



SQUARES, TRIANGLES, AND CONTINUED FRACTIONS

Berit Nilsen Givens

Department of Mathematics and Statistics, Cal Poly Pomona, Pomona, California
 bngivens@cpp.edu

Mits Kobayashi

Department of Mathematics, Dartmouth College, Hanover, New Hampshire
 mits.kobayashi@dartmouth.edu

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Abstract

Let T_n be the n th triangular number and S_m be the m th square number. A *triangular-square number* is a number of the form $T_n = S_m$. In 2024, Kasatani and the second author showed that solutions to $T_n = S_m$ can be obtained by dissecting and rearranging geometric figures. In this paper, we consider the generalization $T_n = dS_m$. We demonstrate an algorithm for recursively generating the solutions to $T_n = dS_m$ using geometric tilings and dissections. Along the way, we use continued fractions and convergents to $\sqrt{2d}$ to guide the algorithm and make connections to Pell's equation $x^2 - 2dy^2 = 1$.

1. Introduction

The first author taught an introductory number theory class early in her career and used the textbook *A Friendly Introduction to Number Theory* by Joseph Silverman [5]. This excellent textbook is suitable for students with very little background in proofs. In her rush at the beginning of the term, she glanced at the exercises in Chapter 1 and thought, “Hmm, those seem doable.” She assigned several exercises. She soon began to receive emails from students who could not solve Exercise 1. The problem asks students to find *triangular-square numbers*, that is, numbers that are both square and triangular. The exercise further asks that students identify patterns and asks whether they believe there are a finite or an infinite quantity of such numbers. In the context of the chapter, the tools available to the students are to draw these numbers as shapes or to use the formulas $S_m = m^2$ and $T_n = \frac{n(n+1)}{2}$, respectively, for the m th square and the n th triangular numbers.

As it turns out, the problem of finding triangular-square numbers is equivalent to solving the Pell's equation $x^2 - 2y^2 = 1$. Indeed, the equation $m^2 = \frac{n(n+1)}{2}$ is

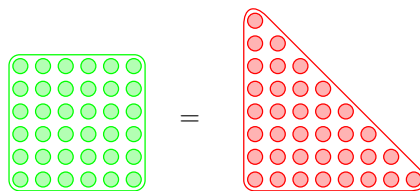


Figure 1: A geometric depiction of the triangular square number $36 = S_6 = T_8$.

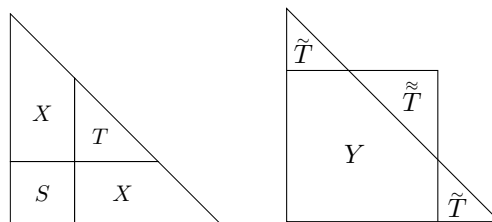


Figure 2: Starting from a triangular square number $S = T$, we generate a new triangular square number $S_+ = \tilde{\tilde{T}} + Y = 2\tilde{T} + Y = T_+$.

equivalent to $(2n+1)^2 - 2(2m)^2 = 1$. The theory of finding integer solutions to Pell's equations of the form $x^2 - dy^2 = 1$, for d a non-square integer, is well-developed. However, it would be satisfying to have a solution that uses only what is known to students by the end of Chapter 1 of [5].

In [3], a geometric algorithm is presented that generates all the triangular-square numbers. In particular, given a triangular-square number, we can arrange this number of dots on a square lattice either in the shape of an isosceles right triangle or a square. The example $S_6 = T_8$ is shown in Figure 1.

We arrange the square S and triangle T inside a larger right isosceles triangle as in the left image of Figure 2. The extension contains triangles of size $\tilde{T} = T + X$ and $\tilde{\tilde{T}} = T + S + 2X$ that are related by $2\tilde{T} = \tilde{\tilde{T}}$. These triangles are in turn arranged as in the right side of Figure 2 to yield a square $S_+ = \tilde{\tilde{T}} + Y$ and a triangle $T_+ = 2\tilde{T} + Y$, so that $S_+ = T_+$. This construction can begin at $S = T = 1$ and be continued indefinitely to generate infinitely many triangular-square numbers, and it is shown that this algorithm will find all of them.

The diagrams can be used to determine the indices of the new square and triangle. Given $S_m = T_n$, from the left side of Figure 2 we see that $\tilde{T} = T_{m+n}$ and $\tilde{\tilde{T}} = T_{2m+n}$. From the right side of Figure 2, we see that $S_+ = S_{3m+2n+1}$ and $T_+ = T_{4m+3n+1}$, where the additional 1 in the indices occurs because there is one more dot on the Y side than the $\tilde{T}$ side, as illustrated in Figure 3.

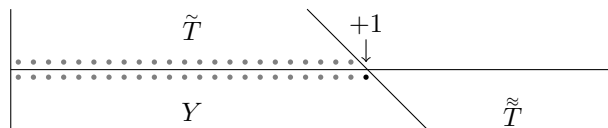


Figure 3: When two triangles meet along the side of a square as shown, an extra dot inside the square is not included in either triangle.

