

AN ADDITIVE VARIANT OF THE GAME EUCLID

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Received: 12/27/25, Revised: 8/2/26, Accepted: 10/14/26, Published: 11/23/26

Abstract

In this article, we introduce and study the following additive variant of the game Euclid. The game is played on pairs of non-negative integers smaller than a fixed positive integer n, sorted in weakly decreasing order. Until the smaller entry of the current position becomes zero, two players make moves alternately. A move consists of changing the current position, say (x_1, x_2), first by replacing x_1 with $x_1 + mx_2 \pmod{n}$ for some positive integer $m \leq \left\lceil \frac{n-x_1}{x_2} \right\rceil$ and then by sorting the resulting pair in weakly decreasing order. When the smaller entry of the current position becomes zero, the game ends; the current player loses whereas the other player wins. We first provide a characterization of P-positions of this game using generalized Fibonacci sequences, which enables us to efficiently determine whether a given position is a P-position. We also show that those positions from which the game ends in a finite non-zero number of forced moves can be described by the Fibonacci sequence. By extending the set of positions to pairs of non-negative real numbers, we then study the question from which positions does the game end in a finite non-zero number of moves.

1. Introduction

The game *Euclid* [1] is a famous combinatorial game defined as follows. The game is played on pairs of non-negative integers, sorted in weakly decreasing order. Let us note that the original formulation of the game does not require that pairs be sorted. We adopted this formulation because fixing the order in this way not only causes no essential changes but also simplifies the following presentation. Until the smaller entry of the current position becomes zero, two players make moves alternately. A move consists of changing the current position, say (x_1, x_2), first by replacing x_1 with $x_1 - mx_2$ for some positive integer $m \leq \left\lfloor \frac{x_1}{x_2} \right\rfloor$ and then by sorting the resulting pair in weakly decreasing order. When the smaller entry of the current position becomes zero, the game ends; the current player loses whereas the other player wins.

DOI: 10.5281/zenodo.24474166

Since the game was introduced in 1969, various properties of the game have been revealed. For example, in the aforementioned article, it is proved that (a) if the initial position (x_1, x_2) satisfies $0 < \frac{x_2}{x_1} < \frac{-1+\sqrt{5}}{2}$, then the first player can eventually win by always moving to a pair (x_1', x_2') satisfying either $x_2' = 0$ or $\frac{x_2'}{x_1'} > \frac{-1+\sqrt{5}}{2}$; and (b) if the initial position (x_1, x_2) satisfies $\frac{x_2}{x_1} > \frac{-1+\sqrt{5}}{2}$, then the second player can eventually win by using the strategy for the first player as described in (a). In [9], another winning strategy that does not require one to calculate the ratio $\frac{x_2}{x_1}$ is presented. In [8], the intimate relationships between the game Euclid and continued fraction is described. In [7], it is shown that P-positions of the game Euclid exhibit a self-similar structure within the Calkin–Wilf tree. A number of variations have also been introduced [2, 3, 5].

In this article, we introduce and study the following additive variant of the game Euclid. The game is played on pairs of non-negative integers smaller than a fixed positive integer n, sorted in weakly decreasing order. Until the smaller entry of the current position becomes zero, two players make moves alternately. A move consists of changing the current position, say (x_1, x_2), first by replacing x_1 with $x_1 + mx_2 \pmod{n}$ for some positive integer $m \leq \left\lceil \frac{n-x_1}{x_2} \right\rceil$ and then by sorting the resulting pair in weakly decreasing order. When the smaller entry of the current position becomes zero, the game ends; the current player loses whereas the other player wins.

After presenting all necessary definitions in the next preliminary section, we first provide a characterization of P-positions of our game using generalized Fibonacci sequences, which enables us to determine whether a given position is a P-position efficiently. We also show that those positions from which the game ends in a finite non-zero number of forced moves can be described by the Fibonacci sequence. By extending the set of positions to pairs of non-negative real numbers, we then study the question from which positions does the game end in a finite non-zero number of moves. We close this article with a wealth of questions.

2. Preliminaries

In this preliminary section, we present all definitions needed in the next main section.

We start with the definition of the game. For each positive integer n, define a two-player alternating game as follows. The *positions* of the game are the elements of the set

$$X_n := \{(x_1, x_2) \in \mathbb{Z}^2 \mid n > x_1 \geq x_2 \geq 0\}.$$

A move from $(x_1, x_2) \in X_n$ to $(x_1', x_2') \in X_n$ is *valid* if $x_2 \neq 0$ and (x_1', x_2') is equal to one of

$$(x_1 + x_2, x_2), (x_1 + 2x_2, x_2), \ldots, (x_1 + (a-1)x_2, x_2), (x_2, x_1 + ax_2 - n)$$

when $a \geq 2$ and to $(x_2, x_1 + x_2 - n)$ when $a = 1$, where $a = \left\lceil \frac{n-x_1}{x_2} \right\rceil$. Other moves

between elements of X_n are not valid. Hence, we have

$$\left\lceil \frac{n - x_1}{x_2} \right\rceil = \left|\{(x_1', x_2') \in X_n \mid (x_1', x_2') \text{ is reachable from } (x_1, x_2)\}\right| \tag{1}$$

for any non-terminal position $(x_1, x_2) \in X_n$. The first player unable to make a valid move loses (which happens if and only if the smaller entry of the current position is zero) and, of course, the other player wins. Because the notation of the set of all positions, X_n, clearly indicates which n is referred to, hereafter, we will not explicitly mention which n is considered.

We next introduce some concepts from the general theory of combinatorial games (see, for example, Part I of [4]), which are all important for the study of our game. A move from a position is called a *forced move* if that move is the only valid move from that position. Those positions from which there is no valid move are referred to as *terminal* positions. The concepts of *P-position* and *N-position* are defined recursively as follows:

- A position is a P-position if it is either a terminal position or a non-terminal position from which only N-positions can be reached.
- A position is an N-position if at least one P-position can be reached from it.

Observe that, after one move, the smaller entry of the position decreases weakly; also, after finitely many moves, the smaller entry of the position decreases strictly. From these, it follows that, because the terminal positions are those positions whose smaller entries are zero, the game ends in a finite number of moves no matter how it is played. Also, X_n being finite, there are only finitely many valid moves from each position. Using these observations, it can be proved that every position is either a P-position or an N-position.

As is well-known in combinatorial game theory, P-positions and N-positions are winning for the previous player and the next player, respectively (hence, the name). In other words, the strategy of always moving to a P-position is a winning strategy. To see this, suppose that one of the two players, say A, has moved to a non-terminal P-position. Then the other player, say B, can move only to an N-position, from which A can again move to a P-position. If the game continues until the end in this way (that is, A always moves to a P-position) then, because B's moves are always to N- (hence, non-terminal) positions, the person who reached a terminal position should be A.

We then recall the definitions of some relevant integer sequences. The *Fibonacci sequence* is an integer sequence $f_1, f_2, f_3, \ldots$ defined inductively by $f_1 = 1$, $f_2 = 1$, and $f_i = f_{i-2} + f_{i-1}$ $(i \geq 3)$. A calculation shows that the ith term $(i \geq 1)$ can be written explicitly as

$$f_i = \frac{\tau^i - (-\tau)^{-i}}{\sqrt{5}}, \tag{2}$$

where τ denotes the golden ratio $\frac{1+\sqrt{5}}{2}$. As one will realize, we can introduce a family of analogous sequences by changing the initial values, which is precisely what we

need in this article. The *generalized Fibonacci sequence with the initial values u and v* [6] is an integer sequence $gf_1(u,v), gf_2(u,v), gf_3(u,v), \dots$ defined inductively by $gf_1(u,v) = u$, $gf_2(u,v) = v$, and $gf_i(u,v) = gf_{i-2}(u,v) + gf_{i-1}(u,v)$ $(i \geq 3)$. Then Equation (2) generalizes to

$$gf_i(u,v) = \frac{\tau^{i-2} - (-\tau)^{-(i-2)}}{\sqrt{5}}u + \frac{\tau^{i-1} - (-\tau)^{-(i-1)}}{\sqrt{5}}v. \tag{3}$$

Note that we have $f_i = gf_{i+3}(-1,1)$ for $i \geq 1$. Suggested by this equation, we shall henceforth write f_{-2}, f_{-1}, and f_0 for $gf_1(-1,1), gf_2(-1,1)$, and $gf_3(-1,1)$, respectively. Then the following can be proved by induction.

