

THE GAME OF FLATTEN

Takanori Hida
Aichi, Japan
takanori.hida@protonmail.com

Received: 8/16/25, Revised: 7/4/26, Accepted: 10/23/26, Published: 11/23/26

Abstract

In this article, we introduce and study the following combinatorial game: given an integer partition, two players alternately flatten it over and over, until it cannot be flattened any more. Utilizing the Pierce expansion of real numbers, we first provide a characterization of P-positions of this game. Making use of it, we then calculate the limits (as $n \to \infty$) of the probability that a partition of n into two parts (resp. an element of $\{(p, q) \mid n \geq p \geq q \geq 1\}$), chosen at random, is a P-position. We also take up the question of how long the game lasts if both players make moves optimally (in other words, always try to move to P-positions whenever possible) and furnish an upper bound.

1. Introduction

A *partition* of a positive integer n is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\sum_{i=1}^{k} \lambda_i = n$. A wealth of questions already present themselves: How many partitions of n are there? How many partitions of n exist whose *parts* (i.e., λ_i) are all odd, prime, at most a certain number, etc? This type of question received special attention and has been studied for a long time. The theory developed has found applications not only in combinatorics and number theory but also in other areas of research, including statistical mechanics and theoretical computer science. For more details, we refer the reader to [1].

Combinatorial game theory, on the other hand, is a relatively young branch of mathematics and theoretical computer science. Among various types of games studied in this field, our concern is on two-player games of the ensuing type: The game is played on the set of *positions* following a certain set of *rules*, that specifies for each position which moves to other positions are valid or not. The rules do not make any distinction between players. Once a position is chosen and provided as *initial position*, two players, Player I and Player II, start making moves alternately.

DOI: 10.5281/zenodo.24474225

(Player I makes the first move.) The first player unable to make a move loses, and, of course, the other player wins. For a concise account of the theory, see Part I of [5].

The game of *Euclid* [2] is a famous example of a combinatorial game. It is played on pairs of non-negative integers. A valid move consists of replacing the larger of two integers with any smaller non-negative number obtained by subtracting a positive multiple of the smaller number from the larger number. Since its introduction, Euclid and its variants have been studied by many authors [3, 4, 6, 7, 11]; the intimate relationship between Euclid and the continued fraction representation is now well-known [8].

Given the aforementioned intimate relationship between Euclid and the continued fraction representation, it is natural to ask if there is another game that is related to a number representation system. We studied this question and then found the following game, which we call *Flatten*: Given an integer partition, two players alternately flatten it over and over, until it cannot be flattened any more. More precisely, they alternately increase the number of occurrences of the smallest part by decreasing some larger parts; they can introduce a part that is smaller than the (currently) smallest part only when other parts are all equal to that smallest part. As we will see, this game is intimately related to the number representation system introduced by Pierce [9].

The organization of this article is as follows. After presenting definitions and some of their properties in the next preliminary section, we first characterize P-positions of this game utilizing the Pierce expansion of real numbers. Making use of the characterization, we then calculate the limits (as $n \to \infty$) of the probability that a partition of n into two parts (resp. an element of $\{(p, q) \mid n \geq p \geq q \geq 1\}$), chosen at random, is a P-position. We also take up the question of how long the game lasts if both players make moves optimally (in other words, always try to move to P-positions whenever possible) and furnish an upper bound. We close this article with a number of questions.

2. Preliminaries

We first introduce some notation and terminology on integer partitions. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer, we write $s(\lambda)$ for the smallest part λ_k of it. Also, we let $r(\lambda)$ and $|\lambda|$ denote the sums $\sum_{i=1}^{k}(\lambda_i - s(\lambda))$ and $\sum_{i=1}^{k}\lambda_i$, respectively. When we write a part that has an exponent, say p, that should be read not as the pth power of that part but as an abbreviation for p times occurrences of that part. So, for example, $(5^3, 1)$,$(5^2, 5, 1)$, $(5^1, 5^1, 5^1, 1^1)$, and $(5, 5, 5, 1)$ all represent the same partition. The *multiplicity* of an integer i in the partition λ is the total number of occurrences of i in λ as a part and is denoted by $m_\lambda(i)$. Hence, if $\lambda_{j_1} > \lambda_{j_2} > \cdots > \lambda_{j_\ell}$ are the distinct parts of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, then the partition $(\lambda_{j_1}^{m_\lambda(\lambda_{j_1})}, \lambda_{j_2}^{m_\lambda(\lambda_{j_2})}, \ldots, \lambda_{j_\ell}^{m_\lambda(\lambda_{j_\ell})})$ is the same as λ.

As we wrote in the abstract and the introduction, the present article is about

the game in which, given an integer partition, two players alternately flatten it over and over, until it cannot be flattened any more. Since this description is not precise enough to build solid arguments based on it, using the notation introduced above, we define the game formally as follows.

Definition 1. Positions of the game are partitions of positive integers. A move from $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ to $(\lambda'_1, \lambda'_2, \ldots, \lambda'_{k'})$ is valid if they are partitions of the same integer, k' is strictly larger than k, and the following three conditions are satisfied:

- $\lambda'_i \leq \lambda_i$ for all $i \in \{1, 2, \ldots, k\}$.
- $\lambda'_i = s(\lambda)$ for all $i \in \{k, k+1, \ldots, k'-1\}$.
- $\lambda'_{k'} < s(\lambda)$ only when $\lambda'_i = s(\lambda)$ for all $i \in \{1, 2, \ldots, k'-1\}$.

Let us call this game *Flatten*. Observe that there is no valid move from λ if and only if the parts of λ are all identical. In the following, such a position is referred to as a *terminal* position.

Since the above definition of Flatten is somewhat technical, let us present one example in order to help the reader understand it.