Thus, we have the recurrence relations

$$\begin{aligned} m_+ &= 3m + 2n + 1, \\ n_+ &= 4m + 3n + 1. \end{aligned}$$

Finally, it is shown that starting with $(m, n) = (1, 1)$, repeatedly using the recurrence relations finds each triangular square $S_m = T_n$.

In what follows, we build upon this solution. In Section 2 we use a similar method to find all solutions to $T_n = T_k S_m$. Then, recognizing the ad hoc nature of the results, we switch in Section 3 to an algorithm based on continued fractions for solving $T_n = d S_m$. We see that the geometry allows us to give meaning to the convergents of a continued fraction via results regarding solutions to a Pell's equation $x^2 - 2dy^2 = 1$.

2. The Geometric Generalization to $T_n = T_k S_m$

The method of solving $T = S$ in [3] has a natural generalization of the form $T = T_k S$. That is, we can find numbers that are both triangular and a triangular number of squares using a similar geometric decomposition.

Fix an index k and consider T_k squares S_m arranged as on the left side of Figure 4. When we overlay a triangle T_n over the triangular array of squares, we see two types of smaller triangles extending outside one of the shapes. First, there are $k + 1$ smaller triangles of the form T_a , where $a = n - km$, that are inside the original triangle and outside the squares. Second, there are k triangles of the form T_c , where $c = (k + 1)m - n - 1$, that are inside the squares but outside the original triangle. Thus $(k + 1)T_a = kT_c$. We overlap the triangles as in the right side of Figure 4.

Using an inclusion/exclusion style of counting, we label each region with $+1$ for each T_a triangle in which it lies, and -1 for each T_c triangle containing it. By adding $+1$ and -1 according to the labels, the result is that each region is ultimately labelled with either $+1$, 0 , or -1 . We see that the result is of the form $T_{2a-c} = T_k S_{c-a}$.

To summarize, we start with a formula of the form $T = T_k S$, and after two steps of overlaying and “cancelling out” shapes as described above, we get a new

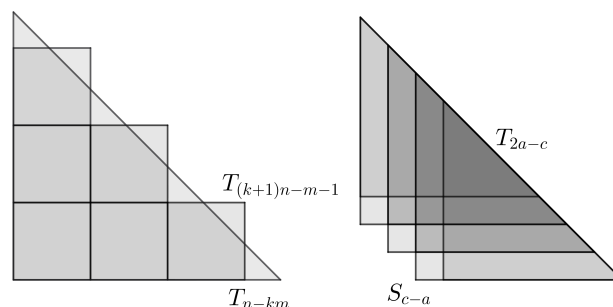
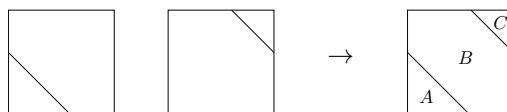

 Figure 4: Diagrams for the generalization to $T = T_k S$ with $k = 3$.


Figure 5: Visualizing the process as a tiling of a square into three parts.

formula of the form $T' = T_k S'$, where the T' and S' are strictly smaller than the original T and S . By carefully tracking the indices at each stage, we get $T' = T_{n'} = T_{(2k+1)n-(2k(k+1))m+k}$ and $S' = S_{m'} = S_{(2k+1)m-2n-1}$. Finally, we can solve for m and n in terms of m' and n' to get

$$\begin{aligned} n &= 2k(k+1)m' + (2k+1)n' + k, \\ m &= (2k+1)m' + 2n' + 1. \end{aligned}$$

We can conclude that these recursive formulas find all solutions using the same argument as in [3]. Essentially, given any solution to $T_n = T_k S_m$, we can invert the recursive formulas to get a previous solution. We would obtain a decreasing sequence of pairs of nonnegative integers, which must terminate in a finite number of steps. It is not hard to show that it terminates only at $(0,0)$. Thus, starting from the trivial solution $n = m = 0$, these recursive formulas give all solutions to $T_n = T_k S_m$.

Another way to visualize the process is to observe that each of our figures consists of tiles that are either squares or parts of squares with diagonal cuts in one of two places. We combine the cuts into a single square to define three subtiles that we label alphabetically from lower left to upper right, as shown in Figure 5.

It will be convenient to use m to represent lengths of squares and a or c to represent lengths of triangles, to help us track what kind of shapes we are considering.

The formula $T = T_k S$ results in a particularly tractable solution and regular diagrams. Unfortunately, the general case $T = dS$ does not always result in straight-

forward decompositions as in Figure 4. The inclusion/exclusion method of counting depends heavily on having a number of squares that can be arranged nicely. For a general d , we must consider a variety of different cases and the method soon becomes convoluted. In the following sections, we present a different method of attack which is inspired by the geometric diagrams here.

3. The Continued Fraction Algorithm

Starting in this section, we assume the reader has some familiarity with continued fractions. For a gentle introduction, we recommend Chapter IV of Davenport's book [2]. For further reading, Davenport recommends [1, Chapters 32–34] (in English) and Perron's book [4] (in German).