Lemma 1. (i) *The identity $f_i f_{i+4} - f_{i+2}^2 = (-1)^{i+1}$ holds for any integer $i \geq -2$.*

(ii) *The inequality $f_{i+3} + f_{i+2}f_{i+1} - f_{i+3}f_i \geq 0$ holds for any odd integer $i \geq -2$.*

(iii) *The inequality $f_{i+3} + f_{i+3}f_i - f_{i+2}f_{i+1} \geq 0$ holds for any even integer $i \geq -2$.*

Lastly, we set up notation and terminology on continued fraction. For any integers $\ell \geq 0$ and $a_0 \in \mathbb{Z}, a_1, \dots, a_\ell \in \mathbb{Z}_{>0}$, we write $[a_0; a_1, \dots, a_\ell]$ for the expression

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_\ell}}}.$$

For each rational number $r \in \mathbb{Q}$, there are precisely two such expressions that have a value equal to r; they are of the form $[a_0; a_1, \dots, a_\ell]$ and $[a_0; a_1, \dots, a_\ell - 1, 1]$ for $\ell = 0$ or for $\ell \geq 1$ and $a_\ell > 1$. We take the former as the *continued fraction representation* of r. The integers $a_0, a_1, \dots, a_\ell$ are called the *elements* of the continued fraction representation. We let $S(r)$ denote the sum of the elements of the continued fraction representation of r.

3. Results

In this section, we shall present our results on the additive variant of the game Euclid.

Since the general theory of combinatorial game tells us that the strategy of always moving to a P-position brings victory, it is important that we have an efficient way of finding out which position is a P-position. Therefore, in the following first theorem, we shall provide a characterization of P-positions that enables us to efficiently determine whether a given position is a P-position.

Theorem 1. *For any position $(x_1, x_2) \in X_n$, the following two statements are equivalent.*

(i) *It is a P-position.*

(ii) *The smallest integer $i \geq 2$ satisfying $gf_i(x_1 - n, x_2 - n) \leq -n$ is even.*

Proof. Take a position $(x_1, x_2) \in X_n$ arbitrarily. Let us divide the argument into two cases.

`Case 1:` (x_1, x_2) is terminal. In this case, both statements are evidently true; hence, the asserted equivalence is valid.

`Case 2:` (x_1, x_2) is non-terminal. Let us first introduce two important integer sequences. Define integers a_1 and x_3 by the equation

$$a_1 = \left\lceil \frac{n - x_1}{x_2} \right\rceil \quad \text{and} \quad x_3 = x_1 + a_1 x_2 - n.$$

(Note that these definitions make sense because x_2 is non-zero.) Observe then that $a_1 > 0$ and $x_2 > x_3 \geq 0$. Note also that (x_2, x_3) is eventually reached from (x_1, x_2) no matter how the game is played. If x_3 is non-zero, define integers a_2 and x_4 by the equations

$$a_2 = \left\lceil \frac{n - x_2}{x_3} \right\rceil \quad \text{and} \quad x_4 = x_2 + a_2 x_3 - n.$$

Then the above observations with each subscript incremented by 1 are both valid. If x_4 is non-zero, define integers a_3 and x_5 by the equations

$$a_3 = \left\lceil \frac{n - x_3}{x_4} \right\rceil \quad \text{and} \quad x_5 = x_3 + a_3 x_4 - n.$$

If we continue defining positive integers $a_1, a_2, a_3, \ldots$ and non-negative integers $x_3, x_4, x_5, \ldots$ in this way then, because the integer sequence $x_2, x_3, x_4, \ldots$ is strictly decreasing, we eventually reach a $k \geq 1$ such that $x_{k+2} = x_k + a_k x_{k+1} - n$ is zero.

Let us then make two observations. The first one is that the equation

$$a_i = \left|\{(x, x') \in X_n \mid (x, x') \text{ is reachable from } (x_i, x_{i+1})\}\right| \tag{4}$$

holds for any positive integer $i \leq k$, which follows immediately from Equation (1). The other observation is that for any position $(x_1, x_2) \in X_n$, the generalized Fibonacci sequence with the initial values $x_1 - n$ and $x_2 - n$ is a weakly decreasing sequence of negative integers, which can readily be verified by induction. In other words, the inequalities

$$0 > gf_1(x_1 - n, x_2 - n) \geq gf_2(x_1 - n, x_2 - n) > gf_3(x_1 - n, x_2 - n) > \cdots \tag{5}$$

hold. Note that we have the strict inequality $gf_i(x_1 - n, x_2 - n) > gf_{i+1}(x_1 - n, x_2 - n)$ for all but $i = 1$.

Having finished preparations, we now divide the argument further into two subcases.

Subcase 2.1: The game starting from (x_1, x_2) ends in a finite number of forced moves (equivalently, by Equation (4), all of $a_1, a_2, \ldots, a_k$ are equal to 1). It is readily inferred that the position (x_1, x_2) is a P-position if and only if k is even. Also, since we have $gf_i(x_1 - n, x_2 - n) = x_i - n$ for any positive integer $i \leq k + 2$, which can be proved readily by induction, the smallest integer $i \geq 2$ satisfying $gf_i(x_1 - n, x_2 - n) \leq -n$ should be $k + 2$. Because k and $k + 2$ have the same parity, it follows from these that the asserted equivalence is valid in this subcase.

Subcase 2.2: The game starting from (x_1, x_2) does not end in a finite number of forced moves (equivalently, by Equation (4), not all of $a_1, a_2, \ldots, a_k$ are equal to 1). We first prove the claim that if there is more than one valid move from a position then that position is an N-position. To do so, let $(x, x') \in X_n$ be a position from which there is more than one valid move. Equation (1) then implies that $a \coloneqq \left\lceil \frac{n-x}{x'} \right\rceil$ is larger than 1. The reachable positions from (x, x') are thus

$$(x + x', x'), (x + 2x', x'), \ldots, (x + (a - 1)x', x'), (x', x + ax' - n).$$

If $(x', x + ax' - n)$ is a P-position, then (x, x') is an N-position because there is a move from it to the P-position $(x', x + ax' - n)$. If, on the other hand, $(x', x + ax' - n)$ is an N-position, then $(x + (a - 1)x', x')$ is a P-position because there is only one valid move from $(x + (a - 1)x', x')$, and that move is to the N-position $(x', x + ax' - n)$. As the P-position $(x + (a - 1)x', x')$ is reachable from (x, x'), it follows that (x, x') is an N-position also in this case.

From the above claim, it can be inferred that (x_1, x_2) is a P-position if and only if the smallest integer $i \geq 1$ such that there is more than one valid move from (x_i, x_{i+1}) (equivalently, by Equation (4), $a_i > 1$) is even. It thus remains to show that the latter is true if and only if the smallest integer $i \geq 2$ satisfying $gf_i(x_1 - n, x_2 - n) \leq -n$ is even. To do so, let i_0 denote the smallest integer $i \geq 1$ satisfying $a_i > 1$. Then induction shows that $gf_i(x_1 - n, x_2 - n) = x_i - n$ for any positive integer $i \leq i_0 + 1$, which implies $gf_{i_0+2}(x_1 - n, x_2 - n) = x_{i_0} + x_{i_0+1} - 2n$. As we have $x_{i_0+1} > 0$ and $\left\lceil \frac{n-x_{i_0}}{x_{i_0+1}} \right\rceil = a_{i_0} > 1$, it follows that

$$gf_{i_0+1}(x_1 - n, x_2 - n) > -n > gf_{i_0+2}(x_1 - n, x_2 - n).$$

Combined with Inequalities (5), the above inequalities indicate the smallest integer $i \geq 2$ satisfying $gf_i(x_1 - n, x_2 - n) \leq -n$ is $i_0 + 2$. Since i_0 and $i_0 + 2$ have the same parity, it follows that the smaller integer $i \geq 1$ satisfying $a_i > 1$ is even if and only if the smallest integer $i \geq 2$ satisfying $gf_i(x_1 - n, x_2 - n) \leq -n$ is even, as desired. □

Unlike the original recursive definition of P-positions, the established result provides us with a concrete number theoretic description of P-positions. Moreover, it is useful in determining whether a given position is a P-position; indeed, the algorithm based on it is clearly more efficient than the algorithm based on the original recursive definition of P-positions. However, it does not immediately answer the geometric question of what the shape of the set of P-positions looks like. Therefore, using the algorithm based on it, we made a figure of X_{100} in which each cell (x_1, x_2)

