' > 1$
$(-1, -2, -1)$	5	$z' > x' + 4$	$y' = x' + 1$	$x' + y' - z' > 0$
$(-1, -1, -2)$	5	$z' > x' + 3$	$y' = x' + 2$	$x' + y' - z' > 2$
$(0, -1, 0)$	6	$z' > x' + 4$	$y' = x' + 1$	$x' + y' - z' > 1$
$(0, -1, -1)$	5	$z' > x' + 3$	$y' = x' + 1$	$x' + y' - z' > 2$
$(0, 0, -1)$	6	$z' > x' + 3$	$y' = x' + 2$	$x' + y' - z' > 3$

Table 15: Options of G for Lemma 32.

$G' \in U$ and $\text{SG}(G') = 3$.

Case 8: $y + z - x \equiv 0 \pmod{8}$, $z > x + 5$, $y = x + 2$, and $x + y - z > 1$.

Case 8a: $G \notin [E, O, O]$. Then $\text{SG}(G + (0, -1, -1)) = 3$.

Case 8b: $G \notin [O, O, E]$ and $x + y - z = 2$. Consider $G' = G + (-1, -1, -2)$. Then $G' \in R$ if and only if $y = 2x + 1$, so $x = 1$. There are only finitely many such cases with $G \in U$, all of which have been checked. So we can assume $G' \in U$, and then $\text{SG}(G') = 3$.

Case 8c: $G \notin [O, O, E]$ and $x + y - z > 2$. Then $G' = G + (-1, -2, -1) \in U$ and $\text{SG}(G') = 3$.

Case 9: $y + z - x \equiv 0 \pmod{8}$, $z = x + y - 2$, $y > x + 2$, and $y \leq 2x$.

Case 9a: $G \notin [E, O, O]$. Then $\text{SG}(G + (0, -1, -1)) = 3$.

Case 9b: $G \notin [O, O, E]$. Then $\text{SG}(G + (-1, -1, -2)) = 3$. Thus, there is always an option whose SG value is 3.

So G has options whose SG values are 0, 1, 2, and 3, and no option whose SG value is 4, and thus $\text{SG}(G) = 4$. $\square$

Lemma 32. Let $G \in U$, and suppose that $y + z - x \equiv 7 \pmod{8}$, $z > x + 4$, $x + 2 = y$, and $x + y - z > 2$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .

Proof. In this case, $n = 5$. The options of G are as shown in Table 15. Notice that no option of G can be in $L \cup D$. If some option of G is in R , then $y \geq 2x$, so $x + 2 \geq 2x$, so $x \leq 2$. There are only finitely many positions in U which satisfy $x \leq 2$, and they have all been checked already. So we can assume every option of G is in U .

Consider the options satisfying $y' + z' - x' \equiv 5 \pmod{8}$. All these options have SG values of 1.

Now we will consider the following cases to show that some option has an SG value of 2.

Case 1: $G \notin [E, E, O]$. Then let $G' = G + (0, 0, -1)$. Then $y' + z' - x' \equiv 2 \pmod{4}$, $G' \notin [E, E, E]$, and $z' > x' + 2$. Since $y' > x' + 1$ and $z' < x' + y' - 3$, neither of the excluding conditions apply, so $\text{SG}(G') = 2$.

Case 2: $G \in [E, E, O]$. Then let $G' = G + (-1, -1, -1)$. Then $y' + z' - x' \equiv 2 \pmod{4}$, $G' \notin [E, E, E]$, and $z' > x' + 2$. Since $y' > x' + 1$, the first excluding condition does not apply. Suppose $z' = x' + y' - 2$. Then $z = x + y - 3$, so $y + z - x = 2x + 1 \equiv 7 \pmod{8}$, so $x \equiv 3 \pmod{4}$. But this contradicts x being even. Thus, neither of the excluding conditions hold and so $\text{SG}(G') = 2$.

Now we will show that some option has an SG value of 3. Let $G' = G + (0, -1, 0)$. Note that $G \notin [E, O, E]$ since $y = x + 2$. Then G' satisfies $y' + z' - x' \equiv 6 \pmod{8}$, $G' \notin [E, E, E]$, $z' > x' + 4$, and $y' \leq x' + 1$, and so $\text{SG}(G') = 3$.

Now we will show that some option has an SG value of 4. Let $G' = G + (-1, 0, 0)$. Note that $G \notin [O, E, E]$ since $y = x + 2$. Then G' satisfies $y' + z' - x' \equiv 0 \pmod{4}$, $G' \notin [E, E, E]$, and $z' > x' + 2$. Since $y' = x' + 3$, none of the excluding conditions for this case can apply, so $\text{SG}(G') = 4$.

Now, we will show that no option which is in U has an SG value of 5. Consider all the cases in U which have SG values of 5. Note that all our options satisfy $z' > x' + 3$ and $y' + z' - x' \not\equiv 7 \pmod{8}$.

Case 1: There is an option satisfying $G' = [x', x' + 1, x' + 5]$ with $x' \equiv 2 \pmod{8}$. Then $y' + z' - x' = x' + 6 \equiv 0 \pmod{8}$ and $y' = x' + 1$, and these conditions do not apply to any of our options.

Case 2: There is an option satisfying $G' = [x', x' + 1, x' + 6]$ with $x' \equiv 1 \pmod{8}$. Then $y' + z' - x' = x' + 7 \equiv 0 \pmod{8}$ and $y' = x' + 1$, and these conditions do not apply to any of our options.