3.1. The Example $T = T_k S$ Redone

In Section 2, we showed how the geometric decomposition of a triangular number of squares and one triangle yields recursive formulas for finding m and n with $T_n = T_k S_m$. Here, we introduce a different method and use it to again solve $T_n = T_k S_m$. Then we describe the new method in full generality. We will approximate n and m as real numbers, and use a continued fraction to guide a new algorithm to dissect the figures of squares and triangles. Then, we discretize to recover the integer solutions.

Suppose that we seek integer solutions to $T_n = dS_m$. Then $n(n+1) = 2dm^2$, or $n \approx m\sqrt{2d}$. Accordingly, we will now consider $n = \sqrt{2d}$ asymptotically (with $m = 1$). Next, we determine how many copies of S_m can be placed along the bottom edge of T_n . This is clearly $\lfloor \sqrt{2d} \rfloor$, with an amount $a = \sqrt{2d} - \lfloor \sqrt{2d} \rfloor$ left over. Next, we place as many copies as possible of T_a along the bottom edge of S_m . In other words, we seek the integer j so that $m = aj + r$, $r < a$. Continuing in this way, it is clear that our computations are equivalent to finding the continued fraction expansion of $\sqrt{2d}$.

As an example, suppose $d = T_k = \frac{k(k+1)}{2}$. The continued fraction for $\sqrt{2d} = \sqrt{k(k+1)}$ is $[k; \overline{2, 2k}]$. Using the coefficients in the continued fraction, we get

$$\begin{aligned}\sqrt{2d} &= k \cdot 1 + (\sqrt{2d} - k), \\ 1 &= 2(\sqrt{2d} - k) + ((2k+1) - 2\sqrt{2d}), \\ \sqrt{2d} - k &= 2k((2k+1) - 2\sqrt{2d}) + ((4k+1)\sqrt{2d} - (4k^2 + 3k)).\end{aligned}$$

If $n = \sqrt{2d}$ and $m = 1$, the equations above are then rewritten as

$$n = km + a, \quad a = \sqrt{2d} - k, \quad (1)$$

$$m = 2a + m', \quad m' = (2k+1) - 2\sqrt{2d}, \quad (2)$$

$$a = 2km' + a', \quad a' = (4k+1)\sqrt{2d} - (4k^2 + 3k). \quad (3)$$

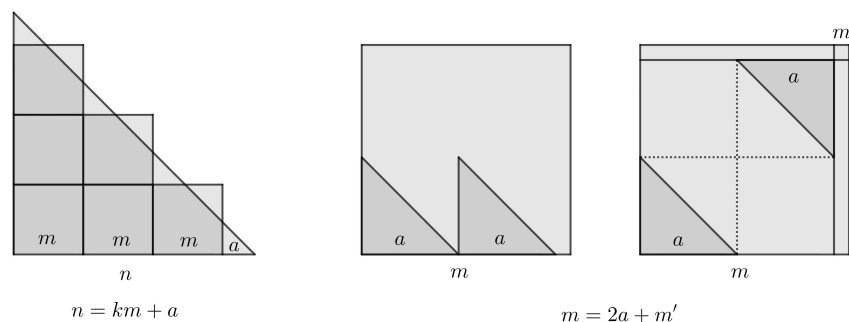


Figure 6: An illustration of the first two equations from the continued fraction algorithm.

Each of these equations represents a corresponding diagram of overlapping triangles and squares. For example, the expression $n = km + a$ corresponds to the left image in Figure 6, because k full copies of the side m of the square fit along the side n of the triangle, with a length a left over. As seen in the diagram, the a represents the side of a smaller triangle.

Next, the equation $m = 2a + m'$ represents the fact that two of the T_a triangles can be placed side by side inside the square S_m , with leftover side length m' . The two triangles are pictured in the middle image of Figure 6. Using a tiling style diagram, we flip the second triangle over to get the rightmost image of Figure 6, and we see that the leftover side length m' corresponds to a small square in the upper right.

Now we have $a = 2km' + a'$, signifying that we again have a diagram similar to the left of Figure 6. At this point, we see that we have an m' representing the side length of a square, and an a' representing the side length of a protruding triangle, so we can define n' as $n' = km' + a'$.

It is straightforward to verify that $\frac{n'}{m'} = \sqrt{2d}$, indicating that we have successfully moved from one solution of $T_n = T_k S_m$ to a smaller solution $T_{n'} = T_k S_{m'}$.

