



GENERALIZED APOSTOL-BERNOULLI FUNCTIONS AND THE EVALUATION OF CHARACTER SUMS

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Abstract

We introduce two types of generalized Apostol–Bernoulli functions attached to Dirichlet characters and express them by generalized Apostol–Bernoulli numbers. As a corollary, we obtain formulas expressing two families of character sums by generalized Apostol–Bernoulli numbers. These sums are twisted generalizations of sums introduced and studied by Arakawa, Berndt, Ibukiyama, Kaneko, Kim, Ramanujan, and Zaharescu in the context of modular forms and theta function identities.

*– Dedicated to Melvyn Nathanson and Carl Pomerance
 on the occasion of their 80th birthdays*

1. Introduction

Let χ be a Dirichlet character modulo m . We put $\zeta = \exp(2\pi i/m)$. For integers r, s with s prime to m , we define the Gaussian sum

$$\tau(r/s, \chi) = \sum_{j \pmod{m}} \chi(j) \zeta^{jrs^{-1}},$$

where s^{-1} is regarded as an element of $\mathbb{Z}/m\mathbb{Z}$ such that $ss^{-1} \equiv 1 \pmod{m}$, and j runs over a complete residue system modulo m . We will write $\tau(\chi)$ for $\tau(1, \chi)$. We also extend the definition of χ by multiplicativity by defining $\chi(r/s) = \chi(rs^{-1})$.

Let λ be a nonzero complex number. For $x \in \mathbb{Q}$ with denominator prime to m , we define two types of generalized Apostol–Bernoulli functions $\mathcal{A}_{k,\lambda,\chi}(x)$ and $\mathcal{B}_{k,\lambda,\chi}(x)$ by the generating functions

$$\sum_{j=0}^{m-1} \frac{\tau(j+x, \chi) t \lambda^j e^{(j+x)t}}{\lambda^m e^{mt} - 1} = \sum_{k=0}^{\infty} \mathcal{A}_{k,\lambda,\chi}(x) \frac{t^k}{k!},$$

$$\sum_{j=0}^{m-1} \frac{\chi(j+x) t \lambda^j e^{(j+x)t}}{\lambda^m e^{mt} - 1} = \sum_{k=0}^{\infty} \mathcal{B}_{k,\lambda,\chi}(x) \frac{t^k}{k!}.$$

Note that if χ is primitive, then $\mathcal{A}_{k,\lambda,\chi}(x) = \tau(\chi)\mathcal{B}_{k,\lambda,\bar{\chi}}(x)$, and if χ is trivial (i.e., $m = 1$) which we will denote by $\chi = \mathbf{1}$, then $\mathcal{A}_{k,\lambda,\chi}(x)$ and $\mathcal{B}_{k,\lambda,\chi}(x)$ both reduce to the Apostol–Bernoulli polynomials $\mathcal{B}_{k,\lambda}(x)$ defined by

$$\frac{te^{tx}}{\lambda e^t - 1} = \sum_{k=0}^{\infty} \mathcal{B}_{k,\lambda}(x) \frac{t^k}{k!}. \quad (1)$$

That is, $\mathcal{A}_{k,\lambda,\mathbf{1}}(x) = \mathcal{B}_{k,\lambda,\mathbf{1}}(x) = \mathcal{B}_{k,\lambda}(x)$. These definitions lead to the formulas

$$\begin{aligned} \mathcal{A}_{k,\lambda,\chi}(x) &= m^{k-1} \sum_{j=0}^{m-1} \lambda^j \mathcal{B}_{k,\lambda^m} \left(\frac{j+x}{m} \right) \tau(j+x, \chi), \\ \mathcal{B}_{k,\lambda,\chi}(x) &= m^{k-1} \sum_{j=0}^{m-1} \lambda^j \mathcal{B}_{k,\lambda^m} \left(\frac{j+x}{m} \right) \chi(j+x), \end{aligned} \quad (2)$$

or equivalently, for any natural number s prime to m and any integer r , we have

$$\begin{aligned} \mathcal{A}_{k,\lambda,\chi}(r/s) &= \chi(s) m^{k-1} \sum_{j=0}^{m-1} \lambda^j \mathcal{B}_{k,\lambda^m} \left(\frac{sj+r}{sm} \right) \tau(sj+r, \chi), \\ \mathcal{B}_{k,\lambda,\chi}(r/s) &= \bar{\chi}(s) m^{k-1} \sum_{j=0}^{m-1} \lambda^j \mathcal{B}_{k,\lambda^m} \left(\frac{sj+r}{sm} \right) \chi(sj+r). \end{aligned}$$

Remark. Corresponding to $\lambda = 1$, the generalized Bernoulli functions $\mathcal{B}_{k,1,\chi}(x)$ were introduced by Snyder [30] in connection with p -adic Dedekind sums, and the generalized Bernoulli functions $\mathcal{A}_{k,1,\chi}(x)$ were introduced by the present author in [22]. The generalized Bernoulli functions $\mathcal{A}_{k,\lambda,\chi}(x)$ and $\mathcal{B}_{k,\lambda,\chi}(x)$ for $\lambda \neq 1$ are new.

Corresponding to the generalized Bernoulli functions $\mathcal{A}_{k,\lambda,\chi}(x)$ and $\mathcal{B}_{k,\lambda,\chi}(x)$, we have the quasi-periodic generalized Bernoulli functions $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$ and $\overline{\mathcal{B}}_{k,\lambda,\chi}(x)$ defined by

$$\begin{aligned} \overline{\mathcal{A}}_{k,\lambda,\chi}(x) &= \lambda^{-[x]} \mathcal{A}_{k,\lambda,\chi}(x - [x]), \\ \overline{\mathcal{B}}_{k,\lambda,\chi}(x) &= \lambda^{-[x]} \mathcal{B}_{k,\lambda,\chi}(x - [x]), \end{aligned}$$

where $[x]$ denotes the greatest integer not exceeding x . These functions are easier to work with due to their quasi-periodicity. For example, from (2), we can write

$$\begin{aligned} \overline{\mathcal{A}}_{k,\lambda,\chi}(x) &= m^{k-1} \sum_{j \pmod{m}} \lambda^j \overline{\mathcal{B}}_{k,\lambda^m} \left(\frac{j+x}{m} \right) \tau(j+x, \chi), \\ \overline{\mathcal{B}}_{k,\lambda,\chi}(x) &= m^{k-1} \sum_{j \pmod{m}} \lambda^j \overline{\mathcal{B}}_{k,\lambda^m} \left(\frac{j+x}{m} \right) \chi(j+x), \end{aligned} \quad (3)$$

where $\overline{\mathcal{B}}_{k,\lambda^m}(x) = \lambda^{-[x]} \mathcal{B}_{k,\lambda^m}(x - [x])$.

Remark. The Apostol–Bernoulli functions $\overline{\mathcal{B}}_{n,\lambda}(x)$ were introduced by Luo [28] and further studied by He [14] and Bayad [4]. The generalized Apostol–Bernoulli functions $\overline{\mathcal{A}}_{n,\lambda,\chi}(x)$ and $\overline{\mathcal{B}}_{n,\lambda,\chi}(x)$ for $\chi \neq \mathbb{1}$ are new.

We define the generalized Apostol–Bernoulli numbers $\mathcal{A}_{k,\lambda,\chi}$ and $\mathcal{B}_{k,\lambda,\chi}$ by $\mathcal{A}_{k,\lambda,\chi} = \mathcal{A}_{k,\lambda,\chi}(0)$ and $\mathcal{B}_{k,\lambda,\chi} = \mathcal{B}_{k,\lambda,\chi}(0)$, which together with (3) yields

$$\begin{aligned}\mathcal{A}_{k,\lambda,\chi} &= m^{k-1} \sum_{j \pmod{m}} \lambda^j \overline{\mathcal{B}}_{k,\lambda^m} \left(\frac{j}{m} \right) \tau(j, \chi), \\ \mathcal{B}_{k,\lambda,\chi} &= m^{k-1} \sum_{j \pmod{m}} \lambda^j \overline{\mathcal{B}}_{k,\lambda^m} \left(\frac{j}{m} \right) \chi(j).\end{aligned}\tag{4}$$

The main interest of these numbers is that they give the special values at nonpositive integers of some very important Dirichlet L -series. For example, the special values of the Dirichlet L -series

$$\begin{aligned}\mathcal{L}_1(s, \lambda, \chi) &= \sum_{n=1}^{\infty} \frac{\lambda^n \tau(n, \chi)}{n^s}, \\ \mathcal{L}_2(s, \lambda, \chi) &= \sum_{n=1}^{\infty} \frac{\lambda^n \chi(n)}{n^s} \quad (|\lambda| < 1 \text{ or } |\lambda| = 1, \Re(s) > 1),\end{aligned}$$

at nonpositive integers are

$$\begin{aligned}\mathcal{L}_1(1-n, \lambda, \chi) &= \frac{1}{\lambda} \left(-\frac{\mathcal{A}_{n,\lambda,\chi}}{n} - \delta_1 \right), \\ \mathcal{L}_2(1-n, \lambda, \chi) &= \frac{1}{\lambda} \left(-\frac{\mathcal{B}_{n,\lambda,\chi}}{n} - \delta_2 \right) \quad (n \in \mathbb{N}),\end{aligned}$$

where

$$\begin{aligned}\delta_1 &= \begin{cases} \phi(m) & \text{if } n = 1 \text{ and } \chi \text{ is principal,} \\ 0 & \text{otherwise,} \end{cases} \\ \delta_2 &= \begin{cases} 1 & \text{if } n = 1 \text{ and } \chi \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Observe that the special cases $\mathcal{L}_2(s, \lambda, \mathbb{1})$, $\mathcal{L}_2(s, 1, \chi)$, $\mathcal{L}_2(s, 1, \mathbb{1})$ are the well-known polylogarithm, Dirichlet L -function, and Riemann zeta function, respectively.