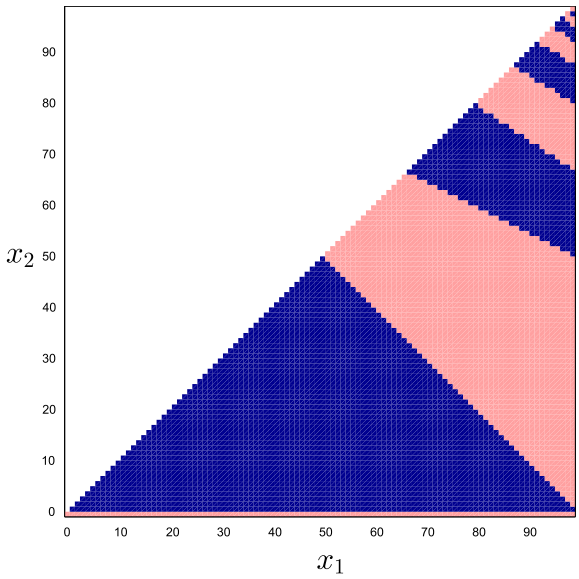


Figure 1: An illustration of X_{100} in which each cell (x_1, x_2) is colored Pink or Navy according as (x_1, x_2) is a P-position or an N-position.

is colored Pink or Navy according as (x_1, x_2) is a P-position or an N-position; see Figure 1.

Corollary 1. *For any position $(x_1, x_2) \in X_n$, if it is chosen as the initial position then there will be at least $|\{i \in \mathbb{Z} \mid i \geq 3, gf_i(x_1 - n, x_2 - n) \geq -n\}|$ forced moves by the end of the game.*

Proof. Take a position $(x_1, x_2) \in X_n$ arbitrarily, and suppose that the game starts from it. We shall prove the consequent of the statement by using notation and results from the proof of Theorem 1.

In Case 1, we have $gf_1(x_1 - n, x_2 - n) \geq gf_2(x_1 - n, x_2 - n) = -n$. From these relations and Inequalities (5), it follows that $|\{i \in \mathbb{Z} \mid i \geq 3, gf_i(x_1 - n, x_2 - n) \geq -n\}| = 0$. Therefore, the consequent of the statement is true in this case.

In Subcase 2.1, the equation $gf_i(x_1 - n, x_2 - n) = x_i - n$ holds for any positive integer $i \leq k+2$, which implies that $gf_{k+1}(x_1-n, x_2-n) > -n = gf_{k+2}(x_1-n, x_2-n)$. From these and Inequalities (5), it follows that $|\{i \in \mathbb{Z} \mid i \geq 3, gf_i(x_1 - n, x_2 - n) \geq -n\}| = k$. Since the game starting from (x_1, x_2) ends in k forced moves no matter how it is played, we see that the consequent of the statement is true also in this subcase.

In Subcase 2.2, because the first $i_0 - 1$ terms of $a_1, a_2, \ldots, a_k$ are all equal to 1, it follows from Equation (4) that the first $i_0 - 1$ moves are all forced. Also, because $gf_{i_0+1}(x_1 - n, x_2 - n) > -n > gf_{i_0+2}(x_1 - n, x_2 - n)$ and because we have

Inequalities (5), the quantity $|\{i \in \mathbb{Z} \mid i \geq 3, gf_i(x_1 - n, x_2 - n) \geq -n\}|$ is equal to $i_0 - 1$. Therefore, the consequent of the statement is true also in this subcase. □

Define a subspace $X_{\mathbb{Q}}$ of $\mathbb{R}^2$ by setting

$$X_{\mathbb{Q}} = \{(x_1, x_2) \in \mathbb{Q}^2 \mid 1 > x_1 \geq x_2 \geq 0\},$$

and consider the following two subsets:

$$\mathcal{P} = \{(x_1, x_2) \in X_{\mathbb{Q}} \mid (nx_1, nx_2) \text{ is a P-position of } X_n \text{ for some } n \geq 1\}, \quad \text{and}$$
$$\mathcal{N} = \{(x_1, x_2) \in X_{\mathbb{Q}} \mid (nx_1, nx_2) \text{ is an N-position of } X_n \text{ for some } n \geq 1\}.$$

Note that these two subsets are disjoint. To see this, assume for the sake of contradiction that the intersection $\mathcal{P} \cap \mathcal{N}$ has an element, say (x_1, x_2). Then there should be positive integers $n, n' \geq 1$ such that (nx_1, nx_2) is a P-position of X_n and $(n'x_1, n'x_2)$ is an N-position of $X_{n'}$. Since the structure of the game on X_n starting from (x_1, x_2) is essentially the same as the structure of the game on $X_{n'n}$ starting from $(n'x_1, n'x_2)$, the position $(n'nx_1, n'nx_2)$ should be a P-position of $X_{n'n}$. As interchanging n and n' in the preceding sentence leads to the contradictory conclusion that the position $(n'nx_1, n'nx_2)$ should be an N-position of $X_{n'n}$, we see that the intersection $\mathcal{P} \cap \mathcal{N}$ is empty.

Concerning these two disjoint subsets, we have the ensuing result. (In what follows, int and ∂ denote the interior operator and the boundary operator, respectively.)

Corollary 2. *For any $(x_1, x_2) \in X_{\mathbb{Q}}$, we have*

(i) $(x_1, x_2) \in \operatorname{int} \mathcal{P}$ *if and only if* $gf_{i_0}(x_1 - 1, x_2 - 1) < -1$ *and* i_0 *is even;*

(ii) $(x_1, x_2) \in \operatorname{int} \mathcal{N}$ *if and only if* $gf_{i_0}(x_1 - 1, x_2 - 1) < -1$ *and* i_0 *is odd; and*

(iii) $(x_1, x_2) \in \partial\mathcal{P} \cup \partial\mathcal{N}$ *if and only if* $gf_{i_0}(x_1 - 1, x_2 - 1) = -1$,

where i_0 denotes the smallest integer $i \geq 2$ satisfying $gf_i(x_1 - 1, x_2 - 1) \leq -1$.

Proof. Because $\mathcal{P} \cap \mathcal{N} = \emptyset$, the three sets, $\operatorname{int} \mathcal{P}$, $\operatorname{int} \mathcal{N}$, and $\partial\mathcal{P} \cup \partial\mathcal{N}$, are pairwise disjoint. It is also plain that the consequents of Parts (i), (ii), and (iii) are mutually exclusive. These two facts combine to show that if we prove the backward implications of Parts (i), (ii), and (iii) then the validity of the forward implications of all three parts follows.

To prove the backward implication of Part (i), take an arbitrary pair $(x_1, x_2) \in X_{\mathbb{Q}}$ satisfying the consequent of Part (i). Then, because the equation $i_0 = 2$ is equivalent to $x_2 = 0$ and implies $gf_{i_0}(x_1 - 1, x_2 - 1) = -1$, we have $i_0 > 2$ (equivalently, $x_2 > 0$). Also, since we have the inequalities

$$gf_{i_0-1}(x_1 - 1, x_2 - 1) > -1 > gf_{i_0}(x_1 - 1, x_2 - 1)$$

and since both $gf_{i_0-1}(u,v)$ and $gf_{i_0}(u,v)$ are continuous in u and v, there exists a sufficiently small positive real number ε such that the inequalities

$$gf_{i_0-1}(x_1'-1,x_2'-1) > -1 > gf_{i_0}(x_1'-1,x_2'-1) \tag{6}$$

hold for any $(x_1',x_2') \in \mathbb{Q}^2$ satisfying $|x_1-x_1'|+|x_2-x_2'|<\varepsilon$. Choose a pair (x_1',x_2') from an open neighborhood $\{(x_1',x_2') \in \mathbb{Q}^2 \mid |x_1-x_1'|+|x_2-x_2'|<\varepsilon\} \cap X_{\mathbb{Q}}$ of (x_1,x_2) in $X_{\mathbb{Q}}$ arbitrarily, and take a positive integer m so that both mx_1' and mx_2' are integers. Then, because the equation $mgf_i(u,v) = gf_i(mu,mv)$ holds for any u,v, and $i \geq 1$, it follows from Inequalities (5) and Equation (6) that i_0 is the smallest integer $i \geq 2$ satisfying $gf_i(mx_1'-m,mx_2'-m) \leq -m$. Thus, by Theorem 1, (mx_1',mx_2') is a P-position of X_m, implying that $(x_1',x_2') \in \mathcal{P}$. As the pair (x_1',x_2') was chosen arbitrarily, it follows that the open neighborhood $\{(x_1',x_2') \in \mathbb{Q}^2 \mid |x_1-x_1'|+|x_2-x_2'|<\varepsilon\} \cap X_{\mathbb{Q}}$ of (x_1,x_2) in $X_{\mathbb{Q}}$ is contained in $\mathcal{P}$; therefore, we have $(x_1,x_2) \in \operatorname{int}\mathcal{P}$.