This whole procedure was carried out thinking of n and m asymptotically, with $n = \sqrt{2d}$ and $m = 1$. However, as these shapes are meant to represent integers, we now must rescale and discretize. Equations (1) and (3) remain unchanged, but Equation (2) must be modified to $m = 2a + m' + 1$. To see the need to add 1, consider Figure 7, which shows an example with $k = 2$ at the step corresponding to $m = 2a + m'$. The asymptotic values are $m = 1, a = \sqrt{6} - 2 \approx 0.45$, and $m' = 5 - s\sqrt{6} \approx 0.1$. One solution to the problem $T_n = 3S_m$ is $n = 24$ and $m = 10$. The discretized values visible in the figure are $m = 10, a = 4$, and $m' = 1$. We see in the figure that when we discretize and count dots instead of lengths, we miss one dot, shown in white at the bottom of the figure.

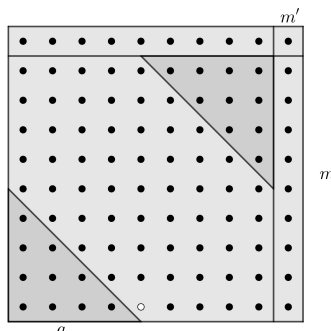


Figure 7: An illustration of one solution to $T = T_2S = 3S$, with $n = 24$, $m = 10$, and $a = 4$.

Now we have the discretized equations with n and m representing integer solutions to $T = T_kS$:

$$\begin{aligned} n &= km + a, \\ m &= 2a + m' + 1, \\ a &= 2km' + a', \\ n' &= km' + a'. \end{aligned}$$

By substituting back from the last equation to the first, we get

$$\begin{aligned} n &= (2k^2 + 2k)m' + (2k + 1)n' + k, \\ m &= (2k + 1)m' + 2n' + 1, \end{aligned}$$

which is indeed the same solution found in Section 2. Starting with the trivial solution $n' = m' = 0$ and iterating these formulas gives all solutions to $T_n = T_kS_m$.

In the following sections, we describe how to use the continued fraction for $\sqrt{2d}$ to generate a series of diagrams of squares and triangles, always culminating in a series of equations that can be combined to give general recursive formulas for solving $T_n = dS_m$. In particular, we answer the following questions:

- For each equation, what criteria determine whether the leftover side length represents a square or a triangle?
- After how many steps will the process yield a solution?
- When discretizing, when must we adjust by adding a constant?

3.2. Parity of Coefficients in the Continued Fraction

It is well-known that the continued fraction of the square root of a non-square integer always has the form $[r_0; \overline{r_1, r_2, \dots, r_k, 2r_0}]$, where $r_1, \dots, r_k$ is a palindromic sequence. Given an equation $T_n = dS_m$, we let $n = \sqrt{2d}$ and $m = 1$. Then we use the terms of the continued fraction to find a sequence of lengths of squares or triangles. Eventually, we reach a square-triangle pair n', m' , where $n' = m'\sqrt{2d}$.

In Section 3.1, the sequence of shapes that we found had sides labeled by n, m, a, m', a' , representing a triangle, square, triangle, square, and triangle, respectively, which we abbreviate as $[t, s; t, s, t, \dots]$. Unfortunately, it turns out that the formulas are not always this well-behaved. Sometimes the images lead us to tile a square with multiple triangles, and the leftover portion represents a triangle rather than a square. Fortunately, the shapes are predictable based on the parities of the terms in the continued fraction. In this section, we give rules for determining the new shape at each step.

At each step, we have two shapes denoted by (X, Y) where $X, Y \in \{s, t\}$. In addition, we have a term r from the continued fraction of $\sqrt{2d}$. In each case, the term and the pair will result in a third shape $Z \in \{s, t\}$, where r copies of shape Y can be placed inside of shape X , resulting in a remainder corresponding to shape Z . We denote this by $(X, Y) \xrightarrow{r} (Y, Z)$. Because we always begin with $T_n = dS_m$, our initial pair of shapes must be (t, s) .

Rule 1. In the sequence of shapes, $(t, s) \xrightarrow{r} (s, t)$ for any integer r . Alternatively we also write $t := rs + t$.

Starting from a triangle and square, the picture will generally look like the left image in Figure 6. The new shape will always be a triangle jutting out of the right side of that image.

Rule 2. In the sequence of shapes,

$$(s, t) \xrightarrow{r} \begin{cases} (t, s) & \text{if } r \text{ is even,} \\ (t, t) & \text{if } r \text{ is odd.} \end{cases}$$

In assignment form, we also write $s := et + s$ and $s := ot + t$.

To illustrate Rule 2, we start by noting that two triangles can be thought of as creating a square as in the far right side of Figure 6. If we have an even number of triangles, they pair off to form squares. We can draw those squares along the diagonal, corner to corner, as in Figure 8. We see that the remaining leftover length is that of a square in the far upper right. Similarly, given an odd number of triangles along the bottom edge of the square, all but one of them pair off to form squares along the diagonal. The last triangle is part of a smaller square, and the leftover length is the side of an inverted smaller triangle in the upper right.