For any natural number l and any integer u with $u \mid l$, denote by l_u the u -primary part of l , that is, the maximum integer which divides l and is prime to u . For any natural number n , we denote by $Y(n)$ the set of primitive Dirichlet characters modulo n . Let μ and ϕ denote the Möbius and Euler phi functions, respectively.

In this paper, we generalize a theorem of Ibukiyama [15, Theorem 2] expressing the generalized Apostol–Bernoulli functions $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$ and $\overline{\mathcal{B}}_{k,\lambda,\chi}(x)$ by generalized Apostol–Bernoulli numbers $\mathcal{A}_{k,\lambda,\chi}$ and $\mathcal{B}_{k,\lambda,\chi}$, respectively.

Theorem 1. *Let χ be a Dirichlet character modulo m . Let l be a natural number prime to m and c be any integer prime to l . We have*

$$(i) \quad \overline{\mathcal{A}}_{k,\lambda^l,\chi}\left(\frac{c}{l}\right) = \frac{\chi(l)}{\phi(l)l^{k-1}\lambda^c} \sum_{u|l} \sum_{\delta \in Y(u)} \frac{\bar{\delta}(c^{-1}m)\bar{\chi}(u)}{\tau(\delta)} \sum_{n|l_u} \mu(n)n^{k-1}\bar{\delta}\chi(n)\mathcal{A}_{k,\lambda^n,\delta\chi},$$

$$(ii) \quad \overline{\mathcal{B}}_{k,\lambda^l,\chi}\left(\frac{c}{l}\right) = \frac{\bar{\chi}(l)}{\phi(l)l^{k-1}\lambda^c} \sum_{u|l} \sum_{\delta \in Y(u)} \delta(c^{-1}) \sum_{n|l_u} \mu(n)n^{k-1}\delta\chi(n)\mathcal{B}_{k,\lambda^n,\delta\chi}.$$

Remark. The assertions (i) and (ii) of Theorem 1 in the special case of $\lambda = 1$ were previously established in [24, Theorem 3] and [19, Theorem 3.1] to evaluate certain families of character sums which are much simpler than the ones presented in this paper.

Let ψ be the primitive Dirichlet character modulo f which induces χ . We put

$$q = \prod_{\substack{p|m \\ p \nmid f \\ p \text{ prime}}} p, \quad R = \frac{m}{fq}.$$

As an application of Theorem 1, we express the generalized Apostol–Bernoulli numbers $\mathcal{A}_{k,\lambda,\chi}$ and $\mathcal{B}_{k,\lambda,\chi}$ by generalized Apostol–Bernoulli numbers $\mathcal{A}_{k,\lambda,\psi}$ and $\mathcal{B}_{k,\lambda,\psi}$, respectively, where we note that $\mathcal{A}_{k,\lambda,\psi} = \tau(\psi)\mathcal{B}_{k,\lambda,\bar{\psi}}$.

Theorem 2. *Let χ, ψ, m, f, q, R be as above. We have*

$$(i) \quad \mathcal{A}_{k,\lambda,\chi} = \left(\frac{m}{f}\right)^k \sum_{d|q} \frac{\mu(d)\psi(d)}{d^k} \mathcal{A}_{k,\lambda^{Rq/d},\psi},$$

$$(ii) \quad \mathcal{B}_{k,\lambda,\chi} = \sum_{d|q} \mu(d)\psi(d)d^{k-1}\mathcal{B}_{k,\lambda^d,\psi}.$$

In view of Theorems 1 and 2, we can express the generalized Apostol–Bernoulli functions $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$ and $\overline{\mathcal{B}}_{k,\lambda,\chi}(x)$ by generalized Apostol–Bernoulli numbers attached to primitive Dirichlet characters. See Corollary 7 for such a formulation.

Let a, b be nonnegative integers and α be any integer. As a corollary to Theorem 1, we obtain formulas expressing the character sums

$$M_{a,b}(h, \lambda, \chi) = \sum_{j=1}^{m-1} \frac{\chi(j)}{(\lambda^h \zeta^{hj} - 1)^a (\lambda \zeta^j - 1)^b},$$

$$c_{a,b}(h, \alpha, \chi) = \sum_{j=1}^{m-1} \cot^a\left(\pi \frac{hj}{m}\right) \cot^b\left(\pi \left(\frac{j}{m} + \frac{\alpha}{h}\right)\right) \chi(j)$$

by generalized Apostol–Bernoulli numbers $\mathcal{B}_{k,\lambda,\chi}$. This generalizes our earlier work in [21] where we expressed $M_{a,b}(h, 1, \chi)$ and $c_{a,b}(h, 0, \chi)$ by generalized Bernoulli numbers $B_{k,\chi} := \mathcal{B}_{k,1,\chi}$. We note that the sums $M_{1,1}(h, e^{2\pi i\alpha/h}, \chi)$ and $c_{1,1}(h, \alpha, \chi)$ without the character were studied by Berndt–Kim–Zaharescu [9] where the corresponding reciprocity formulas were obtained.

The sum $M_{a,b}(h, \lambda, \chi)$ is a natural generalization of sums introduced and studied by Berndt [7] and Arakawa–Ibukiyama–Kaneko [3] in the context of the theory of modular forms. For related examples, see [3, 16, 17, 18, 20, 29]. The sum $c_{a,b}(h, \alpha, \chi)$ is a variation of trigonometric sums first investigated by Ramanujan in connection to certain theta function identities. For related examples, see [5, 6, 8, 9, 10, 12].

The layout of this paper is as follows. In Section 2, we prove preliminary results such as multiplication formulas for the generalized Apostol–Bernoulli functions $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$ and $\overline{\mathcal{B}}_{k,\lambda,\chi}(x)$. We also establish the relationship between the two types of generalized Apostol–Bernoulli numbers $\mathcal{A}_{k,\lambda,\chi}$ and $\mathcal{B}_{k,\lambda,\chi}$ given by Theorem 2. In Section 3, we prove the main theorem of this paper (Theorem 1) which directly expresses $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$ and $\overline{\mathcal{B}}_{k,\lambda,\chi}(x)$ by generalized Apostol–Bernoulli numbers $\mathcal{A}_{k,\lambda,\chi}$ and $\mathcal{B}_{k,\lambda,\chi}$, along with an alternative formulation in terms of $\mathcal{B}_{k,\lambda,\overline{\psi}}$ and $\mathcal{B}_{k,\lambda,\psi}$ (Corollary 7). In Section 4, we express the character sums $M_{a,b}(h, \lambda, \chi)$ and $c_{a,b}(h, \alpha, \chi)$ by generalized Apostol–Bernoulli functions $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$, then using Theorem 1, give examples expressing $M_{a,b}(h, \lambda, \chi)$ and $c_{a,b}(h, \alpha, \chi)$ by generalized Apostol–Bernoulli numbers.

2. Preliminaries

We keep the notation used previously. In this section, we review the ordinary Apostol–Bernoulli polynomials $\mathcal{B}_{k,\lambda}(x)$, derive Raabe-type multiplication formulas for the generalized Apostol–Bernoulli functions $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$ and $\overline{\mathcal{B}}_{k,\lambda,\chi}(x)$, and express the generalized Apostol–Bernoulli numbers $\mathcal{A}_{k,\lambda,\chi}$ and $\mathcal{B}_{k,\lambda,\chi}$ by generalized Apostol–Bernoulli numbers attached to primitive Dirichlet characters.

Recall the ordinary Apostol–Bernoulli polynomials $\mathcal{B}_{n,\lambda}(x)$ defined by (1). When $\lambda = 1$, $\mathcal{B}_{n,\lambda}(x)$ reduce to the classical Bernoulli polynomials $B_n(x)$, and when $\lambda \neq 1$, we have

$$\mathcal{B}_{0,\lambda}(x) = 0, \mathcal{B}_{1,\lambda}(x) = \frac{1}{\lambda - 1}, \mathcal{B}_{2,\lambda}(x) = \frac{2x(\lambda - 1) - 2\lambda}{(\lambda - 1)^2}, \dots,$$

where the functions $\mathcal{B}_{n,\lambda}(x)$ are polynomials in x and rational functions in λ . This

can be seen from the closed form expressions ([27, Remark 2.6])

$$\mathcal{B}_{n,\lambda}(x) = \begin{cases} \sum_{k=1}^n k \binom{n}{k} \sum_{j=0}^{k-1} (-1)^j \lambda^j (\lambda - 1)^{-j-1} j! S(k-1, j) x^{n-k} & \text{if } \lambda \neq 1, \\ \sum_{k=0}^n \frac{1}{k+1} \sum_{j=0}^k (-1)^j \binom{k}{j} (x+j)^n & \text{if } \lambda = 1, \end{cases}$$

where $S(n, k)$ are Stirling numbers of the second kind defined by

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n.$$

For additional properties of Apostol–Bernoulli polynomials $\mathcal{B}_{k,\chi}(x)$, we refer the interested reader to [1, 11].