The backward implication of Part (ii) can be proved analogously.

It remains to show the backward implication of Part (iii). To do so, take an arbitrary pair $(x_1,x_2) \in X_{\mathbb{Q}}$ satisfying $gf_{i_0}(x_1-1,x_2-1) = -1$. In the following, we divide the argument into two cases according as $x_2 = 0$ or not.

In the first case (that is, $x_2 = 0$), take a positive integer m so that mx_1 is an integer. Then, (mx_1,mx_2) is a terminal (hence, P-) position of X_m; therefore, $(x_1,x_2) \in \mathcal{P}$. Let x_1'' be an arbitrary rational number satisfying $x_1 < x_1'' < 1$, and take a positive rational number $x_2'' < \min\{x_1'',1-x_1''\}$ arbitrarily. Then, because we have $gf_2(x_1''-1,x_2''-1) > -1 > gf_3(x_1''-1,x_2''-1)$, we can argue as in the proof of $(x_1',x_2') \in \mathcal{P}$ (see the second paragraph of this proof) to show that $(x_1'',x_2'') \in \mathcal{N}$. As this implies that an arbitrarily small open neighborhood of (x_1,x_2) in $X_{\mathbb{Q}}$ contains an element of $\mathcal{N}$, we see that $(x_1,x_2) \notin \operatorname{int}\mathcal{P}$. Hence, $(x_1,x_2) \in \partial\mathcal{P}$ in this case.

In the other case (that is, $x_2 > 0$), we have

$$gf_{i_0-1}(x_1-1,x_2-1) > -1 = gf_{i_0}(x_1-1,x_2-1) > gf_{i_0+1}(x_1-1,x_2-1)$$

with $i_0 > 2$. (That $i_0 > 2$ follows from the fact that $i_0 = 2$ is equivalent to $x_2 = 0$.) Note also that we have $gf_{i_0}(x_1''-1,x_2-1) > gf_{i_0}(x_1-1,x_2-1)$ for any $x_1'' > x_1$, which is immediate from Equation (3). From these and the continuity of $gf_{i_0-1}(u,v)$, $gf_{i_0}(u,v)$, and $gf_{i_0+1}(u,v)$ in u and v, it follows that there exists a sufficiently small positive real number $\varepsilon < 1-x_1$ such that the inequalities

$$gf_{i_0}(x_1''-1,x_2-1) > -1 > gf_{i_0+1}(x_1''-1,x_2-1)$$

hold for any $x_1'' \in \mathbb{Q}$ satisfying $x_1 < x_1'' < x_1+\varepsilon$. By arguing as in the proof of the relation $(x_1',x_2') \in \mathcal{P}$ (see the second paragraph of this proof), we can then show that $(x_1,x_2) \in \mathcal{P}$ when i_0 is even and $(x_1,x_2) \in \mathcal{N}$ when i_0 is odd; also, $(x_1'',x_2) \in \mathcal{N}$ when i_0 is even and $(x_1'',x_2) \in \mathcal{P}$ when i_0 is odd, where x_1'' is an arbitrary rational number satisfying $x_1 < x_1'' < x_1+\varepsilon$. From these, it can readily be inferred that $(x_1,x_2) \in \partial\mathcal{P}$ when i_0 is even and $(x_1,x_2) \in \partial\mathcal{N}$ when i_0 is odd. Therefore, the backward implication is true also in this case. □

Let us note that for any $n \geq 1$ and $(x_1, x_2) \in X_n$, the point $\left(\frac{x_1}{n}, \frac{x_2}{n}\right) \in X_{\mathbb{Q}}$ satisfies the consequent of Part (iii) if and only if the game starting from (x_1, x_2) ends in a finite number of forced moves, which can be seen from the proof of Theorem 1. Here, "a finite number" includes zero.

In the game Euclid, those positions from which the game ends in a finite non-zero number of forced moves can be described by the Fibonacci sequence [9]. But then what about our game? Can those positions in our game also be described by the Fibonacci sequence? In the next theorem, we shall answer this question affirmatively.

Theorem 2. *Let ℓ be an arbitrary positive integer. Then for any position $(x_1, x_2) \in X_n$, the following two statements are equivalent.*

(i) *The game starting from (x_1, x_2) ends in ℓ forced moves.*

(ii) *We have*

$$\frac{x_1 - (1 + (-1)^{\ell-1} f_{\ell-1})n}{(-1)^{\ell} f_{\ell+1}} = \frac{x_2 - (1 + (-1)^{\ell-2} f_{\ell-2})n}{(-1)^{\ell-1} f_{\ell}} \in I(\ell, n),$$

where

$$I(\ell, n) = \begin{cases} \mathbb{Z} \cap \left(\frac{f_{\ell-1}}{f_{\ell+1}} n, \frac{f_{\ell}}{f_{\ell+2}} n\right] & \text{if } \ell \text{ is odd,} \\ \mathbb{Z} \cap \left[\frac{f_{\ell}}{f_{\ell+2}} n, \frac{f_{\ell-1}}{f_{\ell+1}} n\right) & \text{if } \ell \text{ is even.} \end{cases}$$

Proof. Let $(x_1, x_2) \in X_n$ be an arbitrary position. There are then two cases to consider.

`Case 1:` (x_1, x_2) is terminal. In this case, Statement (i) is clearly false. Also, because Lemma 1 (ii) and (iii) imply

$$\frac{x_2 - (1 + (-1)^{\ell-2} f_{\ell-2})n}{(-1)^{\ell-1} f_{\ell}} = \frac{(f_{\ell-2} - 1)n}{f_{\ell}} \leq \frac{f_{\ell-1}}{f_{\ell+1}} n$$

if ℓ is odd and

$$\frac{x_2 - (1 + (-1)^{\ell-2} f_{\ell-2})n}{(-1)^{\ell-1} f_{\ell}} = \frac{(1 + f_{\ell-2})n}{f_{\ell}} \geq \frac{f_{\ell-1}}{f_{\ell+1}} n$$

if ℓ is even, the last relation from Statement (ii) cannot be valid. Both statements being false, the asserted equivalence is valid.

`Case 2:` (x_1, x_2) is non-terminal. Let us note that, because X_n has a non-terminal position, the positive integer n cannot be 1.

We first prove the claim that Statement (i) for the chosen position (x_1, x_2) is equivalent to the statement that there exists a positive integer $x < n$ satisfying $x_1 = \varphi_1(x), x_2 = \varphi_2(x)$, and the inequalities

$$n > \varphi_1(x) \geq \varphi_2(x) > \cdots > \varphi_{\ell}(x) > \varphi_{\ell+1}(x) > 0. \tag{7}$$

To prove the forward implication, assume that Statement (i) for the chosen position (x_1, x_2) is true, and define integers $a_1, a_2, \ldots, a_k$ and $x_3, x_4, \ldots, x_{k+2}$ as in the proof of Theorem 1. Then, since there is only one valid move from (x_1, x_2) by the assumption, it follows from Equation (4) that $a_1 = 1$, and the only valid move from (x_1, x_2) is to (x_2, x_3). If (x_2, x_3) is not terminal then, because there is only one valid move from (x_2, x_3) by the assumption, it follows from Equation (4) that $a_2 = 1$, and the only valid move from (x_2, x_3) is to (x_3, x_4). By continuing in this manner, we get $\ell = k$ and

$$\begin{aligned} x_3 &= x_1 + x_2 - n, \\ x_4 &= x_2 + x_3 - n, \\ &\vdots \\ x_{\ell+1} &= x_{\ell-1} + x_\ell - n, \\ x_{\ell+2} &= x_\ell + x_{\ell+1} - n. \end{aligned} \tag{8}$$