Example 1. The move from $(8, 7, 6)$ to $(6^3, 3)$ is valid; moves to other positions, e.g., $(6^3, 2, 1)$ and $(8, 6^2, 1)$, are all invalid. Moves from $(6^3, 3)$ to (3^7), $(5^3, 3^2)$, $(6, 5, 4, 3^2)$, $(6^2, 3^3)$, $(4^3, 3^3)$, $(5, 4, 3^3)$, and $(6, 3^5)$ are all valid; moves to other positions, e.g., $(3^5, 2^3)$ and $(6^2, 3^2, 2, 1)$, are all invalid. From the position $(5, 4, 3^4)$, there is only one valid move, which is to (3^7). Since there is no valid move from (3^7), we conclude that Player I wins if the game goes as follows:

$$(8, 7, 6) \overset{\text{I}}{\longmapsto} (6^3, 3) \overset{\text{II}}{\longmapsto} (5, 4, 3^4) \overset{\text{I}}{\longmapsto} (3^7).$$

Here and in what follows, a move from one position to another position is represented by an arrow with a label specifying the player who made that move. Also, when we simply write "a move", it means a valid move. A calculation shows that if the initial position is $(8, 7, 6)$, then there are fourteen scenarios, including the above one. They are illustrated in Figure 1 in the appendix so that the reader can readily see the idea of flattening an integer partition.

In the first move of the above game, Player I moves from $(8, 7, 6)$ to $(6^3, 3)$ because that is the only valid move from $(8, 7, 6)$. Hereafter, we call such a move a *forced move*.

Let us note that, because there are only finitely many partitions of a fixed positive integer and because valid moves are only to longer partitions of the same positive integer, the game of Flatten satisfies the following two conditions:

(*Finiteness condition*) From each position, only finitely many positions can be reached.

(*Ending condition*) From any initial position, no matter how the game is played, a terminal position is eventually reached. (In particular, no loop happens.)

We next introduce two concepts that play key roles in our study of Flatten. The first one, P-position and N-position, is from combinatorial game theory and is defined recursively as follows:

- A position is a *P-position* if it is either a terminal position or a non-terminal position from which only N-positions can be reached.
- A position is an *N-position* if at least one P-position can be reached from it.

Because this definition is a recursive one, in general, there can be a position for which we cannot decide whether it is a P-position or an N-position. However, if a game satisfies both the finiteness and ending conditions, then it can be proved by induction that every position is either a P-position or an N-position.

Remark 1. The names "P-positions" and "N-positions" comes from the following fact, which can be seen from Proposition 2 from the next section: P-positions are those positions that are winning for the *P*revious player (i.e., the player who just moved), whereas N-positions are those positions that are winning for the *N*ext player to move.

The other key concept is the following apparatus for representing real numbers [9]. For each rational number $\alpha \in (0,1] \cap \mathbb{Q}$, one can find a strictly increasing finite sequence $a_1, a_2, \ldots, a_\ell$ ($\ell \geq 1$) of positive integers satisfying the equation

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots + \frac{(-1)^{\ell+1}}{a_1 a_2 \cdots a_\ell}.$$

There is, in general, more than one such integer sequence. However, if we require that the sequence additionally satisfy the condition that either $\ell \geq 2$ and $a_{\ell-1} < a_\ell - 1$ or $\ell = 1$, then uniqueness follows. Let us call the alternating series $\sum_{i=1}^{\ell} \frac{(-1)^{i+1}}{a_1 a_2 \cdots a_i}$ corresponding to that unique sequence the *Pierce expansion of* α, and denote it by $\langle a_1, a_2, \ldots, a_\ell \rangle$. The integer ℓ will be referred to as the *length* of the expansion. For each irrational number $\alpha \in (0,1] \setminus \mathbb{Q}$, one can find precisely one strictly increasing infinite sequence $a_1, a_2, \ldots$ of positive integers satisfying the equation $\alpha = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{a_1 a_2 \cdots a_i}$. Let us call the alternating series $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{a_1 a_2 \cdots a_i}$ the *Pierce expansion of* α, and denote it by $\langle a_1, a_2, \ldots \rangle$.

The ensuing fact is crucial to the main theorem. Hence, even though it is probably well-known, we shall include its proof here.

Proposition 1. *For any positive integer $k \geq 1$ and rational number $\alpha \in (0,1] \cap \mathbb{Q}$, the following two statements are equivalent:*

(i) $\frac{1}{k} > \alpha > \langle k, k+1, k+2, \ldots \rangle$.

(ii) *There exists an* $i \in \{1, 2, \ldots, \ell\}$ *satisfying* $a_i > k + i - 1$*, and the smallest such* i *is even, where* ℓ *and* a_i *are defined by* $\alpha = \langle a_1, a_2, \ldots, a_\ell \rangle$.

Proof. It will be sufficient to prove that for any $\ell \geq 1$, the ensuing claim on ℓ is true: for any positive integer $k \geq 1$ and a rational number $\alpha \in (0, 1] \cap \mathbb{Q}$ having length ℓ Pierce expansion, Statements (i) and (ii) are equivalent. Let us do so by induction on ℓ.