We state the Raabe-type multiplication formulas for ordinary Apostol–Bernoulli functions (see [25, Equation (2.3)]):

$$\overline{\mathcal{B}}_{k,\lambda}(nx) = n^{k-1} \sum_{j \pmod{n}} \lambda^j \overline{\mathcal{B}}_{k,\lambda^n} \left(x + \frac{j}{n} \right) \quad (n \in \mathbb{N}). \quad (5)$$

When $\lambda = 1$, we get Raabe’s multiplication formula for ordinary Bernoulli functions.

As a natural generalization of Equation (5), we have the Raabe-type multiplication formulas for generalized Apostol–Bernoulli functions $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$ and $\overline{\mathcal{B}}_{k,\lambda,\chi}(x)$.

Proposition 3. *Let χ, m be as above. Let $n \in \mathbb{N}$ with $(n, m) = 1$ and $x \in \mathbb{Q}$ with denominator relatively prime to m . Then, we have*

$$\begin{aligned} (i) \quad \overline{\mathcal{A}}_{k,\lambda,\chi}(nx) &= n^{k-1} \overline{\chi}(n) \sum_{j \pmod{n}} \lambda^j \overline{\mathcal{A}}_{k,\lambda^n,\chi} \left(x + \frac{j}{n} \right), \\ (ii) \quad \overline{\mathcal{B}}_{k,\lambda,\chi}(nx) &= n^{k-1} \chi(n) \sum_{j \pmod{n}} \lambda^j \overline{\mathcal{B}}_{k,\lambda^n,\chi} \left(x + \frac{j}{n} \right). \end{aligned}$$

Proof. We only prove the case for $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$, as the case for $\overline{\mathcal{B}}_{k,\lambda,\chi}(x)$ can be proved similarly. By definition, we have

$$\overline{\mathcal{A}}_{k,\lambda,\chi}(nx) = m^{k-1} \sum_{l \pmod{m}} \lambda^l \overline{\mathcal{B}}_{k,\lambda^m} \left(\frac{l+nx}{m} \right) \tau(l+nx, \chi).$$

Replacing l by nl (which is permissible since $(n, m) = 1$) and applying the multiplication formula (5), we get

$$\overline{\mathcal{A}}_{k,\lambda,\chi}(nx) = n^{k-1} \overline{\chi}(n) m^{k-1} \sum_{l \pmod{m}} \sum_{j \pmod{n}} \lambda^{l+mj} \overline{\mathcal{B}}_{k,\lambda^{mn}} \left(\frac{l+x+\frac{mj}{n}}{m} \right) \tau(l+x, \chi).$$

Since $\tau(l+x, \chi) = \tau(l+x+mj/n, \chi)$, we have

$$\overline{\mathcal{A}}_{k,\lambda,\chi}(nx) = n^{k-1} \overline{\chi}(n) \sum_{j \pmod{n}} \lambda^{mj} \overline{\mathcal{A}}_{k,\lambda^n,\chi} \left(x + \frac{mj}{n} \right).$$

Thus the assertion of the lemma follows by replacing j by $m^{-1}j$, where $mm^{-1} \equiv 1 \pmod{n}$. $\square$

Remark. When χ is trivial, both assertions of Proposition 3 reduce to (5).

We now express the generalized Apostol–Bernoulli numbers $\mathcal{A}_{k,\lambda,\chi}$ and $\mathcal{B}_{k,\lambda,\chi}$ by generalized Apostol–Bernoulli numbers $\mathcal{A}_{k,\lambda,\psi}$ and $\mathcal{B}_{k,\lambda,\psi}$, respectively. We note that $\mathcal{A}_{k,\lambda,\psi} = \tau(\psi) \mathcal{B}_{k,\lambda,\overline{\psi}}$.

Theorem 2. Let χ, ψ, m, f, q, R be as above. We have

$$\begin{aligned} (i) \quad \mathcal{A}_{k,\lambda,\chi} &= \left(\frac{m}{f} \right)^k \sum_{d|q} \frac{\mu(d)\psi(d)}{d^k} \mathcal{A}_{k,\lambda^{Rq/d},\psi}, \\ (ii) \quad \mathcal{B}_{k,\lambda,\chi} &= \sum_{d|q} \mu(d)\psi(d)d^{k-1} \mathcal{B}_{k,\lambda^d,\psi}. \end{aligned}$$

Proof. We first prove the assertion (i). Let the multiplicative function g be defined by $g(n) = \mu((n, q))\phi((n, q))\overline{\psi}(n)$. By [26, Theorem B], we have

$$\begin{cases} \tau(\alpha, \chi) = 0 & \text{if } R \nmid \alpha, \\ \tau(Rn, \chi) = R\mu(q)\psi(q)\tau(\psi)g(n) & \text{for } n = 1, 2, \dots \end{cases} \quad (6)$$

From (4) and applying (6), we get

$$\begin{aligned} \mathcal{A}_{k,\lambda,\chi} &= m^{k-1} \sum_{j \pmod{fq}} \lambda^{Rj} \overline{\mathcal{B}}_{k,\lambda^m} \left(\frac{Rj}{m} \right) \tau(Rj, \chi) \\ &= m^{k-1} R\mu(q)\psi(q)\tau(\psi) \sum_{j \pmod{fq}} \lambda^{Rj} \mu((j, q))\phi((j, q)) \overline{\mathcal{B}}_{k,\lambda^m} \left(\frac{j}{fq} \right) \overline{\psi}(j). \end{aligned}$$

Writing $j = q\mu + \nu$ and summing over $\mu \pmod{f}$ and $\nu \pmod{q}$, together with (3), we deduce that

$$\begin{aligned} \mathcal{A}_{k,\lambda,\chi} &= \left(\frac{m}{f} \right)^{k-1} R\mu(q)\tau(\psi) \sum_{\nu \pmod{q}} \lambda^{R\nu} \mu((\nu, q))\phi((\nu, q)) \overline{\mathcal{B}}_{k,\lambda^{m/f},\overline{\psi}} \left(\frac{\nu}{q} \right) \\ &= \left(\frac{m}{f} \right)^{k-1} R\mu(q)\tau(\psi) \sum_{d|q} \sum_{\substack{\nu \pmod{q} \\ (\nu, q)=d}} \lambda^{R\nu} \mu(d)\phi(d) \overline{\mathcal{B}}_{k,\lambda^{m/f},\overline{\psi}} \left(\frac{\nu}{q} \right) \\ &= \left(\frac{m}{f} \right)^{k-1} R\mu(q)\tau(\psi) \sum_{d|q} \mu(d)\phi(d) \sum_{\substack{\xi \pmod{q/d} \\ (\xi, q/d)=1}} (\lambda^{Rd})^\xi \overline{\mathcal{B}}_{k,\lambda^{m/f},\overline{\psi}} \left(\frac{\xi}{q/d} \right). \end{aligned}$$

Applying the assertion (ii) of Theorem 1 to evaluate the innermost sum, noting that $\sum_{g=0}^{q/d-1} \delta(c^{-1}) = 0$ for any $\delta \in Y(u)$ with $u|_d^q$, unless $\delta = \mathbf{1}$, we obtain

$$\begin{aligned} \mathcal{A}_{k,\lambda,\chi} &= \left(\frac{m}{f}\right)^{k-1} R\mu(q)\tau(\psi) \sum_{d|q} \mu(d)\phi(d) \\ &\quad \times \left(\frac{\psi(q/d)}{(q/d)^{k-1}} \sum_{n|_d^q} \mu(n)n^{k-1}\bar{\psi}(n)\mathcal{B}_{k,(\lambda^{Rd})^n,\bar{\psi}} \right) \\ &= R^k \mu(q)\psi(q)\tau(\psi) \sum_{dns=q} \mu(dn)\phi(d)\bar{\psi}(dn)(dn)^{k-1}\mathcal{B}_{k,\lambda^{Rdn},\bar{\psi}} \\ &= R^k \mu(q)\psi(q)\tau(\psi) \sum_{e|q} \sum_{g|e} \mu(e)\phi(g)\bar{\psi}(e)e^{k-1}\mathcal{B}_{k,\lambda^{Re},\bar{\psi}} \\ &= R^k \mu(q)\psi(q)\tau(\psi) \sum_{e|q} \mu(e)\bar{\psi}(e)e^k \mathcal{B}_{k,\lambda^{Re},\bar{\psi}}, \end{aligned}$$

where we used the fact that $\sum_{g|e} \phi(g) = e$ in the penultimate step. The assertion (i) then follows from replacing e by q/e and noting that $\mathcal{A}_{k,\lambda^{Rq/e},\bar{\psi}} = \tau(\psi)\mathcal{B}_{k,\lambda^{Rq/e},\bar{\psi}}$.