Observe that $n > x_1 \geq x_2 > x_3 > \cdots > x_{\ell+1} > x_{\ell+2} = 0$. Next, define linear functions $\varphi_i(x)$ $(i = 1, 2, \ldots, \ell + 2)$ by

$$\varphi_i(x) = (-1)^{\ell-i+1} f_{\ell-i+2} x + (1 + (-1)^{\ell-i} f_{\ell-i}) n.$$

Then a calculation shows that $\varphi_{\ell+2}(x) = 0$, $\varphi_{\ell+1}(x) = x$, and $\varphi_{i+2}(x) = \varphi_i(x) + \varphi_{i+1}(x) - n$ for any x and positive integer $i \leq \ell$. Using these equations and Equations (8), one can show by decreasing induction on i that the equation $\varphi_i(x_{\ell+1}) = x_i$ holds for any positive integer $i \leq \ell + 2$. Therefore, the positive integer $x = x_{\ell+1} < n$ satisfies $x_1 = \varphi_1(x)$, $x_2 = \varphi_2(x)$, and Inequalities (7). Having proved the forward implication, we next assume that a positive integer $x < n$ satisfies $x_1 = \varphi_1(x)$, $x_2 = \varphi_2(x)$, and Inequalities (7). Then, using the equations $\varphi_{i+2}(x) = \varphi_i(x) + \varphi_{i+1}(x) - n$ $(i \leq \ell)$, it can readily be verified that Statement (i) for $(\varphi_1(x), \varphi_2(x)) = (x_1, x_2)$ is true.

We then claim that for any positive integer $i \leq \ell$, the set

$$I_i \coloneqq \begin{cases} \{x \in \mathbb{Z} \cap [1, n) \mid n > \varphi_1(x) \geq \varphi_2(x)\} & \text{if } i = 1, \\ \{x \in \mathbb{Z} \cap [1, n) \mid n > \varphi_i(x) > \varphi_{i+1}(x)\} & \text{if } i \geq 2 \end{cases}$$

satisfies the equation

$$I_i = \begin{cases} \mathbb{Z} \cap \left(\dfrac{f_{\ell-1}}{f_{\ell+1}} n, \dfrac{f_\ell}{f_{\ell+2}} n \right] & \text{if } i = 1 \text{ and } \ell \text{ is odd}, \\ \mathbb{Z} \cap \left[\dfrac{f_\ell}{f_{\ell+2}} n, \dfrac{f_{\ell-1}}{f_{\ell+1}} n \right) & \text{if } i = 1 \text{ and } \ell \text{ is even}, \\ \mathbb{Z} \cap \left(\dfrac{f_{\ell-i+1}}{f_{\ell-i+3}} n, \dfrac{f_{\ell-i}}{f_{\ell-i+2}} n \right) & \text{if } i \geq 2 \text{ and } \ell - i \text{ is odd}, \\ \mathbb{Z} \cap \left(\dfrac{f_{\ell-i}}{f_{\ell-i+2}} n, \dfrac{f_{\ell-i+1}}{f_{\ell-i+3}} n \right) & \text{if } i \geq 2 \text{ and } \ell - i \text{ is even}. \end{cases} \tag{9}$$

To prove this equation, there are the following four cases to consider: (a) $i = 1$ and ℓ is odd, (b) $i = 1$ and ℓ is even, (c) $i \geq 2$ and $\ell - i$ is odd, and (d) $i \geq 2$ and $\ell - i$ is even. However, since the proofs for these four cases run parallel to each other, we shall provide a proof for the last case only. In this case, as we have $n > \varphi_i(x) = -f_{\ell-i+2}x + (1 + f_{\ell-i})n$, the inequality $x > \frac{f_{\ell-i}}{f_{\ell-i+2}}n$ holds. Also, $-f_{\ell-i+2}x + (1 + f_{\ell-i})n = \varphi_i(x) > \varphi_{i+1}(x) = f_{\ell-i+1}x + (1 - f_{\ell-i-1})n$ implies $\frac{f_{\ell-i+1}}{f_{\ell-i+3}}n = \frac{f_{\ell-i-1}+f_{\ell-i}}{f_{\ell-i+1}+f_{\ell-i+2}}n > x$. Therefore, Equation (9) is correct in this case.

Observe that for any positive integer $x < n$, because $\varphi_{\ell+1}(\cdot)$ is the identity function, it satisfies Inequalities (7) if and only if it belongs to $\bigcap_{i=1}^{\ell} I_i$. Also, Equation (9) combines with Lemma 1 (i) to imply $I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_\ell$, which in turn implies $\bigcap_{i=1}^{\ell} I_i = I_1$. From these and the first claim, we thus see that Statement (i) for the chosen position (x_1, x_2) is equivalent to the statement that there exists an $x \in I_1$ such that $x_1 = \varphi_1(x)$ and $x_2 = \varphi_2(x)$. Since $x_1 = \varphi_1(x)$ and $x_2 = \varphi_2(x)$ are equivalent to

$$x = \frac{x_1 - (1 + (-1)^{\ell-1} f_{\ell-1})n}{(-1)^{\ell} f_{\ell+1}} \quad \text{and} \quad x = \frac{x_2 - (1 + (-1)^{\ell-2} f_{\ell-2})n}{(-1)^{\ell-1} f_{\ell}},$$

respectively, and since $I_1 = I(\ell, n)$ by Equation (9), the latter statement is equivalent to Statement (ii), finishing the proof. □

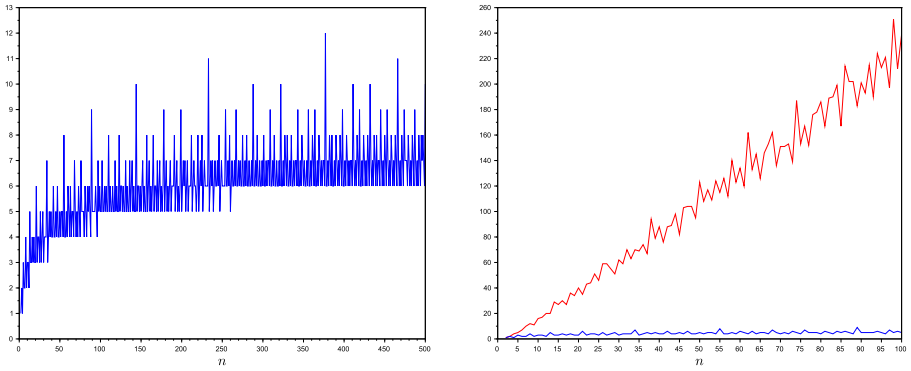
The following is an immediate consequence of the above theorem.

Corollary 3. *For any positive integer ℓ, the number of positions of X_n such that the game starting from that position ends in ℓ forced moves is $\left\lfloor \frac{f_\ell}{f_{\ell+2}}n \right\rfloor - \left\lfloor \frac{f_{\ell-1}}{f_{\ell+1}}n \right\rfloor$ if ℓ is odd and $\left\lceil \frac{f_{\ell-1}}{f_{\ell+1}}n \right\rceil - \left\lceil \frac{f_\ell}{f_{\ell+2}}n \right\rceil$ if ℓ is even.*

Using this corollary, for each $n = 2, 3, \ldots, 500$, we plotted the largest positive integer ℓ such that there exists a position of X_n from which the game ends in ℓ forced moves as Figure 2 (a). The graph does not grow rapidly, but it is certainly unbounded; indeed, for each positive integer ℓ, the above corollary shows that if a positive integer $n \geq 2$ satisfies $\left|\frac{f_{\ell-1}}{f_{\ell+1}} - \frac{f_\ell}{f_{\ell+2}}\right| > \frac{1}{n}$, then there is at least one position of X_n from which the game ends in ℓ forced moves. For the purpose of comparison, in Figure 2 (b), we also plotted not only the above largest positive integer ℓ but also the length of the longest play starting from some position of X_n for each $n = 2, 3, \ldots, 100$. (Let us note that the length of the longest play starting from a non-terminal position $(x_1, x_2) \in X_n$ is $\sum_{i=1}^{k} a_i$, which can be inferred from the proof of Theorem 1.)

Observe that after a forced move, the smaller entry decreases strictly. It follows that if the game starting from (x_1, x_2) ends in a finite non-zero number of forced moves then the total number of forced moves by the end of the game is at most x_2, which is smaller than n. We thus have the ensuing corollary.

