



VARIATIONS ON SHELDON PRIMES

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Abstract

A *Sheldon prime* is a prime p_n such that the product of the digits of p_n (in base 10) is the index n of p_n in the sequence of prime numbers and such that, furthermore, the reversal of p_n (also in base 10) is the prime p_m , where m is the reversal of n in base 10. It was recently proved by Pomerance and Spicer that 73 is the only Sheldon prime. We generalize this notion by allowing possibly different bases b, b' such that the product of the digits of p_n in base b is n , the reversal of p_n in base b is p_m , and m and n are reversals of each other in base b' . We present an heuristic that there should be infinitely many quadruples (b, b', p_n, p_m) with the above properties and present some examples.

– *To Professors Mel Nathanson and Carl Pomerance at their 80th birthdays*

1. Introduction

Let p_n be the n th prime. The lead character Sheldon Cooper of the sitcom *The Big Bang theory* claimed that 73 is a perfect prime since it has the following properties:

- (1) $73 = p_{21}$ and $7 \cdot 3 = 21$;
- (2) $37 = p_{12}$, 37 is the reversal of 73 and 12 is the reversal of 21.

Call the first property the *Product property* and the second property the *Reversal property*. Pomerance and Spicer [1] proved that 73 is the only prime with the above two properties. They also proved that given a base $b \geq 2$, primes $p = p_n$ such that n is the product of the digits of p_n in base b are bounded in terms of b . The challenge that Pomerance and Spicer had to deal with is that for the base $b = 10$, the bound

is quite large (10^{45}) so brute force computations are not feasible to exhaust the above range and therefore clever arguments based on smooth numbers and explicit prime number estimates had to come to the rescue.

Here, we attempt to extend this to other bases. We propose the following. Let $b \geq 2$, $b' \geq 2$ be integers and then ask that for pairs of primes p_n , p_m such that Property 1 and Property 2 above hold for the bases b and b' , respectively. That is, we want primes p_n , p_m such that

- (1) the product of the digits of p_n in base b is n ;
- (2) the reversal of the number p_n in base b is the prime p_m which must fulfill the additional condition that m is the reversal of n in base b' .

When additionally $b = b'$, we say that b is a *perfect Sheldon base*. Otherwise, we say that the pair (b, b') is a *Sheldon amicable pair*.

2. Examples

Some computations reveal the following Sheldon amicable pairs (together with the examples of the primes p_n , p_m):

$$\begin{aligned}
 (b, b', n, m) = & (10, 10, 21, 12), \\
 & (38, 59, 119, 61), \\
 & (94, 31, 255, 225), \\
 & (772, 188, 1729, 6965), \\
 & (25463, 2532, 67056, 3099194), \\
 & (46164, 128875, 128945, 9021251) \\
 & (531, 127, 3074, 3326), \\
 & (81, 36, 616, 161), \\
 & (115935, 27656, 554794, 46296164), \\
 & (5152, 990, 40755, 163391), \\
 & (211, 162, 3082, 667), \\
 & (154395, 8977, 2521606, 72229222), \\
 & (9757, 7003, 329206, 455242), \\
 & (6335, 24578, 565302, 196647), \\
 & (3155, 1155, 390438, 55778), \\
 & (41774, 18154, 5755137, 5791443).
 \end{aligned}$$

Let us test the 13th one. For $n = 329206$, $m = 455242$, we have

$$p_{329206} = 4703557 = (482)(683)_{9757}, \quad \text{and} \quad 482 \times 683 = 329206,$$

so property 1 holds. As for property 2, we have

$$\begin{aligned} p_{455242} &= (683)(482)_{9757} = 6664513, & \text{while also} \\ n &= 329206 = (47)(65)_{7003}, \text{ and } m = 455242 = (65)(47)_{7003}, \end{aligned}$$

so property 2 holds. Note that if b is given then there are only finitely many n such that p_n has property 1 in base b by the result of Pomerance and Spicer. Reverting such p_n in base b we get finitely many possibilities for p_m and certainly finitely many possibilities for b' such that the reversal of n in base b' is m . So, if we want infinitely many examples we should allow both b and b' to vary.

So, we wonder if there are infinitely many Sheldon amicable pairs (b, b') .

3. A Conjecture and an Heuristic

Conjecture 1. *There are infinitely many Sheldon amicable pairs.*

Next we offer some heuristic for the above conjecture. In fact, we shall provide evidence to the fact that

$$\#\{b \leq T : (b, b') \text{ is a Sheldon pair for some } b' > b\} \gg \log \log T. \quad (1)$$

Let us get to work. Let x be a large integer. Let

$$I := [\log \log x, 2 \log \log x].$$

Consider integers $a \in I$ and let c be an integer in the interval

$$\left[\frac{x}{a}, \frac{2x}{a} \right].$$

Then $n = ac \in [x, 2x]$. Consider the primes p_n for such n . We assume that there is b such that the base b representation of p_n is $(ac)_b$. Thus, $p_n = ab + c$. For this, we first want c to be coprime to a . Let $r \in \{1, \dots, a-1\}$ be any residue class modulo a such that $c \equiv r \pmod{a}$. Asking for c to be congruent to r modulo a in $[x/a, 2x/a]$ yields $(1 + o(1))x/a^2$ such values of c . For each such c , the prime p_{ac} should be congruent to c modulo a ; hence, r modulo a . By Dirichlet's theorem on primes in progressions, the number of primes p_n for $n \in [x, 2x]$ which are in a fixed residue class modulo a is the proportion $1/\phi(a)$ of the total number of such primes, which is asymptotic to x . We assume that this is also the case when restricted to indices