We next prove the assertion (ii). From (4), we have

$$\begin{aligned} \mathcal{B}_{k,\lambda,\chi} &= m^{k-1} \sum_{\substack{j \pmod{m} \\ (j,m)=1}} \lambda^j \bar{\mathcal{B}}_{k,\lambda^m} \left(\frac{j}{m}\right) \psi(j) \\ &= m^{k-1} \sum_{d|q} \mu(d) \sum_{r \pmod{m/d}} \lambda^{dr} \bar{\mathcal{B}}_{k,\lambda^m} \left(\frac{dr}{m}\right) \psi(dr). \end{aligned}$$

Writing $r = (m/fd)\mu + \nu$ and summing over $\mu \pmod{f}$ and $\nu \pmod{m/fd}$, together with (3) and the multiplication formula Proposition 3, we get

$$\begin{aligned} \mathcal{B}_{k,\lambda,\chi} &= \left(\frac{m}{f}\right)^{k-1} \psi(m/f) \sum_{d|q} \mu(d) \sum_{\nu \pmod{m/fd}} \lambda^{d\nu} \bar{\mathcal{B}}_{k,\lambda^{m/f},\psi} \left(\frac{\nu}{m/fd}\right) \\ &= \sum_{d|q} \mu(d)\psi(d)d^{k-1}\mathcal{B}_{k,\lambda^d,\psi}. \end{aligned}$$

□

3. Expressing $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$ and $\overline{\mathcal{B}}_{k,\lambda,\chi}(x)$ by Generalized Apostol–Bernoulli Numbers

Fix a Dirichlet character χ modulo m . From (3), for any natural number l prime to m and any integer c , we have

$$\begin{aligned}\overline{\mathcal{A}}_{k,\lambda,\chi}\left(\frac{c}{l}\right) &= \chi(l)m^{k-1} \sum_{j \pmod{m}} \lambda^j \overline{B}_{k,\lambda^m}\left(\frac{l j + c}{lm}\right) \tau(l j + c, \chi), \\ \overline{\mathcal{B}}_{k,\lambda,\chi}\left(\frac{c}{l}\right) &= \overline{\chi}(l)m^{k-1} \sum_{j \pmod{m}} \lambda^j \overline{B}_{k,\lambda^m}\left(\frac{l j + c}{lm}\right) \chi(l j + c).\end{aligned}$$

The aim of this section is to obtain formulas expressing $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$ and $\overline{\mathcal{B}}_{k,\lambda,\chi}(x)$ by generalized Bernoulli numbers using only elementary methods from algebra and number theory.

For any natural number l and any integer u with $u \mid l$, denote by l_u the u -primary part of l , that is, the maximum integer which divides l and is prime to u . For any Dirichlet character δ , we denote by f_δ the conductor of δ . For any natural number n , we denote by $X(n)$ the set of primitive Dirichlet characters δ such that n is divisible by f_δ , and by $Y(n)$ the set of primitive Dirichlet characters with conductor n . Let μ and ϕ denote the Möbius and Euler phi functions, respectively.

To prove Theorem 1, we prepare several lemmas which are generalizations of Lemmas 1-3 in [15].

Lemma 4. *For a natural number l prime to m and any $\delta \in X(l)$, we get*

$$\begin{aligned}(i) \quad & \delta(m)\overline{\chi}(l/f_\delta)l^{k-1} \sum_{c \pmod{l}} \lambda^c \overline{\mathcal{A}}_{k,\lambda^l,\chi}\left(\frac{c}{l}\right) \tau(c, \delta) = \mathcal{A}_{k,\lambda,\delta\chi}, \\ (ii) \quad & \chi(l)l^{k-1} \sum_{c \pmod{l}} \lambda^c \overline{\mathcal{B}}_{k,\lambda^l,\chi}\left(\frac{c}{l}\right) \delta(c) = \mathcal{B}_{k,\lambda,\delta\chi}.\end{aligned}$$

Proof. Since $\tau(c, \delta) = \tau(l j + c, \delta)$ and $\tau(n, \chi)\tau(n, \delta) = \overline{\chi}(f_\delta)\overline{\delta}(m)\tau(n, \delta\chi)$ ($n \in \mathbb{Z}$), we have

$$\begin{aligned}& \delta(m)\overline{\chi}(l/f_\delta)l^{k-1} \sum_{c \pmod{l}} \lambda^c \overline{\mathcal{A}}_{k,\lambda^l,\chi}\left(\frac{c}{l}\right) \tau(c, \delta) \\ &= (lm)^{k-1} \sum_{c \pmod{l}} \sum_{j \pmod{m}} \lambda^{l j + c} \overline{B}_{k,\lambda^{lm}}\left(\frac{l j + c}{lm}\right) \tau(l j + c, \delta\chi) \\ &= (lm)^{k-1} \sum_{n \pmod{lm}} \lambda^n \overline{B}_{k,\lambda^{lm}}\left(\frac{n}{lm}\right) \tau(n, \delta\chi).\end{aligned} \tag{7}$$

Writing $n = (f_\delta m)\mu + \nu$ and summing over $\mu \pmod{l/f_\delta}$ and $\nu \pmod{f_\delta m}$, noting that the Gaussian sum $\tau(n, \delta\chi)$ is periodic modulo $f_\delta m$ and applying the multipli-

cation formula (5), the right-hand side of (7) becomes

$$\begin{aligned}
 & (lm)^{k-1} \sum_{\mu \pmod{l/f_\delta}} \sum_{\nu \pmod{f_\delta m}} \lambda^{f_\delta m \mu + \nu} \overline{\mathcal{B}}_{k, \lambda^{lm}} \left(\frac{f_\delta m \mu + \nu}{lm} \right) \tau(f_\delta m \mu + \nu, \delta \chi) \\
 &= (lm)^{k-1} \sum_{\nu \pmod{f_\delta m}} \lambda^\nu \left(\sum_{\mu \pmod{l/f_\delta}} (\lambda^{f_\delta m})^\mu \overline{\mathcal{B}}_{k, \lambda^{lm}} \left(\frac{\mu}{l/f_\delta} + \frac{\nu}{lm} \right) \right) \tau(\nu, \delta \chi) \\
 &= (lm)^{k-1} \left(\frac{f_\delta}{l} \right)^{k-1} \sum_{\nu \pmod{f_\delta m}} \lambda^\nu \overline{\mathcal{B}}_{k, \lambda^{f_\delta m}} \left(\frac{\nu}{f_\delta m} \right) \tau(\nu, \delta \chi) \\
 &= \mathcal{A}_{k, \lambda, \delta \chi}.
 \end{aligned}$$

Hence, we get the assertion (i). The assertion (ii) can be proved using similar reasoning, so we omit the proof. $\square$

Following the method of Ibukiyama in [15], we obtain an inversion formula. We fix a natural number l which is prime to m and put $L = \prod_{p|l} p$, where p runs over primes. For any $n \mid l$, denote by l_n the n -primary part of l .

Lemma 5. *For any fixed number $d \in (\mathbb{Z}/l\mathbb{Z})^*$, we get*

$$\begin{aligned}
 (i) \quad & \sum_{n|L} \phi(l_n) \overline{\chi}(l_n n) (l_n n)^{k-1} \lambda^{ne} \overline{\mathcal{A}}_{k, \lambda^{l_n n}, \chi} \left(\frac{e}{l_n} \right) = \sum_{\delta \in X(l)} \frac{\overline{\chi}(f_\delta) \overline{\delta}(d^{-1}m)}{\tau(\delta)} \mathcal{A}_{k, \lambda, \delta \chi}, \\
 (ii) \quad & \sum_{n|L} \phi(l_n) \overline{\chi}(l_n n) (l_n n)^{k-1} \lambda^{ne} \overline{\mathcal{B}}_{k, \lambda^{l_n n}, \chi} \left(\frac{e}{l_n} \right) = \sum_{\delta \in X(l)} \delta(d^{-1}) \mathcal{B}_{k, \lambda, \delta \chi}.
 \end{aligned}$$

where e is an integer determined by $ne \equiv d \pmod{l_n}$.