Note first that if $\alpha \in (0, 1] \cap \mathbb{Q}$ has length one Pierce expansion then, no matter which k is chosen, neither of the two statements is correct. Thus, the claim is true if $\ell = 1$. Next, observe that if $\frac{1}{k} > \alpha = \langle a_1, a_2 \rangle > \langle k, k+1, k+2, \ldots \rangle$, then a_1 should be k, which combines with the requirement $a_2 > a_1 + 1$ of the Pierce expansion to imply that the inequality $a_i > k + i - 1$ holds not for $i = 1$ but for $i = 2$. Conversely, if $a_1 = k$ and $a_2 > k + 1$, then a calculation shows that $\alpha = \langle a_1, a_2 \rangle$ is strictly between $\frac{1}{k}$ and $\langle k, k+1, k+2, \ldots \rangle$. Therefore, the claim is true if $\ell = 2$. For the induction step, assume that we have verified that the claim is true if ℓ is at most an $\ell_0 \geq 2$, and arbitrarily take a positive integer $k \geq 1$ and a rational number $\alpha \in (0, 1] \cap \mathbb{Q}$ having length $\ell_0 + 1$ Pierce expansion, say $\langle a_1, a_2, \ldots, a_{\ell_0}, a_{\ell_0+1} \rangle$. Then, a calculation shows that Statement (i) for this k and α is correct if and only if either of the following is true: (a) $a_1 = k$ and $a_2 > k + 1$, or (b) $a_1 = k, a_2 = k + 1$, and $\frac{1}{k+2} > \langle a_3, a_4, \ldots, a_{\ell_0}, a_{\ell_0+1} \rangle > \langle k + 2, k + 3, k + 4, \ldots \rangle$. Also, an application of the induction hypothesis to $k + 2$ and $\langle a_3, a_4, \ldots, a_{\ell_0}, a_{\ell_0+1} \rangle$ shows that (b) is equivalent to saying that there exists an $i \in \{3, 4, \ldots, \ell_0 + 1\}$ satisfying $a_i > k + i - 1$, and the smallest such i is even. These two facts combine to imply that Statements (i) and (ii) for this arbitrarily chosen k and α are equivalent, completing the proof of the induction step. □

3. Results

In this section, we present our results on Flatten. To do so, let us first recall the following well-known fact from combinatorial game theory (see, e.g., [5]).

Proposition 2. *In any game satisfying both the finiteness and ending conditions, the strategy of always moving to P-positions is a winning strategy.*

Since Flatten satisfies both the finiteness and ending conditions (as has been mentioned in the preceding section), we can apply the above proposition to conclude that the strategy of always moving to P-positions is a winning strategy in Flatten. It is then natural to seek a concise characterization of P-positions. To obtain such a characterization, we will need two lemmas.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer, define $\varphi(\lambda)$ to be the

following partition of the same integer:

$$\Big(s(\lambda)^{\left\lceil \frac{|\lambda|}{s(\lambda)} \right\rceil - 1}, |\lambda| - \Big(\left\lceil \frac{|\lambda|}{s(\lambda)} \right\rceil - 1\Big) s(\lambda)\Big) = \Big(\underbrace{s(\lambda), s(\lambda), \ldots, s(\lambda)}_{\left\lceil \frac{|\lambda|}{s(\lambda)} \right\rceil - 1}, |\lambda| - \Big(\left\lceil \frac{|\lambda|}{s(\lambda)} \right\rceil - 1\Big) s(\lambda)\Big).$$

(Note that when $s(\lambda)$ divides $|\lambda|$, we have $\varphi(\lambda) = (s(\lambda)^{\left\lceil \frac{|\lambda|}{s(\lambda)} \right\rceil})$.) Then, the next property can be proved.

Lemma 1. *For any partition λ of a positive integer, the following three attributes of $r(\lambda)$ hold:*

(i) *$r(\lambda) = 0$ if and only if there is no valid move from λ.*

(ii) *$r(\lambda) \in \{1, 2, \ldots, s(\lambda)\}$ if and only if there is precisely one valid move from λ, which is to $\varphi(\lambda)$.*

(iii) *$r(\lambda) > s(\lambda)$ if and only if there is more than one valid move from λ, including the one to $\varphi(\lambda)$.*

It follows in particular that from any λ satisfying $r(\lambda) > 0$, the position $\varphi(\lambda)$ can be reached. Henceforth, this fact will be freely used without explicit mention.

The second lemma is crucial in obtaining a characterization.

Lemma 2. *If there is more than one valid move from λ, then one of them is to a P-position.*

Proof. Suppose that there is more than one valid move from $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Then, $r(\lambda) > s(\lambda)$ by Lemma 1. Take an arbitrary weakly decreasing sequence $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_k \geq s(\lambda)$ of positive integers so that both $\lambda'_i \leq \lambda_i$ for all $i \in \{1, 2, \ldots, k\}$ and $\sum_{i=1}^{k} \lambda'_i = \sum_{i=1}^{k} \lambda_i - \big(\left\lceil \frac{r(\lambda)}{s(\lambda)} \right\rceil - 1\big) s(\lambda)$ hold, and define a new partition λ' of $|\lambda|$ by the equation $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_k, s(\lambda)^{\left\lceil \frac{r(\lambda)}{s(\lambda)} \right\rceil - 1})$. It can then be verified that $\varphi(\lambda) = \varphi(\lambda')$.

Clearly, $r(\lambda') = \sum_{i=1}^{k} (\lambda'_i - s(\lambda))$ holds. By combining this equation with $\sum_{i=1}^{k} \lambda'_i = \sum_{i=1}^{k} \lambda_i - \big(\left\lceil \frac{r(\lambda)}{s(\lambda)} \right\rceil - 1\big) s(\lambda)$ and $r(\lambda) = \sum_{i=1}^{k} (\lambda_i - s(\lambda))$, we get $r(\lambda') = r(\lambda) - \big(\left\lceil \frac{r(\lambda)}{s(\lambda)} \right\rceil - 1\big) s(\lambda)$. As $r(\lambda) > s(\lambda)$, we conclude that $r(\lambda') \in \{1, 2, \ldots, s(\lambda)\}$. We also have $s(\lambda) = s(\lambda')$. Applying Lemma 1 to these, we find that the only move available from λ' is to $\varphi(\lambda')$. Hence, if $\varphi(\lambda')$ is an N-position then, because the position λ' should be a P-position by the recursive definition of P-position, the move from λ to λ' is the desired one. On the other hand, if $\varphi(\lambda')$ is a P-position then, because we have $\varphi(\lambda) = \varphi(\lambda')$, the move from λ to $\varphi(\lambda)$ is the desired one. □

Having finished preparations, we can prove the main theorem.