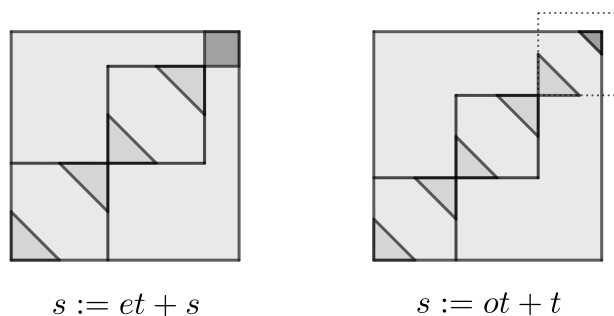


Figure 8: Illustrations of the rules $(s, t) \xrightarrow{r} (t, s)$ when r is even and $(s, t) \xrightarrow{r} (t, t)$ when r is odd.

Rule 3. In the sequence of shapes,

$$(t, t) \xrightarrow{r} \begin{cases} (t, t) & \text{if } r \text{ is even,} \\ (t, s) & \text{if } r \text{ is odd.} \end{cases}$$

The assignment form is $t := et + t$ and $t := ot + s$.

To illustrate Rule 3, we again pair up the triangles to form squares as in Figure 9. When there are an even number of squares, the remaining length corresponds to a small triangle on the rightmost bottom edge. When there are an odd number of triangles, we form squares and then slide the extra triangle into the corner of the bigger triangle. The remaining length corresponds to a small square as shown in Figure 9.

These three rules uniquely determine the sequence of shapes obtained from the continued fraction for $\sqrt{2d}$. Given a continued fraction $[r_0; r_1, r_2, \dots]$, we now create a sequence of shapes $[X_{-1}, X_0; X_1, X_2, \dots]$, where $X_{-1} = t, X_0 = s$, and $(X_{j-1}, X_j) \xrightarrow{r_j} (X_j, X_{j+1})$. For example, $\sqrt{32} = [5; \overline{1, 1, 1, 10}]$. Using Rules 1–3, we get $(t, s) \xrightarrow{5} (s, t) \xrightarrow{1} (t, t) \xrightarrow{1} (t, s) \xrightarrow{1} (s, t) \xrightarrow{10} (t, s) \dots$. The resulting sequence is $[t, s; t, t, s, t, s, t, s, t, s]$. Any continued fraction $[r_0; r_1, r_2, \dots]$ and sequence of shapes $[t, s; X_1, X_2, \dots]$ corresponds to the series of assignments $t := r_0s + X_1$, $s := r_1X_1 + X_2$, and so on. When working out a particular example, we tend to use $m, m', m'', \dots$ to represent the side length of squares, and $a, a', a'', \dots$ and $c, c', c'', \dots$ for side lengths of triangles, in alignment with the notation for the tiling illustrations. The variables $n, n', n'', \dots$ are used for side lengths of triangles that solve the equation $T_n = dS_m$.

The rules make some facts plain. For instance, it is impossible to have the subsequence (s, s) in our continued fraction of shapes. Second, notice that the rules

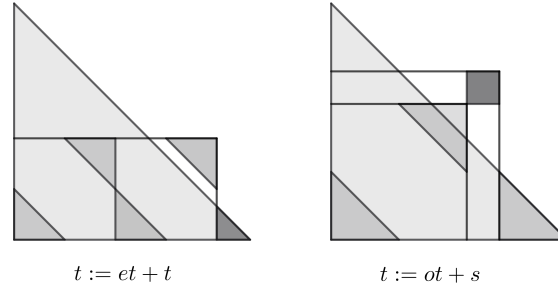


Figure 9: Illustrations of the rules $(t, t) \xrightarrow{r} (t, t)$ when r is even and $(s, t) \xrightarrow{r} (t, s)$ when r is odd.

are reversible. Given an output (Y, Z) and a term r from the sequence of terms of the continued fraction, there is only one possible (X, Y) with $(X, Y) \xrightarrow{r} (Y, Z)$. In fact, we have an even stronger reversing property.

Proposition 1. *If $(X, Y) \xrightarrow{r} (Y, Z)$, then $(Z, Y) \xrightarrow{r} (Y, X)$.*

Combining Proposition 1 with the palindromic property of continued fractions for numbers of the form $\sqrt{d}$ for an integer d results in a similar palindromic property of the sequence of shapes:

Proposition 2. *Suppose $\sqrt{2d} = [r_0; \overline{r_1, \dots, r_{k-1}, 2r_0}]$. Then*

1. *the sequence of shapes $[t, s; X_1, X_2, \dots]$ satisfies $X_{k-j} = X_{k+j}$ for $0 \leq j \leq k$;*
2. *$X_{2k} = s$; and*
3. *the sequence of shapes is periodic with period at most $2k$.*

Proof. For the first assertion, we induct on j . The case $j = 0$ is trivial. Now fix $j < k$ and suppose that $X_{k-\ell} = X_{k+\ell}$ for $0 \leq \ell \leq j$. Then $(X_{k-j-1}, X_{k-j}) \xrightarrow{r_{k-j}} (X_{k-j}, X_{k-j+1})$ and $(X_{k+j-1}, X_{k+j}) \xrightarrow{r_{k+j}} (X_{k+j}, X_{k+j+1})$. By the palindromic property of the continued fraction of $\sqrt{2d}$, $r_{k-j} = r_{k+j}$. By the inductive hypothesis $X_{k-j} = X_{k+j}$ and $X_{k-j+1} = X_{k+j-1}$. Finally, by applying Proposition 1 and reversing the formula for X_{k-j-1} , we see that $X_{k+j+1} = X_{k-j-1}$.