Proof. We prove this lemma by taking the sum over $\delta \in X(l)$ of both sides of the formula in Lemma 4. For any integer c with $0 \leq c \leq l-1$, there exists a unique $n \mid L$ such that $n \mid c$ and $(c, L/n) = 1$. For such c , we have $\sum_{\delta \in X(l)} \delta(c) = \sum_{\delta \in X(l_n)} \delta(c)$, since $\delta(c) = 0$ whenever $(f_\delta, n) > 1$. Denote by $A(n)$ the following set of integers.

$$A(n) = \{c \in \mathbb{Z} : 0 \leq c \leq l-1, n \mid c, (c, L/n) = 1\}.$$

Then, applying Lemma 4, noting that $\tau(c, \delta) = \overline{\delta}(c) \tau(\delta)$, we get

$$\begin{aligned}
 & \sum_{\delta \in X(l)} \frac{\overline{\chi}(f_\delta) \overline{\delta}(d^{-1}m)}{\tau(\delta)} \mathcal{A}_{k, \lambda, \delta \chi} \\
 &= \overline{\chi}(l) l^{k-1} \sum_{\delta \in X(l)} \sum_{c \pmod{l}} \overline{\delta}(d^{-1}c) \lambda^c \overline{\mathcal{A}}_{k, \lambda^l, \chi} \left(\frac{c}{l} \right) \\
 &= \overline{\chi}(l) l^{k-1} \sum_{n|L} \sum_{c \in A(n)} \sum_{\delta \in X(l_n)} \overline{\delta}(d^{-1}c) \lambda^c \overline{\mathcal{A}}_{k, \lambda^l, \chi} \left(\frac{c}{l} \right).
 \end{aligned} \tag{8}$$

Observe that

$$\sum_{\delta \in X(l_n)} \bar{\delta}(d^{-1}c) = \begin{cases} \phi(l_n), & \text{if } d \equiv c \pmod{l_n}, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $C(n)$ the following set of integers:

$$C(n) = \{c \in \mathbb{Z} : 0 \leq c \leq l-1, n \mid c, (c, l/n) = 1, \text{ and } c \equiv d \pmod{l_n}\}.$$

Then, from (8), we get

$$\sum_{\delta \in X(l)} \frac{\bar{\chi}(f_\delta) \bar{\delta}(d^{-1}m)}{\tau(\delta)} \mathcal{A}_{k,\lambda,\delta\chi} = \bar{\chi}(l) l^{k-1} \sum_{n|L} \phi(l_n) \sum_{c \in C(n)} \lambda^c \overline{\mathcal{A}}_{k,\lambda^l,\chi} \left(\frac{c}{l} \right). \quad (9)$$

Let e be an integer such that $ne \equiv d \pmod{l_n}$. Then $(e, l_n) = 1$ since $(d, l) = 1$. Hence, it follows that

$$C(n) = \{n(l_na + e) : a \in \mathbb{Z}, 0 \leq a \leq l/(l_n n) - 1\}.$$

By the multiplication formula Proposition 3, we obtain

$$\begin{aligned} \bar{\chi}(l) l^{k-1} \sum_{c \in C(n)} \lambda^c \overline{\mathcal{A}}_{k,\lambda^l,\chi} \left(\frac{c}{l} \right) &= \bar{\chi}(l) l^{k-1} \sum_{a \pmod{l/(l_n n)}} \lambda^{n(l_na + e)} \overline{\mathcal{A}}_{k,\lambda^l,\chi} \left(\frac{n(l_na + e)}{l} \right) \\ &= \bar{\chi}(l_n n) (l_n n)^{k-1} \lambda^{ne} \overline{\mathcal{A}}_{k,\lambda^{l_n n},\chi} \left(\frac{e}{l_n} \right). \end{aligned}$$

Thus, from (9), we get

$$\sum_{\delta \in X(l)} \frac{\bar{\chi}(f_\delta) \bar{\delta}(d^{-1}m)}{\tau(\delta)} \mathcal{A}_{k,\lambda,\delta\chi} = \sum_{n|L} \phi(l_n) \bar{\chi}(l_n n) (l_n n)^{k-1} \lambda^{ne} \overline{\mathcal{A}}_{k,\lambda^{l_n n},\chi} \left(\frac{e}{l_n} \right).$$

Hence, we get the assertion (i). The assertion (ii) can be proved using similar reasoning, so we omit the proof. $\square$

Lemma 6. *We fix a natural number l prime to m and an integer c prime to l . We define L and l_n for $n \mid l$ in the same way as in Lemma 5. Then, we get*

$$\begin{aligned} (i) \quad \phi(l) \bar{\chi}(l) l^{k-1} \lambda^c \overline{\mathcal{A}}_{k,\lambda^l,\chi} \left(\frac{c}{l} \right) &= \sum_{n|L} \mu(n) \bar{\chi}(n) n^{k-1} \sum_{\delta \in X(l_n)} \frac{\bar{\chi}(f_\delta) \bar{\delta}(c^{-1}mn)}{\tau(\delta)} \mathcal{A}_{k,\lambda,\delta\chi}, \\ (ii) \quad \phi(l) \chi(l) l^{k-1} \lambda^c \overline{\mathcal{B}}_{k,\lambda^l,\chi} \left(\frac{c}{l} \right) &= \sum_{n|L} \mu(n) \chi(n) n^{k-1} \sum_{\delta \in X(l_n)} \delta(c^{-1}n) \mathcal{B}_{k,\lambda,\delta\chi}. \end{aligned}$$

Proof. For $u \mid v \mid L$ and any $d \in (\mathbb{Z}/l\mathbb{Z})^*$, we put

$$g_\lambda(u, v, d) = \phi(l/l_u) \bar{\chi}(lv/(l_u u)) (lv/(l_u u))^{k-1} \lambda^{vw/u} \overline{\mathcal{A}}_{k, \lambda^{lv/(l_u u)}, \chi} \left(\frac{w}{l/l_u} \right),$$

where w is determined by $(v/u)w \equiv d \pmod{(l/l_u)}$. Also, we put

$$f_\lambda(v, d) = \sum_{\delta \in X(l/l_v)} \frac{\bar{\chi}(f_\delta) \bar{\delta}(d^{-1}m)}{\tau(\delta)} \mathcal{A}_{k, \lambda, \delta \chi}.$$

Next, we apply Lemma 5 for $(v, l/l_v)$ instead of (L, l) . Noting that $(l/l_v)_n = l_n/l_v$ for any $n \mid v$, we get

$$\begin{aligned} & \sum_{n \mid v} \phi(l_n/l_v) \bar{\chi}((l_n/l_v)n) ((l_n/l_v)n)^{k-1} \lambda^{ne} \overline{\mathcal{A}}_{k, \lambda^{(l_n/l_v)n}, \chi} \left(\frac{e}{l_n/l_v} \right) \\ &= \sum_{\delta \in X(l/l_v)} \frac{\bar{\chi}(f_\delta) \bar{\delta}(d^{-1}m)}{\tau(\delta)} \mathcal{A}_{k, \lambda, \delta \chi}, \end{aligned}$$

where e is determined by $ne \equiv d \pmod{(l_n/l_v)}$. For each $n \mid v$, we define u by $nu = v$. Then $l_n/l_v = l/l_u$, and we get

$$\sum_{u \mid v} g_\lambda(u, v, d) = f_\lambda(v, d).$$

For any $u \mid v \mid L$, we put $G_\lambda(u) = g_\lambda(u, L, c)$ and $F_\lambda(u) = (L/v)^{k-1} \bar{\chi}(v^{-1}L) \times f_{\lambda^{L/v}}(v, L^{-1}uc)$. Observe that $g_\lambda(u, L, c) = (L/v)^{k-1} \bar{\chi}(v^{-1}L) g_{\lambda^{L/v}}(u, v, L^{-1}vc)$, where $L^{-1}vc$ is regarded as an element of $(\mathbb{Z}/l_v\mathbb{Z})^*$. Hence, we get

$$\begin{aligned} \sum_{u \mid v} G_\lambda(u) &= (L/v)^{k-1} \bar{\chi}(v^{-1}L) \sum_{u \mid v} g_{\lambda^{L/v}}(u, v, L^{-1}vc) \\ &= (L/v)^{k-1} \bar{\chi}(v^{-1}L) f_{\lambda^{L/v}}(v, L^{-1}vc) \\ &= F_\lambda(v). \end{aligned}$$

Applying the Möbius inversion formula for $v = L$, we get

$$G_\lambda(L) = \sum_{n \mid L} \mu(n) F_\lambda \left(\frac{L}{n} \right).$$

Thus, we obtain

$$\phi(l) \bar{\chi}(l) l^{k-1} \lambda^c \overline{\mathcal{A}}_{k, \lambda^l, \chi} \left(\frac{c}{l} \right) = \sum_{n \mid L} \mu(n) \bar{\chi}(n) n^{k-1} \sum_{\delta \in X(l_n)} \frac{\bar{\chi}(f_\delta) \bar{\delta}(c^{-1}mn)}{\tau(\delta)} \mathcal{A}_{k, \lambda^n, \delta \chi}.$$

Hence, we get the assertion (i). The assertion (ii) can be proved using similar reasoning, so we omit the proof. $\square$

We are now in position to prove Theorem 1 given in the Introduction.