Theorem 1. *The partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a P-position if and only if the following inequality holds:*

$$\frac{s(\lambda)}{|\lambda|} > (-1)^k (k-1)! \left(e^{-1} - \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \right).$$

Proof. If the initial position is not a terminal position, then there are two scenarios: (a) the game continues until the end with two players making forced moves alternately, or (b) either of the two players eventually has multiple choices of moves. Hence, by Lemma 2, the P-positions consist of the ensuing three disjoint sets:

$$P_1 = \{ \text{terminal positions} \},$$
$$P_2 = \left\{ \begin{array}{c} \text{non-terminal positions from which the game ends in} \\ \text{a non-zero even number of forced moves} \end{array} \right\},$$
$$P_3 = \left\{ \begin{array}{c} \text{non-terminal positions from which, after an odd number of} \\ \text{forced moves, the next player has multiple choices of moves} \end{array} \right\}.$$

We shall first prove that these three sets can be characterized as follows: for any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we have

(i) $\lambda \in P_1$ if and only if $\frac{s(\lambda)}{|\lambda|} = \frac{1}{k}$;

(ii) $\lambda \in P_2$ if and only if ℓ is even and $\langle a_1, a_2, \ldots, a_\ell \rangle = \langle k, k+1, k+2, \ldots, k+\ell-2, k+\ell \rangle$;

(iii) $\lambda \in P_3$ if and only if $a_\ell > k + \ell$ and the smallest $i \in \{1, 2, \ldots, \ell\}$ satisfying $a_i > k + i - 1$ is even.

Here, ℓ and a_i are defined by $\frac{s(\lambda)}{|\lambda|} = \langle a_1, a_2, \ldots, a_\ell \rangle$.

To establish these characterizations, observe first that the inequality $\frac{s(\lambda)}{|\lambda|} \leq \frac{1}{k}$ holds for any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Moreover, equality is attained if and only if λ is terminal. Hence, Statement (i) is correct for any λ. To prove that Statements (ii) and (iii) are also correct for any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, let us introduce one more notation here. For any partition λ, write ℓ_λ for the smallest non-negative integer ℓ such that $r(\varphi^\ell(\lambda)) = 0$. Then, it is evident that $\ell_\lambda = 0$ if and only if λ is a terminal position. Also, if $\ell_\lambda = 1$ then, because a terminal (hence, P-) position $\varphi(\lambda)$ can be reached from λ, it follows from the recursive definition of P-position that λ is not a P-position. Therefore, any λ satisfying $\ell_\lambda < 2$ is outside P_2 and P_3. Note also that for any such λ, the Pierce expansion of $\frac{s(\lambda)}{|\lambda|}$ has length one. From these, it is immediate that the two statements are both correct for any λ satisfying $\ell_\lambda < 2$. To verify that the two statements are correct also for those λ satisfying $\ell_\lambda \geq 2$, take

such a $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ arbitrarily, and consider the following ℓ_λ positions

$$\begin{aligned}
\varphi(\lambda) &= (\lambda_k^{m_1}, \lambda_{k+1}) \\
&\vdots \\
\varphi^i(\lambda) &= (\lambda_{k+i-1}^{m_i}, \lambda_{k+i}) \qquad (1 \le i \le \ell_\lambda - 1) \\
&\vdots \\
\varphi^{\ell_\lambda - 1}(\lambda) &= (\lambda_{k+\ell_\lambda - 2}^{m_{\ell_\lambda - 1}}, \lambda_{k+\ell_\lambda - 1}) \\
\varphi^{\ell_\lambda}(\lambda) &= (\lambda_{k+\ell_\lambda - 1}^{m_{\ell_\lambda}}),
\end{aligned} \tag{1}$$

where, for brevity, we wrote λ_{k+i-1} and m_i $(i = 1, 2, \ldots, \ell_\lambda)$ for $s(\varphi^{i-1}(\lambda))$ and $m_{\varphi^i(\lambda)}(\lambda_{k+i-1})$, respectively. Note first that we have the inequalities $\lambda_k > \lambda_{k+1} > \cdots > \lambda_{k+\ell_\lambda - 1} \ge 1$ and $m_1 < m_2 < \cdots < m_{\ell_\lambda - 1} < m_{\ell_\lambda} - 1$. Observe also that Equations (1) imply $|\lambda| = m_i \lambda_{k+i-1} + \lambda_{k+i}$ (equivalently, $\frac{\lambda_{k+i-1}}{|\lambda|} = \frac{1}{m_i}\left(1 - \frac{\lambda_{k+i}}{|\lambda|}\right)$) for any positive integer $i \le \ell_\lambda - 1$ and $|\lambda| = m_{\ell_\lambda} \lambda_{k+\ell_\lambda - 1}$ (equivalently, $\frac{\lambda_{k+\ell_\lambda - 1}}{|\lambda|} = \frac{1}{m_{\ell_\lambda}}$). Combining these equations, we obtain

$$\begin{aligned}
\frac{s(\lambda)}{|\lambda|} &= \frac{\lambda_k}{|\lambda|} \\
&= \frac{1}{m_1}\left(1 - \frac{1}{m_2}\left(1 - \cdots - \frac{1}{m_{\ell_\lambda - 1}}\left(1 - \frac{1}{m_{\ell_\lambda}}\right)\right)\right) \\
&= \langle m_1, m_2, \ldots, m_{\ell_\lambda - 1}, m_{\ell_\lambda} \rangle.
\end{aligned}$$

Since it can readily be seen from Equations (1) and Lemma 1 that

- the move from λ to $\varphi(\lambda)$ is a forced move if and only if $m_1 = k$,
- if $\ell_\lambda > 2$ then for any positive integer $i \le \ell_\lambda - 2$, the move from $\varphi^i(\lambda)$ to $\varphi^{i+1}(\lambda)$ is a forced move if and only if $m_{i+1} = m_i + 1$,

and

- the move from $\varphi^{\ell_\lambda - 1}(\lambda)$ to $\varphi^{\ell_\lambda}(\lambda)$ is a forced move if and only if $m_{\ell_\lambda} = m_{\ell_\lambda - 1} + 2$,

it follows from these and the uniqueness of the Pierce expansion that Statements (ii) and (iii) are correct for any λ satisfying $\ell_\lambda \ge 2$.