The second assertion now follows trivially from the first. For the third assertion, consider that $r_j = r_{j+2pk}$ for any p , and both $X_{2k} = X_0$ and $X_{2k-1} = X_{-1}$. By a similar inductive argument, $X_j = X_{2pk+j}$. $\square$

Corollary 1. *Let $\sqrt{2d}$ have a continued fraction with period k . In the sequence of shapes $[t, s; X_1, X_2, \dots]$, either $X_k = s$ or $X_{2k} = s$.*

We define the *shape period* to be k if $X_k = s$ and $2k$ if $X_k = t$. Note that the shape period is not exactly the same as the period of the sequence of shapes. For example, $\sqrt{41} = [6; \overline{2, 2, 12}]$, resulting in the sequence of shapes $[t, s; t, s, t, s, t, s, \dots]$. In this case, the period of the sequence of shapes is 2, but the shape period is 6.

3.3. Discretizing

The discretization rules rely on the idea portrayed in Figure 7. Suppose we interpret the asymptotic figures as defining boundaries of the counts of points at the centers of the unit squares. Then a square tile with asymptotic length m also contains S_m dots. Moreover, if the diagonal cuts m at αm for nonintegral α , the $A = T_a$ and $C = T_c$ tiles have $a = \lfloor \alpha m \rfloor$ and $c = \lfloor (1 - \alpha)m \rfloor$. We claim that $m = a + c + 1$. Indeed, using $x = \lfloor x \rfloor + \{x\}$, we have

$$\begin{aligned} a + c &= \lfloor \alpha m \rfloor + \lfloor (1 - \alpha)m \rfloor \\ &= \alpha m - \{\alpha m\} + (1 - \alpha)m - \{(1 - \alpha)m\} \\ &= m - (\{\alpha m\} + \{-\alpha m\}) \\ &= m - 1. \end{aligned}$$

Consider Figure 8. Every pair of triangles $2t$ forms a smaller square s' . Accordingly, $s' = 2t + 1$. When the number r of triangles is even, we get $s := rt + s + \frac{r}{2}$, because we adjust by adding a constant 1 for each pair of triangles. When the number of triangles is odd, then we have $\frac{r-1}{2}$ smaller squares s' and an additional even smaller square s'' . Each of these squares contributes an adjustment of +1. Thus $s := rt + t + \frac{r-1}{2} + 1$ when k is odd. Summarizing, the discretization for Rule 2 is

$$s := \begin{cases} rt + s + \frac{r}{2} & \text{when } r \text{ is even,} \\ rt + t + \frac{r+1}{2} & \text{when } r \text{ is odd.} \end{cases} \quad (4)$$

Now consider Figure 9. Every pair of triangles $2t$ forms a smaller square s' . Accordingly, $s' = 2t + 1$. In other words, when discretizing we must adjust by adding +1 for each pair of triangles. The discretized shape assignments corresponding to Rule 3 are

$$t := \begin{cases} rt + t + \frac{r}{2} & \text{when } r \text{ is even,} \\ rt + s + \frac{r-1}{2} & \text{when } r \text{ is odd.} \end{cases} \quad (5)$$

3.4. The Example $T_n = 7S_m$

As an example, consider $d = 7$. Then $\sqrt{2d} = [3; \overline{1, 2, 1, 6}]$. Using Rules 1–3, we get the sequence of shapes $[t, s; \overline{t, t, t, s}]$. These correspond to the equations

$$\begin{array}{ll} n = 3m + a, & n = 3m + a, \\ m = a + c, & m = a + c + 1, \\ a = 2c + a', & a = 2c + a' + 1, \\ c = a' + m', & c = a' + m', \\ a' = 6m' + a'', & a' = 6m' + a'', \end{array}$$

where the asymptotic version is on the left and the discretized version is on the right.

If we let $n = \sqrt{14}$ and $m = 1$, then in the last of the asymptotic equations we have $m' = 15 - 4\sqrt{14}$ and $a'' = 27\sqrt{14} - 101$. Let $n' = 3m' + a'' = 15\sqrt{14} - 56$. Then $n'/m' = \sqrt{14}$, as desired. We now substitute back through the sequence of discretized equations in order to express n and m in terms of n' and m' , getting

$$\begin{aligned} m &= 15m' + 4n' + 2, \\ n &= 56m' + 15n' + 7. \end{aligned}$$

Starting from the trivial solution $(n, m) = (0, 0)$ and iterating, we get the sequence of solutions $(m, n) = (0, 0), (2, 7), (60, 224), (1798, 6727), \dots$

Returning to our original objective, observe that we have a geometric interpretation for each equation. For example, we can start with an equation corresponding to the end of a shape period, and assign a value of 0 to all indices below it. In our current example, we may set indices that have two or more primes to 0. Then working upwards, we have also $m' = a' = c = 0$, but $a = 1$ and $m = 2$, which introduces figures $T_a = T_1 = 1$ and $S_m = S_2 = 4$. Next we have $n = 7$, so we have $T_n = T_7 = 28$. Indeed, we have $T_7 = 7S_2 = 28$.