Corollary 4. *For any position $(x_1, x_2) \in X_n$, the following two statements are equivalent.*



(a) (b)

Figure 2: (a) For each $n = 2, 3, \ldots, 500$, the largest positive integer ℓ such that there exists a position of X_n from which the game ends in ℓ forced moves is plotted in blue. (b) For each $n = 2, 3, \ldots, 100$, the same largest positive integer ℓ as in (a) and the length of the longest play starting from some position of X_n are plotted in blue and red, respectively.

(i) *The game starting from* (x_1, x_2) *ends in a finite non-zero number of forced moves.*

(ii) *There exists a positive integer* $\ell < n$ *such that we have*

$$\frac{x_1 - (1 + (-1)^{\ell-1} f_{\ell-1})n}{(-1)^{\ell} f_{\ell+1}} = \frac{x_2 - (1 + (-1)^{\ell-2} f_{\ell-2})n}{(-1)^{\ell-1} f_{\ell}} \in I(\ell, n),$$

where the set $I(\ell, n)$ *is as in the statement of Theorem 2.*

So far, we have considered only games played on pairs of non-negative integers. However, as one may realize, the definition of the game remains valid even if we extend the set of positions to pairs of non-negative real numbers. Before exploring such games, let us note that for any pair $(x_1, x_2) \in \mathbb{R}^2$ satisfying $n > x_1 \geq x_2 \geq 0$, the structure of the game starting from (x_1, x_2) in which players take remainders modulo n is essentially the same as the structure of the game starting from $\left(\frac{x_1}{n}, \frac{x_2}{n}\right)$ in which players take remainders modulo 1. In what follows, without loss of generality, we thus consider games played on

$$X_{\mathbb{R}} \coloneqq \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 > x_1 \geq x_2 \geq 0\}$$

in which players take remainders modulo 1.

As we have already explained, games played on X_n eventually end no matter which position is chosen as the initial position. However, a game played on $X_{\mathbb{R}}$ can

continue forever unless it starts from certain positions. It is then natural to try to describe those positions, in other words, elements of the set

$$E \coloneqq \left\{ (x_1, x_2) \in X_{\mathbb{R}} \;\middle|\; \begin{array}{r} \text{the game starting from } (x_1, x_2) \text{ eventually ends} \\ \text{no matter how it is played} \end{array} \right\}.$$

Note that we have $E \cap \mathbb{Q}^2 = X_{\mathbb{R}} \cap \mathbb{Q}^2 = X_{\mathbb{Q}}$. Also, an element (x_1, x_2) of $E \setminus \mathbb{Q}^2$ is terminal if and only if $x_1 \notin \mathbb{Q}$ and $x_2 = 0$. Therefore, it is sufficient to consider only those elements of $E \setminus \mathbb{Q}^2$ that are non-terminal.

We have not found a necessary and sufficient condition for a non-terminal position $(x_1, x_2) \in X_{\mathbb{R}} \setminus \mathbb{Q}^2$ to be in $E \setminus \mathbb{Q}^2$; however, we at least found a necessary condition. So, let us present it below. (That the stated condition is not a sufficient condition is illustrated later; see Example 1 (iii).)

Theorem 3. *If a non-terminal position $(x_1, x_2) \in X_{\mathbb{R}} \setminus \mathbb{Q}^2$ belongs to $E \setminus \mathbb{Q}^2$, then there exist a unique pair (r_1, r_2) of positive rationals satisfying*

(i) $r_1 x_1 + r_2 x_2 = 1$;

(ii) *the elements of the continued fraction representation of $\frac{r_2}{r_1}$ is weakly increasing; and*

(iii) $r_1 \le r_2$, $\frac{1}{3} < r_1 \le 1$, *and* $\frac{1}{2} < r_2$.

Moreover, the game starting from (x_1, x_2) ends in at most $S(\frac{r_2}{r_1})$ moves no matter how it is played.

Proof. With each non-terminal position $(x_1, x_2) \in X_{\mathbb{R}}$, as in the proof of Theorem 1, we associate (finite or infinite) sequences $a_1, a_2, a_3, \ldots$ and $x_3, x_4, x_5, \ldots$ as follows. Define an integer a_1 and a real number x_3 by the equations

$$a_1 = \left\lceil \frac{1 - x_1}{x_2} \right\rceil \quad \text{and} \quad x_3 = x_1 + a_1 x_2 - 1.$$

Note that these definitions make sense because x_2 is non-zero. Then it can readily be verified that $a_1 > 0$ and $x_2 > x_3 \ge 0$. It can also be inferred from the rules of the game that (x_2, x_3) is eventually reached from (x_1, x_2) no matter how the game is played. If x_3 is non-zero, define an integer a_2 and a real number x_4 by the equations

$$a_2 = \left\lceil \frac{1 - x_2}{x_3} \right\rceil \quad \text{and} \quad x_4 = x_2 + a_2 x_3 - 1.$$

As before, we have $a_2 > 0$ and $x_3 > x_4 \ge 0$. Moreover, (x_3, x_4) is eventually reached from (x_2, x_3) no matter how the game is played. If x_4 is non-zero, define an integer a_3 and a real number x_5 by the equations

$$a_3 = \left\lceil \frac{1 - x_3}{x_4} \right\rceil \quad \text{and} \quad x_5 = x_3 + a_3 x_4 - 1.$$

Continue defining positive integers $a_1, a_2, a_3, \ldots$ and non-negative real numbers $x_3, x_4, x_5, \ldots$ in this way until a $k \geq 1$ with $x_{k+2} = 0$ is reached. Note that for any non-terminal position $(x_1, x_2) \in X_{\mathbb{R}}$, such a $k \geq 1$ exists if and only if $(x_1, x_2) \in E$. This fact will be freely used without explicit mention in the remainder of this proof.

To prove the theorem, let us first validate the claim that for any non-terminal position $(x_1, x_2) \in E \setminus \mathbb{Q}^2$, there exists a unique pair (r_1, r_2) of positive rationals that satisfies Item (i). We shall do so by induction on the length k of the associated sequence $x_3, x_4, \ldots, x_{k+2}$. If a non-terminal position $(x_1, x_2) \in E \setminus \mathbb{Q}^2$ has length 1 associated sequence, then $x_3 = 0$. As $x_3 = x_1 + a_1 x_2 - 1$, we have $x_1 + a_1 x_2 = 1$, showing that the existential part of the claim is correct for this position. Assume then that the existential part of the claim is correct for any non-terminal position of $E \setminus \mathbb{Q}^2$ whose associated sequence has length k, and take an arbitrary non-terminal position $(x_1, x_2) \in E \setminus \mathbb{Q}^2$ whose associated sequence $x_3, x_4, \ldots, x_{k+3}$ has length $k+1$. Then the position (x_2, x_3) is non-terminal and has length k associated sequence $x_4, x_5, \ldots, x_{k+3}$. Since $(x_1, x_2) \in E \setminus \mathbb{Q}^2$, we have $(x_2, x_3) \in E$. If $(x_2, x_3) \in \mathbb{Q}^2$ then $x_1 = -a_1 x_2 + x_3 + 1 \in \mathbb{Q}$; hence $(x_1, x_2) \in \mathbb{Q}^2$, contrary to the assumption. Therefore, $(x_2, x_3) \in E \setminus \mathbb{Q}^2$. We can then apply the induction hypothesis to get a pair (r_1, r_2) of positive rationals satisfying $r_1 x_2 + r_2 x_3 = 1$. Substituting the defining equation of x_3 into this equation, we obtain $r_1 x_2 + r_2(x_1 + a_1 x_2 - 1) = 1$. Hence, $\frac{r_2}{1+r_2} x_1 + \frac{r_1 + r_2 a_1}{1+r_2} x_2 = 1$. As r_1, r_2, and a_1 are all positive, this completes the proof of the induction step. To see the uniqueness, assume that for a non-terminal position $(x_1, x_2) \in E \setminus \mathbb{Q}^2$, there exist pairs $(r_1, r_2), (r_1', r_2')$ of positive rationals satisfying $r_1 x_1 + r_2 x_2 = 1 = r_1' x_1 + r_2' x_2$. If $r_1 \neq r_1'$ then $x_1 = \frac{r_2' - r_2}{r_1 - r_1'} x_2$ and thus $(r_1 \frac{r_2' - r_2}{r_1 - r_1'} + r_2) x_2 = 1$. Therefore, x_2, and hence $x_1 = \frac{r_2' - r_2}{r_1 - r_1'} x_2$, should be rational, contrary to the assumption $(x_1, x_2) \notin \mathbb{Q}^2$. As $r_2 \neq r_2'$ also leads to the same contradictory conclusion, uniqueness follows.