The characterizations (i), (ii), and (iii) being established, we can now prove the assertion as follows. Using those characterizations, it can be shown that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a P-position (equivalently, either $\lambda \in P_1$ or $\lambda \in P_2 \cup P_3$) if and only if either of the following is true: (a) $\frac{s(\lambda)}{|\lambda|} = \frac{1}{k}$, or (b) there exists an $i \in \{1, 2 \ldots, \ell\}$ satisfying $a_i > k + i - 1$, and the smallest such i is even, where ℓ and a_i are defined by $\frac{s(\lambda)}{|\lambda|} = \langle a_1, a_2, \ldots, a_\ell \rangle$. Because (b) is equivalent to the inequalities

$\frac{1}{k} > \frac{s(\lambda)}{|\lambda|} > \langle k, k+1, k+2, \dots \rangle$ by Proposition 1 and because no $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ satisfies the inequality $\frac{s(\lambda)}{|\lambda|} > \frac{1}{k}$, it follows that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a P-position if and only if $\frac{s(\lambda)}{|\lambda|} > \langle k, k+1, k+2, \dots \rangle$. The proof will thus be complete if we substantiate the identity $\langle k, k+1, k+2, \dots \rangle = (-1)^k (k-1)! \big(e^{-1} - \sum_{j=0}^{k-1} \frac{(-1)^j}{j!}\big)$ for $k \geq 1$. To do so, take a positive integer $k \geq 1$ arbitrarily. Then, because

$$\frac{\langle k, k+1, k+2, \dots \rangle}{(k-1)!} = \frac{1}{k!} - \frac{1}{(k+1)!} + \frac{1}{(k+2)!} - \frac{1}{(k+3)!} + \cdots,$$

we have

$$\frac{\langle k, k+1, k+2, \dots \rangle}{(k-1)!} + \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{(k-1)!} = e^{-1}$$

when k is even and

$$-\frac{\langle k, k+1, k+2, \dots \rangle}{(k-1)!} + \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{(k-1)!} = e^{-1}$$

when k is odd. From these, it is immediate that every $k \geq 1$ satisfies the equation $\langle k, k+1, k+2, \dots \rangle = (-1)^k (k-1)! \big(e^{-1} - \sum_{j=0}^{k-1} \frac{(-1)^j}{j!}\big)$, as desired. □

Corollary 1. *If the initial position is λ, then the game terminates at the position $\big(\big(\frac{|\lambda|}{m}\big)^m\big)$ however it is played, where m is the last integer appearing in the Pierce expansion of $\frac{s(\lambda)}{|\lambda|}$.*

Proof. Let us use the notation from the above proof. If $\ell_\lambda < 2$ then, as $s(\lambda)$ divides $|\lambda|$ in this case, we have $\frac{s(\lambda)}{|\lambda|} = \langle \frac{|\lambda|}{s(\lambda)} \rangle$. Since the terminal position is $\varphi^{\ell_\lambda}(\lambda) = (s(\lambda)^{\frac{|\lambda|}{s(\lambda)}}) = (\frac{|\lambda|}{|\lambda|/s(\lambda)}^{\frac{|\lambda|}{s(\lambda)}})$, the assertion is correct in this case. If $\ell_\lambda \geq 2$, then the terminal position is $\varphi^{\ell_\lambda}(\lambda) = (\lambda_{k+\ell_\lambda-1}^{m_{\ell_\lambda}})$, where $\frac{s(\lambda)}{|\lambda|} = \langle m_1, m_2, \dots, m_{\ell_\lambda} \rangle$. Since $m_{\ell_\lambda} \lambda_{k+\ell_\lambda-1} = |\varphi^{\ell_\lambda}(\lambda)| = |\lambda|$, the assertion is correct also in this case. □

Corollary 2. *The probability that a partition of n into two parts, chosen at random, is a P-position converges to $1 - \frac{2}{e}$ as $n \to \infty$.*

Proof. An application of the above theorem shows that for any $i \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$, the position $(n-i, i)$ is a P-position if and only if the inequality $i > \frac{n}{e}$ holds. Therefore, the number of P-positions among $\lfloor \frac{n}{2} \rfloor$ partitions of n into two parts is $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{e} \rfloor$. For $n \geq 3$, the probability $\big(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{e} \rfloor\big) / \lfloor \frac{n}{2} \rfloor$ is bounded above and below by $1 - \frac{2(n-e)}{en}$ and $1 - \frac{2n}{e(n-2)}$, respectively, from which the assertion follows. □

Corollary 3. *The probability that an element of $\{(p, q) \mid n \geq p \geq q \geq 1\}$, chosen at random, is a P-position converges to $1 - \frac{1}{e-1}$ as $n \to \infty$.*

Proof. By Theorem 1, a point $(p,q) \in \{(p,q) \mid n \geq p \geq q \geq 1\}$, seen as a position, is a P-position if and only if $\frac{p}{q} < e-1$. Hence, there are $\frac{n(n+1)}{2} - \sum_{j=1}^{\lfloor \frac{n}{e-1} \rfloor} \big(n - \lfloor j(e-1) \rfloor\big)$ such positions. From the inequality

$$\begin{aligned}\sum_{j=1}^{\lfloor \frac{n}{e-1} \rfloor} \big(n - j(e-1)\big) &= n \left\lfloor \frac{n}{e-1} \right\rfloor - \frac{e-1}{2} \cdot \left\lfloor \frac{n}{e-1} \right\rfloor \left(\left\lfloor \frac{n}{e-1} \right\rfloor + 1 \right) \\ &\geq n \left(\frac{n}{e-1} - 1 \right) - \frac{e-1}{2} \cdot \frac{n}{e-1} \left(\frac{n}{e-1} + 1 \right) \\ &= \frac{n^2}{2(e-1)} - \frac{3n}{2},\end{aligned}$$

it follows that the probability $\big(\frac{n(n+1)}{2} - \sum_{j=1}^{\lfloor \frac{n}{e-1} \rfloor} \big(n - \lfloor j(e-1) \rfloor\big)\big) / \frac{n(n+1)}{2}$ is bounded above by $1 - \frac{1}{n+1}\big(\frac{n}{e-1} - 3\big)$. Since a similar argument shows that the probability is bounded below by $1 - \frac{1}{n+1}\big(\frac{n}{e-1} + \frac{2}{e-1} + 1\big)$, we see that the probability converges to $1 - \frac{1}{e-1}$ as $n \to \infty$. □