We have shown that there are options with SG values of 0, 1, 2, 3, and 4, and no options with an SG value of 5. Thus, $\text{SG}(G) = 5$. $\square$

Lemma 33. *Let $G \in U$, and suppose that $y + z - x \equiv 7 \pmod{8}$, $z = x + y - 3$, $y > x + 2$, and $y \leq 2x$. If Theorem 13 holds for all options of G , then Theorem 13 holds for G .*

Proof. In this case, $n = 5$. The options of G are as shown in Table 16. Note that no option is in $L \cup D$, and the only option which can be in R is $G + (-1, 0, 0)$. So every other option must be in U .

Suppose $z \leq x + 7$. Then $y - 3 = z - x \leq 7$ so $y \leq 10$, so $x \leq 10$. But we have checked the SG values of all positions in U which satisfy $x \leq 10$. Thus, for convenience we may assume that $z > x + 7$ and so each option satisfies $z' > x' + 6$.

Note that $y + z - x = 2y - 3 \equiv 7 \pmod{8}$. Thus, $2y \equiv 2 \pmod{8}$ and so

Move	$y' + z' - x' \pmod{8}$			
$(-1, -1, -1)$	6	$x' + y' - z' = 2$	$y' > x' + 2$	$y' \leq 2x' + 1$
$(-1, 0, 0)$	0	$x' + y' - z' = 2$	$y' > x' + 3$	$y' \leq 2x' + 2$
$(-1, -2, -1)$	5	$x' + y' - z' = 1$	$y' > x' + 1$	$y' \leq 2x'$
$(-1, -1, -2)$	5	$x' + y' - z' = 3$	$y' > x' + 2$	$y' \leq 2x' + 1$
$(0, -1, 0)$	6	$x' + y' - z' = 2$	$y' > x' + 1$	$y' \leq 2x' - 1$
$(0, -1, -1)$	5	$x' + y' - z' = 3$	$y' > x' + 1$	$y' \leq 2x' - 1$
$(0, 0, -1)$	6	$x' + y' - z' = 4$	$y' > x' + 2$	$y' \leq 2x'$

Table 16: Options of G for Lemma 33.

$y \equiv 1 \pmod{4}$. Also note that every option which satisfies $y' + z' - x' \equiv 5 \pmod{8}$ has an SG value of 1.

Since y is odd, $G \notin [E, E, O]$. Then let $G' = G + (0, 0, -1)$. Then G' satisfies $y' + z' - x' \equiv 2 \pmod{4}$, $G' \notin [E, E, E]$, and $z' > x' + 2$. Then since $y' > x' + 2$ and $z' = x' + y' - 4$, neither of the excluding conditions apply and so $\text{SG}(G') = 2$.

Now, we will show that some option has an SG value of 3.

Case 1: $G \notin [O, O, O]$. Then let $G' = G + (-1, -1, -1)$. Then G' satisfies $y' + z' - x' \equiv 6 \pmod{8}$, $G' \notin [E, E, E]$, $z' > x' + 2$, and $z' = x' + y' - 2$. Thus, $\text{SG}(G') = 3$.

Case 2: $G \in [O, O, O]$. Then let $G' = G + (0, -1, 0)$. Then G' satisfies $y' + z' - x' \equiv 6 \pmod{8}$, $G' \notin [E, E, E]$, $z' > x' + 2$, and $z' = x' + y' - 2$. Thus, $\text{SG}(G') = 3$.

Let $G' = G + (-1, 0, 0)$. We will show that $\text{SG}(G') = 4$. Since y is odd, $G' \notin [E, E, E]$. Then $y' + z' - x' \equiv 0 \pmod{4}$ and $z' > x' + 2$. Since $z' > x' + 6$, none of the excluding conditions apply and so $\text{SG}(G') = 4$.

Every option of G which is in U satisfies $z' > x' + 6$ and $y' + z' - x' \not\equiv 7 \pmod{8}$, and so none of them can have an SG value of 5.

So there are options with SG values of 0, 1, 2, 3, and 4, and no options with an SG value of 5. Thus, $\text{SG}(G) = 5$. $\square$

4. Conclusion

To find the Sprague-Grundy value of a 4-pile game (a, b, c, d) of P&R, we can let $[x, y, z] = [d, b - c + d, a - c + d]$ and use Theorem 13 to look up which case the position falls in. Comparing some of our important values in P&R Helper to the corresponding values in our actual P&R piles, we have seen the following:

$$\begin{aligned} z < x + y & \text{ if and only if } a - b < d, \\ y \leq 2x & \text{ if and only if } b - c \leq d. \end{aligned}$$

We know that the SG value of a 4-pile position in P&R is entirely dependent on the values of $a - b$, $b - c$, and d , so it makes sense that the main regions that determine our cases simply come from comparing these values. We also have

$$\begin{aligned}y + z - x &= a + b - 2c + d, \\y + 2z - 2x &= 2a + b - 3c + d.\end{aligned}$$

One might venture to find a concrete reason why these values, especially the first one, proved so crucial in determining the SG values.

One observation of note is that compared to 3-pile P&R, the rules for SG values for 4 piles are a huge jump in complexity. We go from simply checking the parity of the piles in order to a massive amount of casework. Similarly, the winning and losing positions for P&R are known to be entirely based on parity up to 5 piles, but it is still an open problem for 6 or more piles. People working on outcome classes for 6-pile P&R should expect a jump in complexity at least as large as what is seen in this paper, and possibly involving similar tactics. Some work has been done on the 6-pile positions when the smallest pile is of size 1 [1], as well as some work on when there are arbitrarily many small-sized piles [2]. It is the author's hope that by solving the Sprague-Grundy values for 4-pile P&R, some inspiration will have been provided to the mathematicians who are working on those problems.

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