For a more generic example, we show one more cycle by starting with $n' = 7, m' = 2, a' = 13$. Then $T_{a'} = T_{13} = 91$, $T_c = T_{15} = 120$, $T_a = T_{44} = 990$, $S_m = S_{60} = 3600$, $T_n = T_{224} = 25200$, and so $T_{244} = 7S_{60} = 25200$.

4. Connection to Pell's Equation and Continued Fractions

It is well-known that Pell's equations can be solved using the convergents of continued fractions. In this section, we show how the geometric dissection method corresponds to known results in the theory of continued fractions and the connection to Pell's equation.

First, every Pell's equation $x^2 - dy^2 = 1$, for d a nonsquare, has integer solutions in the form of recursive equations for (x_n, y_n) with a fundamental solution (x_0, y_0) . One method for finding the fundamental solution involves the convergents of the continued fraction for $\sqrt{d}$. If $\sqrt{d} = [r_0; \overline{r_1, \dots, r_{k-1}, 2r_0}]$ then the convergents are the finite continued fractions $\frac{p_j}{q_j} = [r_0; r_1, \dots, r_j]$, with $p_0 = r_0$ and $q_0 = 1$. By convention, $p_{-1} = 1$ and $q_{-1} = 0$. Then for $n \geq 1$,

$$\begin{aligned} p_j &= r_j p_{j-1} + p_{j-2}, \\ q_j &= r_j q_{j-1} + q_{j-2}. \end{aligned} \tag{6}$$

Let k be the period of the continued fraction for $\sqrt{d}$. It turns out that the fundamental solution to $x^2 - dy^2 = 1$ is $(x_1, y_1) = (p_j, q_j)$ where $j = k - 1$ if k is even, and $j = 2k - 1$ if k is odd. Further solutions are given by $(p_{\ell k - 1}, q_{\ell k - 1})$ or $(p_{2\ell k - 1}, q_{2\ell k - 1})$ depending again on the parity of k . Every solution to $x^2 - 2dy^2 = 1$ with even value of y gives a solution to $T_n = dS_m$ by setting $x = 2n + 1$ and $y = 2m$. Thus we can solve $T_n = dS_m$ by instead using the continued fraction convergents to solve $x^2 - 2dy^2 = 1$.

Indeed, we claim that our dissection method is essentially following the convergent method, adding geometry and pictures to illustrate and justify the equations. For each convergent (p_j, q_j) , we define $w_j = \lfloor q_j/2 \rfloor$. In other words, $w_j = q_j/2$ when q_j is even and $(q_j - 1)/2$ when q_j is odd. When (p_j, q_j) is a solution to $x^2 - 2dy^2 = 1$ and q_j is even, then $m = w_j$ is a solution to $T_n = dS_m$. Equation (6) is transformed into an equation for w_j in 8 possible forms, depending on the parities of r_j, q_{j-1} , and q_{j-2} . For example, consider the case when r_j is even and q_{j-1} and q_{j-2} are both odd. Then $q_j = r_j q_{j-1} + q_{j-2}$ is necessarily odd. That means that $q_j = 2w_j + 1$, $q_{j-1} = 2w_{j-1} + 1$, and $q_{j-2} = 2w_{j-2}$. The change of variables results in $w_j = r_j w_{j-1} + w_{j-2} + \frac{r_j + 1}{2}$. All 8 possibilities can be computed similarly, and the results are summarized in Table 1.

Consider the example in the fourth line of Table 1, where q_{j-1}, q_{j-2} are odd and r_j is even. We adopt the convention that w_j represents the side of a triangle when q_j is odd and represents the side of a square when q_j is even. Then 6 of the 8 rows in the table correspond to the discretized rules in Rule 1 and Equations (4) and (5). In each case, the rule $(X, Y) \xrightarrow{r} (Y, Z)$ corresponds to $w_j = r_j w_{j-1} + w_{j-2}$ with w_j corresponding to X , w_{j-1} corresponding to Y , and w_{j-2} corresponding to Z . Then for example, the fourth line of the table corresponds to the discretized version of Rule 3 in Equation (5), with even term r . The first and fifth lines of the table represent the case in which our sequence of shapes has two consecutive squares, which is impossible. The other rules are summarized in the last column of Table 1. Rows 1 and 5 are marked as NA for “not applicable” since they represent two consecutive squares, which is not possible in our shape sequence.