Theorem 1. *Let χ be a Dirichlet character modulo m . Let l be a natural number prime to m and c be any integer prime to l . We have*

$$(i) \quad \overline{\mathcal{A}}_{k,\lambda^l,\chi}\left(\frac{c}{l}\right) = \frac{\chi(l)}{\phi(l)l^{k-1}\lambda^c} \sum_{u|l} \sum_{\delta \in Y(u)} \frac{\bar{\delta}(c^{-1}m)\bar{\chi}(u)}{\tau(\delta)} \sum_{n|l_u} \mu(n)n^{k-1}\bar{\delta}\chi(n)\mathcal{A}_{k,\lambda^n,\delta\chi},$$

$$(ii) \quad \overline{\mathcal{B}}_{k,\lambda^l,\chi}\left(\frac{c}{l}\right) = \frac{\bar{\chi}(l)}{\phi(l)l^{k-1}\lambda^c} \sum_{u|l} \sum_{\delta \in Y(u)} \delta(c^{-1}) \sum_{n|l_u} \mu(n)n^{k-1}\delta\chi(n)\mathcal{B}_{k,\lambda^n,\delta\chi}.$$

Proof. We define L and l_n for $n | l$ in the same way as in Lemma 5. From Lemma 6, we get

$$\begin{aligned} \phi(l)\bar{\chi}(l)l^{k-1}\lambda^c\overline{\mathcal{A}}_{k,\lambda^l,\chi}\left(\frac{c}{l}\right) &= \sum_{n|L} \sum_{\delta \in X(l_n)} \frac{\mu(n)n^{k-1}\bar{\chi}(nf_\delta)\bar{\delta}(c^{-1}mn)}{\tau(\delta)} \mathcal{A}_{k,\lambda^n,\delta\chi} \\ &= \sum_{n|l} \sum_{u|l_n} \sum_{\delta \in Y(u)} \frac{\mu(n)n^{k-1}\bar{\chi}(nu)\bar{\delta}(c^{-1}mn)}{\tau(\delta)} \mathcal{A}_{k,\lambda^n,\delta\chi}. \end{aligned}$$

Observe that $u | l_n$ for $n | l$ is equivalent to $n | l_u$ for $u | l$. Thus, we have

$$\begin{aligned} \phi(l)\bar{\chi}(l)l^{k-1}\lambda^c\overline{\mathcal{A}}_{k,\lambda^l,\chi}\left(\frac{c}{l}\right) &= \sum_{u|l} \sum_{n|l_u} \sum_{\delta \in Y(u)} \frac{\mu(n)n^{k-1}\bar{\chi}(nu)\bar{\delta}(c^{-1}mn)}{\tau(\delta)} \mathcal{A}_{k,\lambda^n,\delta\chi} \\ &= \sum_{u|l} \sum_{\delta \in Y(u)} \frac{\bar{\delta}(c^{-1}m)\bar{\chi}(u)}{\tau(\delta)} \sum_{n|l_u} \mu(n)n^{k-1}\bar{\chi}(n)\bar{\delta}(n)\mathcal{A}_{k,\lambda^n,\delta\chi}. \end{aligned}$$

Hence, we get the assertion (i). The assertion (ii) can be proved using similar reasoning, so we omit the proof. $\square$

Let ψ be the primitive Dirichlet character modulo f which induces χ , and set

$$q = \prod_{\substack{p|m \\ p \nmid f \\ p \text{ prime}}} p, \quad R = \frac{m}{fq}.$$

From Theorems 1 and 2, we express the generalized Apostol–Bernoulli functions $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$ and $\overline{\mathcal{B}}_{k,\lambda,\chi}(x)$ by generalized Apostol–Bernoulli numbers $\mathcal{B}_{k,\lambda,\bar{\psi}}$ and $\mathcal{B}_{k,\lambda,\psi}$, respectively.

Corollary 7. *Let χ, ψ, m, f, q, R be as above. Let l be a natural number prime to*

m and c be any integer prime to l . We have

$$\begin{aligned}
 (i) \quad \overline{\mathcal{A}}_{k,\lambda^l,\chi} \left(\frac{c}{l} \right) &= \frac{\tau(\psi)(m/f)^k \psi(l)}{\phi(l)l^{k-1}\lambda^c} \sum_{u|l} \sum_{\delta \in Y(u)} \bar{\delta}(c^{-1}m/f) \\
 &\quad \times \sum_{n|l_u} \mu(n)n^{k-1} \overline{\delta\psi}(n) \sum_{d|q} \frac{\mu(d)\delta\psi(d)}{d^k} \mathcal{B}_{k,\lambda^{Rnq/d},\overline{\delta\psi}}, \\
 (ii) \quad \overline{\mathcal{B}}_{k,\lambda^l,\chi} \left(\frac{c}{l} \right) &= \frac{\overline{\psi}(l)}{\phi(l)l^{k-1}\lambda^c} \sum_{u|l} \sum_{\delta \in Y(u)} \delta(c^{-1}) \\
 &\quad \times \sum_{n|l_u} \mu(n)n^{k-1} \delta\psi(n) \sum_{d|q} \mu(d)\delta\psi(d) d^{k-1} \mathcal{B}_{k,\lambda^{nd},\delta\psi}.
 \end{aligned}$$

Proof. By Theorem 2, for any $u|l$, $\delta \in Y(u)$ and $n|l_u$, we have

$$\begin{aligned}
 \mathcal{A}_{k,\lambda^n,\delta\chi} &= \tau(\delta\psi) \left(\frac{m}{f} \right)^k \sum_{d|q} \frac{\mu(d)\delta\psi(d)}{d^k} \mathcal{B}_{k,\lambda^{Rnq/d},\overline{\delta\psi}}, \\
 \mathcal{B}_{k,\lambda^n,\delta\chi} &= \sum_{d|q} \mu(d)\delta\psi(d) d^{k-1} \mathcal{B}_{k,\lambda^{nd},\delta\psi}.
 \end{aligned}$$

Substituting these expressions for $\mathcal{A}_{k,\lambda^n,\delta\chi}$ and $\mathcal{B}_{k,\lambda^n,\delta\chi}$ back into Theorem 1, noting that $\tau(\delta\psi) = \delta(f)\psi(u)\tau(\delta)\tau(\psi)$, we get the assertions of the corollary. $\square$

For any primitive Dirichlet character η , we set

$$\begin{aligned}
 J_{k,\eta\psi} &= \left(\frac{m}{f} \right)^k \prod_{\substack{p|q \\ p \text{ prime}}} \left(1 - \frac{\eta\psi(p)}{p^k} \right), \\
 K_{k,\eta\psi} &= \prod_{\substack{p|q \\ p \text{ prime}}} (1 - p^{k-1}\eta\psi(p)).
 \end{aligned} \tag{10}$$

When $\lambda = 1$, by multiplicativity, Corollary 7 reduces to the following special case.

Corollary 8 ([24, Theorem 3], [19, Theorem 3.1]). *Let $\chi, \psi, m, f, q, J, K$ be as above. Let l be a natural number prime to m and c be any integer prime to l . We*

have

$$\begin{aligned}
 (i) \quad \overline{\mathcal{A}}_{k,1,\chi} \left(\frac{c}{l} \right) &= \frac{\tau(\psi)\psi(l)}{\phi(l)l^{k-1}} \sum_{u|l} \sum_{\delta \in Y(u)} \bar{\delta}(c^{-1}m/f) J_{k,\delta\psi} \mathcal{B}_{k,1,\bar{\delta}\psi} \\
 &\quad \times \prod_{\substack{p|l_u \\ p \text{ prime}}} (1 - p^{k-1} \bar{\delta}\psi(p)), \\
 (ii) \quad \overline{\mathcal{B}}_{k,1,\chi} \left(\frac{c}{l} \right) &= \frac{\bar{\psi}(l)}{\phi(l)l^{k-1}} \sum_{u|l} \sum_{\delta \in Y(u)} \delta(c^{-1}) K_{k,\delta\psi} \mathcal{B}_{k,1,\delta\psi} \prod_{\substack{p|l_u \\ p \text{ prime}}} (1 - p^{k-1} \delta\psi(p)).
 \end{aligned}$$

Remark. The assertions (i) and (ii) of Corollary 8 were previously established in [24, Theorem 3] and [19, Theorem 3.1] to evaluate certain families of character sums (see [24, Proposition 14]) which are much simpler than the ones presented in this paper.

4. The Evaluation of $M_{a,b}(h, \lambda, \chi)$ and $c_{a,b}(h, \alpha, \chi)$

In this section, we obtain formulas expressing $M_{a,b}(h, \lambda, \chi)$ and $c_{a,b}(h, \alpha, \chi)$ by generalized Apostol–Bernoulli numbers attached to primitive Dirichlet characters. We now fix the notation.

Let χ be a Dirichlet character modulo $m > 1$, and ψ be the primitive Dirichlet character modulo f which induces χ . (Here, we are not interested in the trivial case of $\chi = \mathbf{1}$.) We put $\zeta = \exp(2\pi i/m)$. Let h be a positive integer prime to m , and λ be a nonzero complex number satisfying $\lambda^{hm} \neq 1$ unless $\lambda^h = 1$. Let a, b be nonnegative integers with $a + b \geq 1$, and α be any integer. Recall the definitions of $M_{a,b}(h, \lambda, \chi)$ and $c_{a,b}(h, \alpha, \chi)$ given by

$$\begin{aligned}
 M_{a,b}(h, \lambda, \chi) &= \sum_{j=1}^{m-1} \frac{\chi(j)}{(\lambda^h \zeta^{hj} - 1)^a (\lambda \zeta^j - 1)^b}, \\
 c_{a,b}(h, \alpha, \chi) &= \sum_{j=1}^{m-1} \cot^a \left(\pi \frac{hj}{m} \right) \cot^b \left(\pi \left(\frac{j}{m} + \frac{\alpha}{h} \right) \right) \chi(j).
 \end{aligned}$$

For positive integers n and k , we denote by $\begin{bmatrix} n \\ k \end{bmatrix}$ the Stirling numbers of the first kind. That is, Stirling's cycle numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ denote the number of permutations of n letters (elements of the symmetric group of degree n) that consist of k disjoint cycles.