We next show the claim that if a non-terminal position $(x_1, x_2) \in X_{\mathbb{R}} \setminus \mathbb{Q}^2$ belongs to $E \setminus \mathbb{Q}^2$ then both x_1 and x_2 are irrational. To see this, let (r_1, r_2) be as in the first claim. Then we have $(x_1, x_2) = (x_1, \frac{1}{r_2} - \frac{r_1}{r_2} x_1)$, which combines with the assumption $(x_1, x_2) \notin \mathbb{Q}^2$ to imply that x_1 is irrational. That x_2 is irrational can be proved analogously.

We then substantiate the claim that for any non-terminal position $(x_1, x_2) \in E \setminus \mathbb{Q}^2$, we have $\frac{r_2}{r_1} = [a_1; a_2, \ldots, a_{k-1}, a_k]$, where $r_1, r_2 > 0$, $k \geq 1$, and $a_1, a_2, \ldots, a_k$ are as above. To do so, define rational numbers b_i and c_i $(i = 1, 2, \ldots, k+2)$ inductively by

$$b_1 = 1, \qquad b_2 = -\frac{r_1}{r_2}, \qquad b_i = b_{i-2} + a_{i-2} b_{i-1} \quad (i \geq 3),$$

$$c_1 = 0, \qquad c_2 = \frac{1}{r_2}, \qquad c_i = c_{i-2} + a_{i-2} c_{i-1} - 1 \quad (i \geq 3).$$

Then one can show by induction that the equation $x_i = b_i x_1 + c_i$ holds for any positive integer $i \leq k+2$. Since $x_{k+2} = 0$, it follows that $b_{k+2} x_1 + c_{k+2} = 0$. Because both b_{k+2} and c_{k+2} are rational and because x_1 is irrational (see the above paragraph), this equation implies $b_{k+2} = 0$ and $c_{k+2} = 0$.

Here, let us prove the subclaim that none of $b_1, b_2, \dots, b_{k+1}$ is zero. To do so, assume for the sake of contradiction that $b_i = 0$ for some positive integer $i \le k+1$, and let i_0 be the smallest integer $i \ge 1$ satisfying $b_i = 0$. Then $i_0 > 2$ and $b_{i_0+1} = b_{i_0-1} \neq 0$. If they are positive (resp. negative), then $b_{i_0+2} = a_{i_0} b_{i_0+1}$ should be positive (resp. negative) because a_{i_0} is a positive integer. Then $b_{i_0+3} = b_{i_0+1} + a_{i_0+1} b_{i_0+2}$ should be positive (resp. negative) because a_{i_0+1} is a positive integer. Continuing in this way, we should be able to prove that b_{k+2} is positive (resp. negative), contrary to the equation $b_{k+2} = 0$.

The proved subclaim guarantees that the ratios $\frac{b_{i+1}}{b_i}$ $(i = 1, 2, \dots, k+1)$ are all defined. As they satisfy

$$\frac{b_2}{b_1} = -\frac{r_1}{r_2} \quad \text{and} \quad \frac{b_{i+1}}{b_i} = a_{i-1} + \frac{b_{i-1}}{b_i} = a_{i-1} + \frac{1}{\frac{b_i}{b_{i-1}}} \quad (i \ge 2),$$

it follows that

$$\begin{aligned} 0 &= \frac{b_{k+2}}{b_{k+1}} \\ &= a_k + \frac{1}{\frac{b_{k+1}}{b_k}} \\ &= a_k + \cfrac{1}{a_{k-1} + \cfrac{1}{\frac{b_k}{b_{k-1}}}} \\ &= \cdots \\ &= a_k + \cfrac{1}{a_{k-1} + \cfrac{1}{\ddots + a_2 + \cfrac{1}{a_1 - \frac{r_2}{r_1}}}}. \end{aligned}$$

By solving this equation for $\frac{r_2}{r_1}$, we obtain $\frac{r_2}{r_1} = [a_1; a_2, \dots, a_{k-1}, a_k]$.

From the substantiated equation, it follows that the continued fraction representation of $\frac{r_2}{r_1}$ is $[a_1; a_2, \dots, a_{k-1}, a_k]$ if either $k = 1$ or $a_k > 1$ and is $[a_1; a_2, \dots, a_{k-2}, a_{k-1} + a_k]$ if $k > 1$ and $a_k = 1$. To show that the pair (r_1, r_2) satisfies Item (ii), it is thus sufficient to prove that the sequence $a_1, a_2, \dots, a_k$ is weakly increasing, which we shall do momentarily.

Assume for the sake of contradiction that $k \ge 2$ and $a_i > a_{i+1}$ for a positive integer i below k. Then, because $x_i \ge x_{i+1} \ge x_{i+2} \ge 0$ and because $x_i + (a_i - 1)x_{i+1} - 1 < 0$ by the definition of a_i, we should have $0 \le x_{i+3} = x_{i+1} + a_{i+1}x_{i+2} - 1 \le x_{i+1} + (a_i - 1)x_{i+2} - 1 \le x_i + (a_i - 1)x_{i+1} - 1 < 0$. This contradiction proves that $a_i \le a_{i+1}$ for all positive integers i below k; in words, the sequence $a_1, a_2, \dots, a_k$ is weakly

increasing. Note that this argument also proves the following: for any position $(x_1, x_2) \in X_{\mathbb{R}} \setminus E$, the associated infinite sequence $a_1, a_2, a_3, \ldots$ is weakly increasing.

The substantiated equation $\frac{r_2}{r_1} = [a_1; a_2, \ldots, a_k]$ implies $\frac{r_2}{r_1} \geq a_1 \geq 1$; hence, $r_1 \leq r_2$. Thus, the pair (r_1, r_2) satisfies the first inequality of Item (iii). To see the last inequality of Item (iii), define non-negative integers $m_{1,i}$ and $m_{2,i}$ $(i = 1, 2, \ldots, k+2)$ inductively by

$$\begin{aligned} &m_{1,1} = 0, \qquad m_{1,2} = 1, \qquad m_{1,i} = m_{1,i-2} + a_{i-2}m_{1,i-1} \quad (i \geq 3),\\ &m_{2,1} = 0, \qquad m_{2,2} = 0, \qquad m_{2,i} = m_{2,i-2} + a_{i-2}m_{2,i-1} + 1 \quad (i \geq 3). \end{aligned}$$

Then one can show by induction that the relations $c_i = m_{1,i}c_2 - m_{2,i}$ and $2m_{1,i} > m_{2,i} > 0$ hold for any positive integer i satisfying $3 \leq i \leq k+2$. Since we have $c_{k+2} = 0$ as has been mentioned already, it follows that $r_2 = \frac{1}{c_2} = \frac{m_{1,k+2}}{m_{2,k+2}} > \frac{1}{2}$. Therefore, the last inequality of Item (iii) is also valid.