We then take up the question of how long the game lasts if both players make moves *optimally*, in other words, always try to move to P-positions whenever possible.

Theorem 2. (i) *For any partition λ of a positive integer, the game with initial position λ, when played optimally, ends in at most $2s(\lambda) - 1$ moves.*

(ii) *For each $s > 0$, there exist infinitely many $s' > s$ such that the game with initial position (s', s), when played optimally, ends in $2s - 1$ moves.*

Proof. For Part (i), take an arbitrary $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, and define ℓ_λ as in the proof of Theorem 1. We note that the game ends at the position $\varphi^{\ell_\lambda}(\lambda)$.

We first claim that when $\ell_\lambda \geq 1$, if both players make moves optimally, then it takes precisely one move to reach $\varphi^{\ell_\lambda}(\lambda)$ from $\varphi^{\ell_\lambda - 1}(\lambda)$. This is clearly true because, $\varphi^{\ell_\lambda}(\lambda)$ being a terminal (hence, P-) position, the optimal move from $\varphi^{\ell_\lambda - 1}(\lambda)$ should be to $\varphi^{\ell_\lambda}(\lambda)$.

We next claim that when $\ell_\lambda \geq 2$, if both players make moves optimally, then for each non-negative integer $i \leq \ell_\lambda - 2$, it takes at most two moves to reach $\varphi^{i+1}(\lambda)$ from $\varphi^i(\lambda)$. As the proofs are analogous to each other, we shall prove only the $i = 0$ case. If there is precisely one valid move from λ or if $\varphi(\lambda)$ is a P-position, then the optimal move from λ should be directly to $\varphi(\lambda)$. If on the other hand there is more than one valid move from λ and $\varphi(\lambda)$ is an N-position, define a new partition λ' of $|\lambda|$ as in the proof of Lemma 2. Then, it can be inferred from the proof of the lemma that the move from λ to λ' is not only a valid move but also an optimal move. It can also be seen from that proof that the move from λ' to $\varphi(\lambda)$ is not only a valid move but also a forced move. Thus, in either case, if both players make moves optimally, then it takes at most two moves to reach $\varphi(\lambda)$ from λ, as claimed.

Having established two claims, we can now prove the stated upper bound as follows. If $\ell_\lambda = 0$, then the game ends in zero ($< 2s(\lambda) - 1$) moves. If $\ell_\lambda = 1$, then the game ends in one ($\leq 2s(\lambda) - 1$) move by the first claim. If $\ell_\lambda \geq 2$, then we have $s(\lambda) = \lambda_k > \lambda_{k+1} > \lambda_{k+2} > \cdots > \lambda_{k+\ell_\lambda - 1} \geq 1$, where the λ_{k+i-1} ($i = 1, 2, \ldots, \ell_\lambda$) are as in the proof of Theorem 1. This implies the inequality $s(\lambda) \geq \ell_\lambda$, which combines with the claims to show that the game with initial position λ, when played optimally, ends in at most $2(\ell_\lambda - 1) + 1 = 2\ell_\lambda - 1 \leq 2s(\lambda) - 1$ moves, completing the proof.

For Part (ii), take an arbitrary integer $j > 3$. Then, it is readily verified that $(j, 1) \overset{\mathrm{I}}{\mapsto} (1^{j+1})$ and $(2j+1, 2) \overset{\mathrm{I}}{\mapsto} (3, 2^j) \overset{\mathrm{II}}{\mapsto} (2^{j+1}, 1) \overset{\mathrm{I}}{\mapsto} (1^{2j+3})$ are both optimal play. It is also readily verified by induction that for any $s \geq 3$, the positions from the right (resp. left) column of the following play are all P-positions (resp. N-positions):

$$\begin{array}{rcl}
 & (j \cdot s! - 1 - s, s) & \overset{\mathrm{I}}{\longmapsto} (2s-1, s^{\frac{j\cdot s!}{s}-2}) \\
\overset{\mathrm{II}}{\longmapsto} & (s^{\frac{j\cdot s!}{s}-1}, s-1) & \overset{\mathrm{I}}{\longmapsto} (s^{s-2}, s-1^{\frac{j\cdot s!}{s-1}-(s-1)}) \\
\overset{\mathrm{II}}{\longmapsto} & (s-1^{\frac{j\cdot s!}{s-1}-1}, s-2) & \overset{\mathrm{I}}{\longmapsto} (s-1^{s-3}, s-2^{\frac{j\cdot s!}{s-2}-(s-2)}) \\
 & \vdots & \\
\overset{\mathrm{II}}{\longmapsto} & (s-i^{\frac{j\cdot s!}{s-i}-1}, s-i-1) & \overset{\mathrm{I}}{\longmapsto} (s-i^{s-i-2}, s-i-1^{\frac{j\cdot s!}{s-i-1}-(s-i-1)}) \\
 & \vdots & \\
\overset{\mathrm{II}}{\longmapsto} & (3^{\frac{j\cdot s!}{3}-1}, 2) & \overset{\mathrm{I}}{\longmapsto} (3^1, 2^{\frac{j\cdot s!}{2}-2}) \\
\overset{\mathrm{II}}{\longmapsto} & (2^{\frac{j\cdot s!}{2}-1}, 1) & \overset{\mathrm{I}}{\longmapsto} (1^{j\cdot s!-1}).
\end{array}$$

Moreover, all moves by Player II are forced. Hence, this length $2s - 1$ play is an optimal one. Since $j > 3$ was chosen arbitrarily, the assertion follows. □

4. Questions

Since the game of Flatten and the results presented are related to several areas, a variety of questions arise naturally. Let us present only some of them.