For $k \in \mathbb{N}$ and $u \in \mathbb{Z}$, we define the character sum function $P_{k,u,\lambda,\chi}(t)$ by

$$P_{k,u,\lambda,\chi}(t) = \sum_{j=0}^{m-1} \frac{\chi(j) \zeta^{ju}}{(1 - \lambda \zeta^j e^t)^k}.$$

When $\lambda = 1$ and χ is primitive, we proved the following in [21, Lemma 3.2].

$$P_{k, ch^{-1}, 1, \chi}(0) = -\frac{1}{(k-1)!} \sum_{j=1}^k \left(\sum_{r=0}^{k-j} \frac{(-1)^r}{r+j} \begin{bmatrix} k \\ r+j \end{bmatrix} \binom{r+j}{j} \left(\frac{c}{h}\right)^r \right) \overline{\mathcal{A}}_{j, 1, \chi} \left(\frac{c}{h}\right) \\ (c \in \mathbb{Z}, 0 \leq c \leq k(h-1)),$$

where h^{-1} is regarded as an element of $(\mathbb{Z}/m\mathbb{Z})^*$. This, in fact, generalizes to the following result, whose proof we omit since it can be proved in a similar manner.

Lemma 9. *Let χ, h, λ, k be as above. For any integer c such that $0 \leq c \leq k(h-1)$, we have*

$$P_{k, ch^{-1}, \lambda, \chi}(0) = -\frac{1}{(k-1)!} \sum_{j=1}^k \left(\sum_{r=0}^{k-j} \frac{(-1)^r}{r+j} \begin{bmatrix} k \\ r+j \end{bmatrix} \binom{r+j}{j} \left(\frac{c}{h}\right)^r \right) \overline{\mathcal{A}}_{j, \lambda, \chi} \left(\frac{c}{h}\right).$$

To evaluate the sums $M_{a,b}(h, \lambda, \chi)$ and $c_{a,b}(h, \alpha, \chi)$, we first express them by quasi-periodic Apostol–Bernoulli functions $\overline{\mathcal{A}}_{k, \lambda, \chi}(x)$.

Theorem 10. *Let $\chi, m, h, \lambda, a, b$ be as above. We have*

$$M_{a,b}(h, \lambda, \chi) = \frac{(-1)^{a+b-1} \overline{\chi}(h)}{(a+b-1)!} \sum_{j=1}^{a+b} \sum_{c=0}^{b(h-1)} C_h(b, c) \\ \times \left(\sum_{r=0}^{a+b-j} \frac{(-1)^r}{r+j} \begin{bmatrix} a+b \\ r+j \end{bmatrix} \binom{r+j}{j} \left(\frac{c}{h}\right)^r \right) \lambda^c \overline{\mathcal{A}}_{j, \lambda^h, \chi} \left(\frac{c}{h}\right),$$

where

$$C_h(b, c) = \begin{cases} 1 & \text{if } b = 0, \\ \sum_{k=0}^{[c/h]} (-1)^k \binom{b}{k} \binom{b-1+c-hk}{b-1} & \text{if } b \geq 1. \end{cases}$$

Proof. Letting $C_h(b, c)$ be defined as in the theorem, we have

$$(1 + \lambda \zeta^j + \cdots + (\lambda \zeta^j)^{h-1})^b = \sum_{c=0}^{b(h-1)} C_h(b, c) (\lambda \zeta^j)^c.$$

Since $\lambda^{hm} \neq 1$ unless $\lambda^h = 1$, we have $\lambda^m \neq 1$ unless $\lambda = 1$ as $(h, m) = 1$. This implies that $\lambda \zeta^j \neq 1$ for $j = 1, 2, \dots, m-1$, and we have

$$\frac{\lambda^h \zeta^{hj} - 1}{\lambda \zeta^j - 1} = \sum_{c=0}^{h-1} (\lambda \zeta^j)^c.$$

Hence,

$$\begin{aligned}
 M_{a,b}(h, \lambda, \chi) &= \sum_{j=1}^{m-1} \frac{\chi(j)}{(\lambda^h \zeta^{hj} - 1)^a (\lambda \zeta^j - 1)^b} \\
 &= \sum_{j=1}^{m-1} \frac{\chi(a) \left(\sum_{c=0}^{h-1} (\lambda \zeta^j)^c \right)^b}{(\lambda^h \zeta^{hj} - 1)^{a+b}} \\
 &= \sum_{c=0}^{b(h-1)} C_h(b, c) \lambda^c \sum_{j=1}^{m-1} \frac{\chi(j) \zeta^{cj}}{(\lambda^h \zeta^{hj} - 1)^{a+b}} \\
 &= (-1)^{a+b} \bar{\chi}(h) \sum_{c=0}^{b(h-1)} C_h(b, c) \lambda^c P_{a+b, ch^{-1}, \lambda^h, \chi}(0).
 \end{aligned}$$

The assertion of the theorem then follows by Lemma 9. $\square$

Theorem 11. Let χ, m, h, a, b, α be as above. Put $\xi = \exp(2\pi i/h)$. We have

$$c_{a,b}(h, \alpha, \chi) = i^{a+b} \sum_{j=0}^a \sum_{k=0}^b 2^{j+k} \binom{a}{j} \binom{b}{k} M_{j,k}(h, \xi^\alpha, \chi).$$

Proof. Since

$$\begin{aligned}
 \cot\left(\pi \frac{hj}{m}\right) &= i \left(\frac{2}{\zeta^{hj} - 1} + 1 \right), \\
 \cot\left(\pi \left(\frac{j}{m} + \frac{\alpha}{h} \right)\right) &= i \left(\frac{2}{\xi^\alpha \zeta^j - 1} + 1 \right),
 \end{aligned}$$

we have, by the binomial formula,

$$c_{a,b}(h, \alpha, \chi) = i^{a+b} \sum_{j=0}^a \sum_{k=0}^b 2^{j+k} \binom{a}{j} \binom{b}{k} M_{j,k}(h, \xi^\alpha, \chi).$$

$\square$

As a corollary, we obtain a formula for cotangent power sums considered by Apostol [2], Berndt [7], Girstmair [12, 13], and the present author [23].

Corollary 12 ([22, Corollary 4.4]). Let χ, m be as above, and n be any positive

integer. We have

$$\begin{aligned} & \sum_{j=1}^{m-1} \cot^n \left(\frac{\pi j}{m} \right) \chi(j) \\ &= -i^n \sum_{k=1}^n \frac{1}{k} \left(\sum_{j=k}^n \frac{(-2)^j \binom{n}{j} [j]_k}{(j-1)!} \right) \mathcal{A}_{k,1,\chi} + \begin{cases} i^n \phi(m) & \text{if } \chi \text{ is principal,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Noting that the sum on the left-hand side of the equation is $c_{n,0}(1, 0; \chi)$, we get, by Theorem 11,

$$\sum_{j=1}^{m-1} \cot^n \left(\frac{\pi j}{m} \right) \chi(j) = i^n \sum_{j=0}^n 2^j \binom{n}{j} M_{j,0}(1, 1, \chi). \quad (11)$$

Since

$$M_{0,0}(1, 1; \chi) = \begin{cases} \phi(m) & \text{if } \chi \text{ is principal,} \\ 0 & \text{otherwise,} \end{cases} \quad (12)$$

and for $j \geq 1$, Theorem 10 yields

$$M_{j,0}(1, 1; \chi) = \frac{(-1)^{j-1}}{(j-1)!} \sum_{k=1}^j \frac{[j]_k}{k} \mathcal{A}_{k,1,\chi}, \quad (13)$$

the corollary follows upon plugging (12), (13) back into (11). $\square$

Since the sums $M_{a,b}(h, \lambda, \chi)$ and $c_{a,b}(h, \alpha, \chi)$ can be expressed by generalized Apostol–Bernoulli functions $\overline{\mathcal{A}}_{k,\lambda,\chi}(x)$ which are expressible by generalized Apostol–Bernoulli numbers by virtue of Theorem 1, so are the sums $M_{a,b}(h, \lambda, \chi)$ and $c_{a,b}(h, \alpha, \chi)$. We now give examples expressing $M_{a,b}(h, \lambda, \chi)$ and $c_{a,b}(h, \alpha, \chi)$ by generalized Apostol–Bernoulli numbers.