Because these rules coincide with the discretization rules for the sequence of shapes, the sequence $\{w_j\}$ must be equal to the values of the side lengths obtained

r_j	q_{j-1}	q_{j-2}	q_j	New formula	Rule
e	e	e	e	$w_j = r_j w_{j-1} + w_{j-2}$	NA
e	e	o	o	$w_j = r_j w_{j-1} + w_{j-2}$	Rule 1
e	o	e	e	$w_j = r_j w_{j-1} + w_{j-2} + \frac{r_j}{2}$	Rule 2
e	o	o	o	$w_j = r_j w_{j-1} + w_{j-2} + \frac{r_j}{2}$	Rule 3
o	e	e	e	$w_j = r_j w_{j-1} + w_{j-2}$	NA
o	e	o	o	$w_j = r_j w_{j-1} + w_{j-2}$	Rule 1
o	o	e	o	$w_j = r_j w_{j-1} + w_{j-2} + \frac{r_j-1}{2}$	Rule 3
o	o	o	e	$w_j = r_j w_{j-1} + w_{j-2} + \frac{r_j+1}{2}$	Rule 2

Table 1: The recursive formulas for $w_j = \lfloor q_j/2 \rfloor$, based on parity of r_j , q_{j-1} , and q_{j-2} , denoted “e” for even and “o” for odd. In the last column, “NA” stands for “not applicable.”

in the triangular square algorithm. The wrinkle is that the sequence of shapes goes in “reverse.” In our sequence of shapes algorithm, we start with larger shapes and use the continued fraction to dissect and create smaller shapes. If a particular value of m corresponds to some w_p above, then the next side length in the sequence of shapes is w_{p-1} , followed by w_{p-2} and so on. However, because of the palindromic nature of the continued fraction, reversing the order causes no issues as long as we start at the end of one period of the sequence of shapes. Our work is summarized in the following theorem.

Theorem 1. *Let $\sqrt{2d}$ have continued fraction $[r_0; \overline{r_1, r_2, \dots, r_{k-1}, 2r_0}]$. Construct the sequence of shapes $[t, s; X_0, X_1, \dots]$ corresponding to Rules 1–3. Let m and n represent an arbitrary solution to $T_n = dS_m$. Let m' and a' correspond to the shapes X_p and X_{p+1} where p is the shape period. Let $n' = r_0 m' + a'$. Then the following statements are true:*

1. *By substituting back through the discretized versions of Rules 1–3, we obtain recursive equations expressing n and m in terms of n' and m' that generate all solutions to $T_n = dS_m$, starting with $(0, 0)$.*
2. *If (n, m) is the N th solution in positive integers to $T_n = dS_m$, the quantity w_{pN-j} is the side length of the shape obtained at stage j of the shape algorithm.*

To see this in action, consider again the equation $T_n = 7S_m$. In Section 3.4, we found that

$$\begin{array}{ll} n = 3m + a, & n = 3m + a, \\ m = a + c, & m = a + c + 1, \\ a = 2c + a', & a = 2c + a' + 1, \\ c = a' + m', & c = a' + m', \\ a' = 6m' + a'', & a' = 6m' + a'', \end{array}$$

and we further found that $(n, m) = (60, 224)$ was the second solution. If we trace through the side lengths, we get $n = 224, m = 60, a = 44, c = 15, a' = 13, m' = 2$, and $a'' = 1$. Recall that $\sqrt{14} = [3; \overline{1, 2, 1, 6}]$ and the sequence of shapes for $\sqrt{14}$ is $[t, s; \overline{t, t, t, s}]$. The shape period is 4. The first 9 convergents $\frac{p_n}{q_n}$ for $n = 0, \dots, 8$ of $\sqrt{14}$ are $\frac{3}{1}, \frac{4}{1}, \frac{11}{3}, \frac{15}{4}, \frac{101}{27}, \frac{116}{31}, \frac{333}{89}, \frac{449}{120}$. The corresponding values of w_j are 0, 0, 1, 2, 13, 15, 44, and 60.

5. Conclusion

Some may wonder if we can generalize further to the case $aS_m = bT_n$. We argue that this reduces to the case $kS = T$. First, we may cancel $g = \gcd(a, b)$ on both sides so it suffices to consider a, b relatively prime. This means $b \mid S_m$. Let $b = s^2t$ where t is squarefree, so $m = stm'$ for some m' . Then $a(stm')^2 = s^2tT_n$, which reduces to $atm'^2 = T_n$. This is of the form $kS_{m'} = T_n$ where $k = at$, as claimed.

By attempting to solve $T_n = dS_m$ using the methods of geometric dissection and rearrangement, we have been able to attach a geometric meaning to the process of using convergents of continued fractions to solve the Pell's equation $x^2 - 2dy^2 = 1$. We look forward to gaining further insights into the continued fraction algorithm by finding geometric identities corresponding to the convergents at various steps of the algorithm. We also are interested in investigating the geometric interpretation of $x^2 - dy^2 = 1$ when d is odd.

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