The first set of questions is about P-positions of Flatten. How many elements of $\{\lambda \mid \lambda \text{ is a partition of } n\}$ are P-positions? What are the answers if we replace the set $\{\lambda \mid \lambda \text{ is a partition of } n\}$ with $P(n, k) \coloneqq \{\lambda \mid \lambda \text{ is a partition of } n \text{ into at most } k \text{ parts}\}$ or $P(n, k) \setminus P(n, k-1)$? Does the probability that a randomly chosen partition of n is a P-position converge as $n \to \infty$? Also, for each $k \geq 3$, does the probability that an element of $P(n, k)$ (resp. $P(n, k) \setminus P(n, k-1)$), chosen at random, is a P-position converge as $n \to \infty$? If the limits exist for all (but finitely many) $k \geq 3$, do those limits converge as $k \to \infty$?

It is well-known [10] that for each positive integer n, partitions of n are in one-to-one correspondence with irreducible complex representations of the symmetric group of degree n. Hence, Flatten can equivalently be viewed as a game played on irreducible complex representations of the symmetric group. Let us note that,

though there are lots of take away type games that can be viewed as games played on integer partitions, or equivalently, irreducible complex representations of the symmetric group, to view Flatten as a game on such representations is more natural because, unlike take away type games, n does not change during each play of Flatten. Does the representation theoretic viewpoint lead us to find applications of Flatten in other areas of mathematics than combinatorial game theory? What kind of algebraic properties are possessed by those irreducible complex representations (or their characters) whose corresponding partitions are P-positions? Can we even characterize P-positions representation theoretically? We at least know that P-positions cannot be characterized only by the dimension of the corresponding representations: since the P-position $(3,2)$ and the N-position $(2,2,1)$ are conjugate, their corresponding representations have the same dimension (explicitly, five by the hook length formula).

It is possible to add restrictions to the rules to make the game more challenging. Consider, for example, the condition that the largest part becomes strictly smaller after moving. This variant is indeed different from Flatten: even though $(7,4)$ is an N-position in Flatten by Theorem 1, Player I cannot always win this variant with initial position $(7,4)$ as exemplified by the play $(7,4) \overset{\text{I}}{\mapsto} (4^2,3) \overset{\text{II}}{\mapsto} (3^3,2) \overset{\text{I}}{\mapsto} (2^5,1) \overset{\text{II}}{\mapsto} (1^{11})$, in which every move is a forced move. Another example of restriction is that after moving from $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, the multiplicity of $s(\lambda)$ becomes strictly larger than the cardinality $|\{i \leq k \mid \lambda_i > s(\lambda)\}|$. This variant is also distinct from Flatten: although $(16^3, 10)$ is an N-position in Flatten by Theorem 1, Player I cannot always win this variant with initial position $(16^3, 10)$ as witnessed by the play $(16^3, 10) \overset{\text{I}}{\mapsto} (10^5, 8) \overset{\text{II}}{\mapsto} (10, 8^6) \overset{\text{I}}{\mapsto} (8^7, 2) \overset{\text{II}}{\mapsto} (2^{29})$, in which every move of Player I is a forced move. If we change positions of the game to partitions of positive integers into at most two distinct parts, we also get a different game: though $(11^5, 8)$ is an N-position in Flatten by Theorem 1, Player I cannot always win this variant with initial position $(11^5, 8)$ as evidenced by the play $(11^5, 8) \overset{\text{I}}{\mapsto} (8^7, 7) \overset{\text{II}}{\mapsto} (7^9)$, in which the move of Player I is a forced move. Having verified that these three variants are indeed different from Flatten, it now makes sense to ask the following questions: What are the P-positions for these variants? What is the probability that a position, chosen at random, is a P-position in these variants? In how many moves do these variants, when played optimally, end?

Acknowledgements. The author is grateful to the reviewer for a number of useful suggestions for improvement in the article.

References

[1] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, Addison-Wesley Publishing Co., Reading, Mass., 1976.

[2] A. J. Cole and A. J. T. Davie, A game based on the Euclidean algorithm and a winning strategy

for it, *Math. Gaz.* **53**(386) (1969), 354–357.

[3] D. Collins, Variations on a theme of Euclid, *Integers* **5**(1) (2005), #G3, 12 pp.

[4] D. Collins and T. Lengyel, The game of 3-Euclid, *Discrete Math.* **308**(7) (2008), 1130–1136.

[5] T. S. Ferguson, *A Course in Game Theory*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020.

[6] N. B. Ho, Variations of the game 3-Euclid, *Int. J. Comb.* (2012), Article ID 406250, 11pp.

[7] M. A. Jones and B. C. Ohlinger, Self-similar structure of $\mathcal{P}$-positions of the game Euclid, *Fibonacci Quart.* **62**(1) (2024), 15–28.

[8] T. Lengyel, A nim-type game and continued fractions, *Fibonacci Quart.* **41**(4) (2003), 310–320.

[9] T. A. Pierce, On an algorithm and its use in approximating roots of algebraic equations, *Amer. Math. Monthly* **36**(10) (1929), 523–525.

[10] B. E. Sagan, *The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions*. Second edition. Graduate Texts in Mathematics, 203. Springer Verlag, New York, 2001.

[11] E. L. Spitznagel Jr., Properties of a game based on Euclid's algorithm, *Math. Mag.* **46**(2) (1973), 87–92.