From the multiplication formula Proposition 3, we note the following useful fact:

$$\sum_{c \pmod{h}} \lambda^c \overline{\mathcal{A}}_{k,\lambda^h,\chi}(c/h) = \frac{\chi(h)}{h^{k-1}} \mathcal{A}_{k,\lambda,\chi}. \quad (14)$$

Proposition 13. *Let $m > 1$ be a natural number prime to 4 and χ be a Dirichlet character modulo m . We denote by δ the unique primitive Dirichlet character modulo 4. Let λ be a nonzero complex number such that $\lambda^{4m} \neq 1$ unless $\lambda^4 = 1$. Put*

$\xi = \exp(2\pi i/4)$. Let α be an arbitrary integer. We have

$$\begin{aligned}
 (i) \quad M_{1,1}(4, \lambda, \chi) &= \sum_{j=1}^{m-1} \frac{\chi(j)}{(\lambda^4 \zeta^{4j} - 1)(\lambda \zeta^j - 1)} \\
 &= -\frac{1}{2} \mathcal{A}_{1,\lambda,\chi} - \frac{\bar{\chi}(4)}{2} \mathcal{A}_{1,\lambda^4,\chi} - \frac{\bar{\delta}(m)\bar{\chi}(4)}{4\tau(\delta)} \mathcal{A}_{1,\lambda,\delta\chi} - \frac{1}{8} \mathcal{A}_{2,\lambda,\chi}, \\
 (ii) \quad M_{2,1}(4, \lambda, \chi) &= \sum_{j=1}^{m-1} \frac{\chi(j)}{(\lambda^4 \zeta^{4j} - 1)^2 (\lambda \zeta^j - 1)} \\
 &= \frac{13}{32} \mathcal{A}_{1,\lambda,\chi} - \frac{\bar{\chi}(2)}{32} \mathcal{A}_{1,\lambda^2,\chi} + \frac{5\bar{\chi}(4)}{8} \mathcal{A}_{1,\lambda^4,\chi} + \frac{\bar{\delta}(m)\bar{\chi}(4)}{4\tau(\delta)} \mathcal{A}_{1,\lambda,\delta\chi} \\
 &\quad + \frac{1}{8} \mathcal{A}_{2,\lambda,\chi} + \frac{\bar{\chi}(4)}{4} \mathcal{A}_{2,\lambda^4,\chi} + \frac{\bar{\delta}(m)\bar{\chi}(4)}{32\tau(\delta)} \mathcal{A}_{2,\lambda,\delta\chi} + \frac{1}{96} \mathcal{A}_{3,\lambda,\chi}, \\
 (iii) \quad c_{1,1}(4, \alpha, \chi) &= \sum_{j=1}^{m-1} \cot\left(\frac{4\pi j}{m}\right) \cot\left(\pi\left(\frac{j}{m} + \frac{\alpha}{4}\right)\right) \chi(j) \\
 &= \frac{\bar{\delta}(m)\bar{\chi}(4)}{\tau(\delta)} \mathcal{A}_{1,\xi^\alpha,\delta\chi} + \frac{1}{2} \mathcal{A}_{2,\xi^\alpha,\chi} - \begin{cases} \phi(m) & \text{if } \chi \text{ is principal,} \\ 0 & \text{otherwise,} \end{cases} \\
 (iv) \quad c_{2,1}(4, \alpha, \chi) &= \sum_{j=1}^{m-1} \cot^2\left(\frac{4\pi j}{m}\right) \cot\left(\pi\left(\frac{j}{m} + \frac{\alpha}{4}\right)\right) \chi(j) \\
 &= -i \left(\frac{5}{4} \mathcal{A}_{1,\xi^\alpha,\chi} - \frac{\bar{\chi}(2)}{4} \mathcal{A}_{1,\xi^{2\alpha},\chi} + \bar{\chi}(4) \mathcal{A}_{1,1,\chi} + \frac{\bar{\delta}(m)\bar{\chi}(4)}{4\tau(\delta)} \mathcal{A}_{2,\xi^\alpha,\delta\chi} \right. \\
 &\quad \left. + \frac{1}{12} \mathcal{A}_{3,\xi^\alpha,\chi} \right) - \begin{cases} i\phi(m) & \text{if } \chi \text{ is principal,} \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

Proof. By Theorem 10 together with the help of (14), we get

$$\begin{aligned}
 M_{1,1}(4, \lambda, \chi) &= -\mathcal{A}_{1,\lambda,\chi} - \frac{1}{8} \mathcal{A}_{2,\lambda,\chi} + \frac{\bar{\chi}(4)}{4} \sum_{c=1}^3 c \lambda^c \overline{\mathcal{A}}_{1,\lambda^4,\chi}(c/4), \\
 M_{2,1}(4, \lambda, \chi) &= \mathcal{A}_{1,\lambda,\chi} + \frac{3}{16} \mathcal{A}_{2,\lambda,\chi} + \frac{1}{96} \mathcal{A}_{3,\lambda,\chi} \\
 &\quad + \bar{\chi}(4) \sum_{c=1}^3 \left(\frac{-3c}{8} + \frac{c^2}{32} \right) \lambda^c \overline{\mathcal{A}}_{1,\lambda^4,\chi}(c/4) \\
 &\quad - \frac{\bar{\chi}(4)}{8} \sum_{c=1}^3 c \lambda^c \overline{\mathcal{A}}_{2,\lambda^4,\chi}(c/4).
 \end{aligned} \tag{15}$$

By Theorem 1, we have

$$\begin{aligned}\overline{\mathcal{A}}_{k,\lambda^4,\chi}(1/4) &= \frac{\chi(4)}{2 \cdot 4^{k-1}\lambda} \left(\mathcal{A}_{k,\lambda,\chi} - 2^{k-1}\overline{\chi}(2)\mathcal{A}_{k,\lambda^2,\chi} + \frac{\overline{\delta}(m)\overline{\chi}(4)}{\tau(\delta)}\mathcal{A}_{k,\lambda,\delta\chi} \right), \\ \overline{\mathcal{A}}_{k,\lambda^4,\chi}(2/4) &= \overline{\mathcal{A}}_{k,\lambda^4,\chi}(1/2) = \frac{\chi(2)}{2^{k-1}\lambda^2} \left(\mathcal{A}_{k,\lambda^2,\chi} - 2^{k-1}\overline{\chi}(2)\mathcal{A}_{k,\lambda^4,\chi} \right), \\ \overline{\mathcal{A}}_{k,\lambda^4,\chi}(3/4) &= \frac{\chi(4)}{2 \cdot 4^{k-1}\lambda^3} \left(\mathcal{A}_{k,\lambda,\chi} - 2^{k-1}\overline{\chi}(2)\mathcal{A}_{k,\lambda^2,\chi} - \frac{\overline{\delta}(m)\overline{\chi}(4)}{\tau(\delta)}\mathcal{A}_{k,\lambda,\delta\chi} \right) \\ &\quad (k \in \mathbb{N}).\end{aligned}$$

Thus, for $j, k \in \mathbb{N}$, we have

$$\begin{aligned}\overline{\chi}(4) \sum_{c=1}^3 c^j \lambda^c \overline{\mathcal{A}}_{k,\lambda^4,\chi}(c/4) \\ = \frac{1}{2 \cdot 4^{k-1}} \left\{ (1 + 3^j)\mathcal{A}_{k,\lambda,\chi} - (1 - 2^{j+1} + 3^j) 2^{k-1}\overline{\chi}(2)\mathcal{A}_{k,\lambda^2,\chi} \right. \\ \left. - 2^{2k+j-1}\overline{\chi}(4)\mathcal{A}_{k,\lambda^4,\chi} + (1 - 3^j) \frac{\overline{\delta}(m)\overline{\chi}(4)}{\tau(\delta)}\mathcal{A}_{k,\lambda,\delta\chi} \right\}.\end{aligned}$$

Using this to evaluate the summations on the right-hand side of (15), we obtain the assertions (i) and (ii).

By Theorem 11, we have

$$\begin{aligned}c_{1,1}(4, \alpha, \chi) &= (-1) \left\{ M_{0,0}(4, \xi^\alpha, \chi) + 2M_{0,1}(4, \xi^\alpha, \chi) + 2M_{1,0}(4, \xi^\alpha, \chi) \right. \\ &\quad \left. + 4M_{1,1}(4, \xi^\alpha, \chi) \right\}, \\ c_{2,1}(4, \alpha, \chi) &= -i \left\{ M_{0,0}(4, \xi^\alpha, \chi) + 2M_{0,1}(4, \xi^\alpha, \chi) + 4M_{1,0}(4, \xi^\alpha, \chi) \right. \\ &\quad \left. + 8M_{1,1}(4, \xi^\alpha, \chi) + 4M_{2,0}(4, \xi^\alpha, \chi) + 8M_{2,1}(4, \xi^\alpha, \chi) \right\}.\end{aligned}\tag{16}$$

Observe that $M_{0,0}(4, \xi^\alpha, \chi) = \phi(m)$ if χ is principal, and 0 otherwise. By Theorem 10 together with the help of (14), we have

$$\begin{aligned}M_{1,0}(4, \xi^\alpha, \chi) &= \overline{\psi}(4)\mathcal{A}_{1,1,\chi}, \\ M_{0,1}(4, \xi^\alpha, \chi) &= \mathcal{A}_{1,\xi^\alpha,\chi}, \\ M_{2,0}(4, \xi^\alpha, \chi) &= -\overline{\chi}(4) \left(\mathcal{A}_{1,1,\chi} + \frac{1}{2}\mathcal{A}_{2,1,\chi} \right).\end{aligned}$$

Plugging these values back into (16), the assertions (iii), (iv) follow from (i), (ii). $\square$

We note that Proposition 13 specializes to [24, Proposition 14] when $\lambda = 1$ and $\alpha = 0$ by using Theorem 2.

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