To prove $\frac{1}{3} < r_1 \leq 1$, define rational numbers b'_i and c'_i $(i = 1, 2, \ldots, k+2)$ inductively by

$$\begin{aligned} &b'_1 = -\frac{r_2}{r_1}, \qquad b'_2 = 1, \qquad b'_i = b'_{i-2} + a_{i-2}b'_{i-1} \quad (i \geq 3),\\ &c'_1 = \frac{1}{r_1}, \qquad c'_2 = 0, \qquad c'_i = c'_{i-2} + a_{i-2}c'_{i-1} - 1 \quad (i \geq 3). \end{aligned}$$

Then induction shows that the equation $x_i = b'_i x_2 + c'_i$ holds for any positive integer $i \leq k+2$. Define also integers $m'_{1,i}$ and $m'_{2,i}$ $(i = 1, 2, \ldots, k+2)$ inductively by

$$\begin{aligned} &m'_{1,1} = 1, \qquad m'_{1,2} = 0, \qquad m'_{1,i} = m'_{1,i-2} + a_{i-2}m'_{1,i-1} \quad (i \geq 3),\\ &m'_{2,1} = -1, \qquad m'_{2,2} = 0, \qquad m'_{2,i} = m'_{2,i-2} + a_{i-2}m'_{2,i-1} + 1 \quad (i \geq 3). \end{aligned}$$

Induction again shows that the relations $c'_i = m'_{1,i}c'_3 - m'_{2,i}$ and $2m'_{1,i} > m'_{2,i} \geq 0$ hold for any positive integer i satisfying $3 \leq i \leq k+2$. Two equations, $x_{k+2} = 0$ and $x_{k+2} = b'_{k+2}x_2 + c'_{k+2}$, and the irrationality of x_2 (see the second claim) imply that $c'_{k+2} = m'_{1,k+2}c'_3 - m'_{2,k+2}$ is zero. Therefore, $\frac{1}{r_1} - 1 = c'_3 = \frac{m'_{2,k+2}}{m'_{1,k+2}} < 2$ and thus $r_1 > \frac{1}{3}$. To prove the remaining inequality, assume for the sake of contradiction that $r_1 > 1$. Then $c'_3 = c'_1 - 1 = \frac{1}{r_1} - 1 < 0$. If $k \geq 2$ then it follows from $c'_3 < 0$ that $c'_4 = a_2c'_3 - 1 < 0$. If $k \geq 3$ then it follows from $c'_3, c'_4 < 0$ that $c'_5 = c'_3 + a_3c'_4 - 1 < 0$. Continuing in this way, we should be able to show $c'_{k+2} < 0$. However, as has been proved above, c'_{k+2} is zero. From these, it follows that $r_1 \leq 1$, completing the proof of Item (iii).

It can be inferred from the rules of the game that it takes at most a_i moves from (x_i, x_{i+1}) to (x_{i+1}, x_{i+2}). Hence, the game ends in at most $\sum_{i=1}^{k} a_i$ moves. Because the continued fraction representation of $\frac{r_2}{r_1}$ is either $[a_1; a_2, \ldots, a_k]$ or $[a_1; a_2, \ldots, a_{k-2}, a_{k-1} + a_k]$ as has been explained above, this proves the last statement of the theorem. □

From the above theorem, one can readily derive the ensuing result.

Corollary 5. *Suppose that a non-terminal position $(x_1, x_2) \in X_{\mathbb{R}} \setminus \mathbb{Q}^2$ fulfills one of the following two conditions:*

- *There does not exist a pair (r_1, r_2) of positive rationals that satisfies Items (i), (ii), and (iii) of Theorem 3.*
- *There exist a pair (r_1, r_2) of positive rationals that satisfies Items (i), (ii), and (iii) of Theorem 3, but the game starting from (x_1, x_2) does not end in $S(\frac{r_2}{r_1})$ moves.*

Then the game starting from (x_1, x_2) continues forever.

Before closing this section, examples are in order.

Example 1. (i) Let ℓ be a positive integer and x be an irrational number from the open interval $\left(\frac{f_{2\lfloor \frac{\ell}{2} \rfloor}}{f_{2\lfloor \frac{\ell}{2} \rfloor + 2}}, \frac{f_{2\lceil \frac{\ell}{2} \rceil - 1}}{f_{2\lceil \frac{\ell}{2} \rceil + 1}}\right)$. Then the game starting from

$$((-1)^{\ell} f_{\ell+1} x + 1 + (-1)^{\ell-1} f_{\ell-1}, (-1)^{\ell-1} f_{\ell} x + 1 + (-1)^{\ell-2} f_{\ell-2})$$

ends in ℓ forced moves, which can be inferred from the proof of Theorem 2.

(ii) Take the pair $(\frac{1}{\pi}, \frac{1}{2\pi})$ as (x_1, x_2). Then, because $r_1 x_1 + r_2 x_2 = 1$ implies $\pi = r_1 + \frac{r_2}{2}$, no pair (r_1, r_2) of positive rationals satisfies $r_1 x_1 + r_2 x_2 = 1$. Hence, by the above corollary, the game starting from $(\frac{1}{\pi}, \frac{1}{2\pi})$ continues forever.

(iii) Take the pair $(\frac{2}{5}(3 - \frac{2}{\pi}), \frac{1}{4\pi})$ as (x_1, x_2). Then we have $\frac{5}{6} x_1 + \frac{8}{3} x_2 = 1$. It is also readily verified that $r_1 = \frac{5}{6}$ and $r_2 = \frac{8}{3}$ satisfy both Items (ii) and (iii) of Theorem 3. However, the game starting from (x_1, x_2) does not end in eight $(= S(\frac{r_2}{r_1}))$ moves as evidenced by the following:

$$\begin{aligned}(x_1, x_2) &\mapsto (x_2, x_3) \mapsto (x_2 + x_3, x_3) \mapsto (x_2 + 2x_3, x_3) \\ &\mapsto (x_2 + 3x_3, x_3) \mapsto (x_2 + 4x_3, x_3) \mapsto (x_2 + 5x_3, x_3) \\ &\mapsto (x_2 + 6x_3, x_3) \mapsto (x_2 + 7x_3, x_3) \mapsto (x_2 + 8x_3, x_3) \mapsto \cdots,\end{aligned}$$

where $x_3 = x_1 + x_2 - 1 = \frac{1}{5}(1 - \frac{11}{4\pi})$. Therefore, by the above corollary, the game starting from $(\frac{2}{5}(3 - \frac{2}{\pi}), \frac{1}{4\pi})$ continues forever.

4. Questions

In this last section, we shall present a number of questions about our game.

It can be inferred from the proof of Theorem 1 that the length of the shortest and longest play starting from a non-terminal position $(x_1, x_2) \in X_n$ are k and $\sum_{i=1}^{k} a_i$, respectively. We at least know $k \leq x_2$, which follows from the inequalities

$x_2 > x_3 > \cdots > x_{k+1} > x_{k+2} = 0$, but are there sharper bounds for them? What can be said about their distribution when (x_1, x_2) ranges over the set of all non-terminal positions of X_n? How do their average values, $\frac{2}{n(n-1)} \sum_{(x_1,x_2)} k$ and $\frac{2}{n(n-1)} \sum_{(x_1,x_2)} \sum_{i=1}^{k} a_i$, grow asymptotically? We note that similar questions can be asked for the length of the *optimal* play, that is, the play in which both players always try to move to P-positions whenever possible.

If we extend the set of positions to $X_{\mathbb{R}}$ then, as we have mentioned earlier, it can happen that the game continues forever. Therefore, it is natural and important to study the set E of positions from which the game eventually ends no matter how it is played. (Note that, because we already know $E \cap \mathbb{Q}^2 = X_{\mathbb{Q}}$, what really concerns us is the set $E \setminus \mathbb{Q}^2$.) Is there a concise characterization of the set $E \setminus \mathbb{Q}^2$? Is the set $E \setminus \mathbb{Q}^2$ measurable and of positive measure? Is the set $E \setminus \mathbb{Q}^2$ dense in $X_{\mathbb{R}}$?

As in the proof of Theorem 3, we can associate with each non-terminal position $(x_1, x_2) \in X_{\mathbb{R}}$ a (finite or infinite) sequence $a_1, a_2, a_3, \ldots$ of positive integers, which is infinite if $(x_1, x_2) \notin E$ and, moreover, is weakly increasing. Because every $(x_1, x_2) \in X_{\mathbb{R}} \setminus E$ is non-terminal, it follows that to each $(x_1, x_2) \in X_{\mathbb{R}} \setminus E$, there corresponds a weakly increasing infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers. Is this correspondence one-to-one? Is every weakly increasing sequence of positive integers realizable as the associated infinite sequence for some $(x_1, x_2) \in X_{\mathbb{R}} \setminus E$? Which $(x_1, x_2) \in X_{\mathbb{R}} \setminus E$ has a strictly increasing associated infinite sequence? Is there a characterization of those positions $(x_1, x_2) \in X_{\mathbb{R}} \setminus E$ whose associated infinite sequence consist of even (resp. odd) integers only? For which $(x_1, x_2) \in X_{\mathbb{R}} \setminus E$, is the associated infinite sequence identical to the Fibonacci sequence (or some other known integer sequences)?

Acknowledgements. The author is grateful to the reviewer for recommending various improvements in exposition. He also gratefully acknowledges the managing editor's assistance in publishing this article.

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