A. Appendix

In this appendix, we present illustrations of fourteen games starting from the position $(8, 7, 6)$, as mentioned in Example 1. Note that our notation of Young diagrams is neither the English nor French notation. We adopted this unusual notation in order to help illustrate the idea of flattening an integer partition.

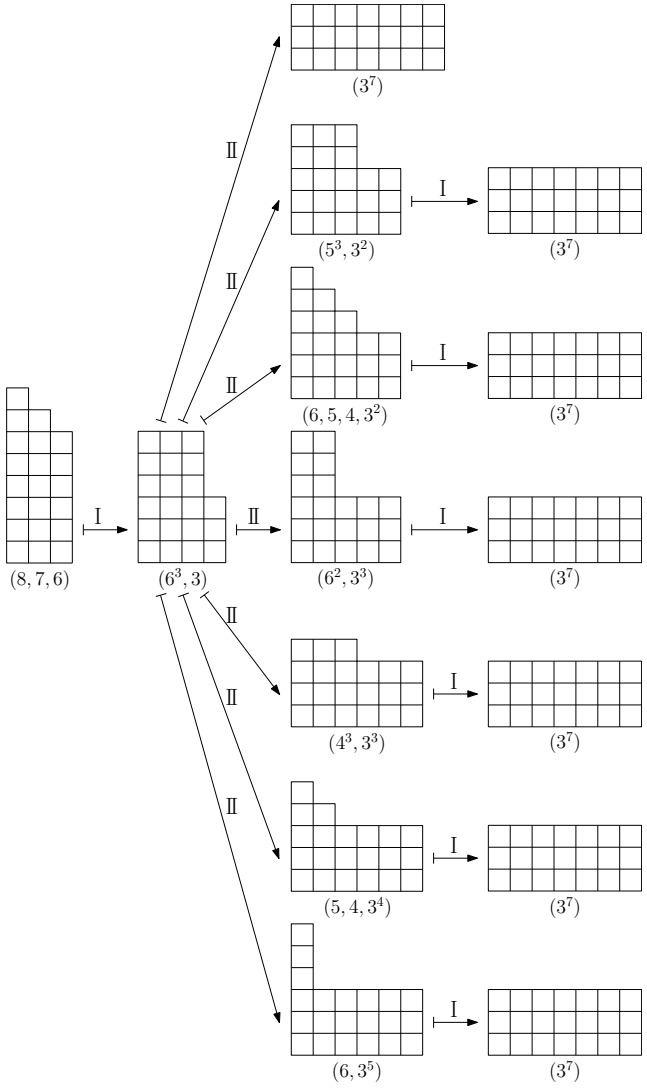

Figure 1: Illustrations of fourteen games starting from the position $(8, 7, 6)$. In the first and the last seven games, Player II wins; in the other six games, Player I wins.

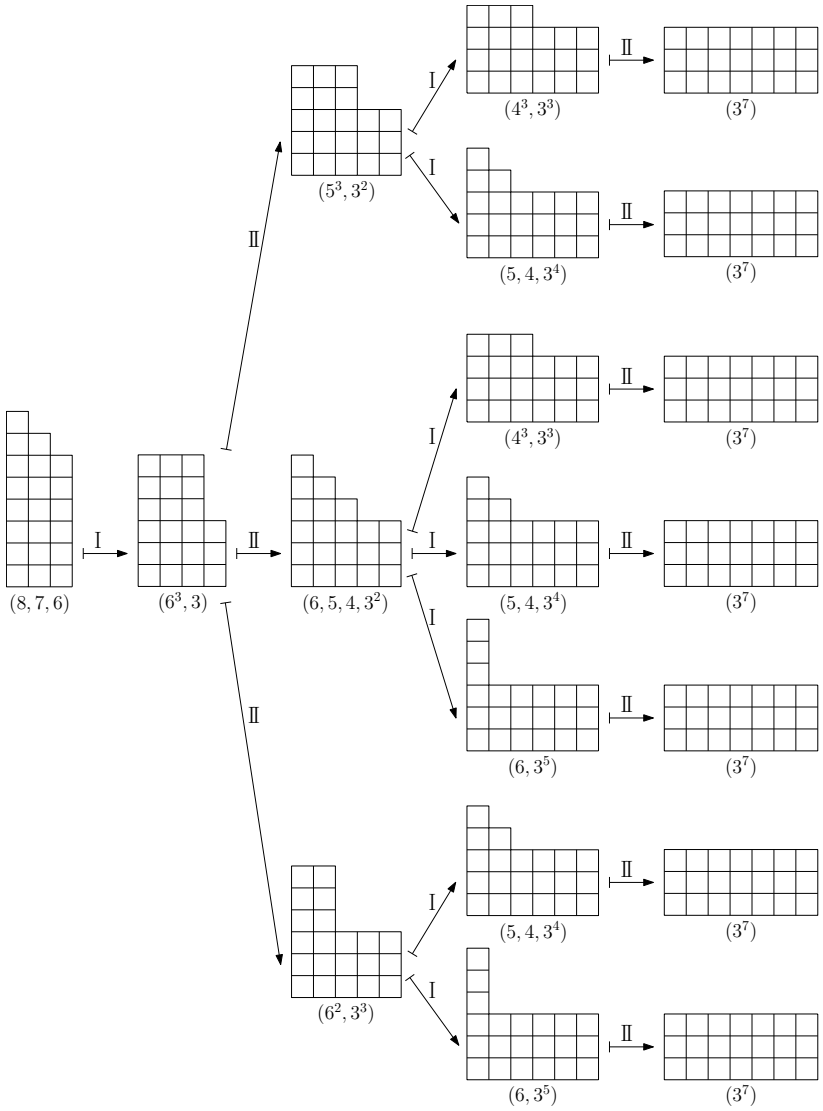

Figure 1 (continued): Illustrations of fourteen games starting from the position $(8,7,6)$. In the first and the last seven games, Player II wins; in the other six games, Player I wins.