n which are of the form ac with $c \equiv r \pmod{a}$. Thus, we assume that the number of $c \in [x/a, 2x/a]$ such that $c \equiv r \pmod{a}$ and $p_{ac} \equiv r \pmod{a}$ is

$$(1 + o(1)) \frac{x}{a^2 \phi(a)} \quad \text{as} \quad x \rightarrow \infty.$$

We now sum up over all the $\phi(a)$ residue classes $r \pmod{a}$ coprime to a to get that the number of such n 's is

$$(1 + o(1)) \frac{x}{a^2}.$$

For such a prime write

$$p_n = p_{ac} = ab + c.$$

Here, $b = (p_n - c)/a = (p_n - n/a)/a$. We have, by the Prime Number theorem,

$$b \sim \frac{p_n}{a} \sim \frac{n \log n}{a} \sim c \log c \in \left(\frac{x \log x}{3 \log \log x}, \frac{2x \log x}{\log \log x} \right) \quad \text{as} \quad x \rightarrow \infty.$$

Then the reverse of p_n in base b is

$$\text{rev}_b(p_n) = cb + a.$$

Asymptotically,

$$\text{rev}_b(p_n) \sim cb \sim c^2 \log c.$$

The probability that this is prime is, by the Prime Number Theorem,

$$\frac{1 + o(1)}{\log \text{rev}_b(p_n)} \sim \frac{1 + o(1)}{2 \log x} \quad \text{as} \quad x \rightarrow \infty.$$

Thus, we assume that for fixed a the number of c such that $p_{ac} \equiv c \pmod{a}$ and in addition with b given by the equation $p_{ac} = ab + c$, we have that $\text{rev}_b(p_n)$ is also prime, is

$$\gg \frac{x}{a^2 \log x}.$$

The constant in the implied inequality can be chosen to be $1/3$ for large x . Now write

$$p_m = cb + a.$$

Then, again by the Prime Number theorem,

$$m \log m \sim p_m \sim cb \sim c^2 \log c, \quad \text{so} \quad \log m \sim 2 \log c \quad \text{as} \quad x \rightarrow \infty.$$

It follows that

$$m \sim \frac{c^2}{2} \in \left(\frac{x^2}{20(\log \log x)^2}, \frac{3x^2}{(\log \log x)^2} \right)$$

for large x . Now we want b' such that

$$n = ub' + v \quad \text{and} \quad m = vb' + u.$$

Certainly, $m > n$ for large x . In order to insure that such a representation exists we see that in case it does then

$$m + n = (b' + 1)(v + u) \quad \text{and} \quad m - n = (b' - 1)(v - u).$$

The left-hand sides $m + n$ and $m - n$ are in the interval

$$\left(\frac{x^2}{21(\log \log x)^2}, \frac{4x^2}{(\log \log x)^2} \right) \quad \text{for large } x.$$

We choose

$$b' \in J := \left(\frac{10x}{\log \log x}, \frac{11x}{\log \log x} \right).$$

With such a b' , we have $(b')^2 > 100x^2/(\log \log x)^2 > m > n$, so m and n have at most 2 digits in base b' . Also for such b' , if $n + m \equiv 0 \pmod{b' + 1}$ and $m - n \equiv 0 \pmod{b' - 1}$, then we get that

$$\frac{m + n}{b' + 1} \in \left(\frac{0.003x}{\log \log x}, \frac{0.4x}{\log \log x} \right),$$

for large x , which shows that $v + u = (m + n)/(b' + 1)$ is smaller than b' for large x . So, both u and v are smaller than b' . It remains to show that u, v resulting from

$$u + v = \frac{n + m}{b' + 1}, \quad v - u = \frac{m - n}{b' - 1}$$

are positive integers. This is so if we choose b' to be even since then $b' + 1, b' - 1$ are both odd while $m + n$ and $m - n$ have the same parity, so the two integers $(m + n)/(b' + 1), (m - n)/(b' - 1)$ have the same parity, showing that

$$\frac{(m + n)/(b' + 1) + (m - n)/(b' - 1)}{2} = u$$

and

$$\frac{(m + n)/(b' + 1) - (m - n)/(b' - 1)}{2} = v$$

are integers. Finally, it remains to show that v is positive. But note that

$$\frac{m + n}{b' + 1} - \frac{m - n}{b' - 1} = \frac{m}{b' + 1} - \frac{m}{b' - 1} + \frac{n}{b' + 1} + \frac{n}{b' - 1} = \frac{n}{b' + 1} + \frac{n}{b' - 1} - \frac{2m}{b'^2 - 1}.$$

Since $n \in [x, 2x]$, it follows that $n/(b' + 1), n/(b' - 1)$ are both $\gg \log \log x$ while on the other hand $m/(b'^2 - 1) = O(1)$, so the expression in the right-hand side above is positive for large x . Hence, it is enough to choose x large, then $b' \in J$ to be even and ask for m and n to be such that both congruences

$$n + m \equiv 0 \pmod{b' + 1} \quad \text{and} \quad m - n \equiv 0 \pmod{b' - 1}$$

hold. And if these congruences hold then indeed $n = \text{rev}_{b'}(m)$. The probability that a pair of random numbers are such that the first one is a multiple of $b' + 1$ and the second one is a multiple of $b' - 1$ is

$$\frac{1}{b'^2 - 1}.$$

So, we may assume that the count of $c \in [x/a, 2x/a]$ leading to n and m with the above properties is

$$\gg \frac{x}{a^2(\log x)(b'^2 - 1)}.$$

We sum up the above expressions over all $a \in I$ and even b' in J resulting a count of

$$\gg \frac{x}{\log x} \sum_{a \in I} \frac{1}{a^2} \sum_{\substack{b' \in J \\ b' \equiv 0 \pmod{2}}} \frac{1}{b'^2 - 1} \gg \frac{x}{\log x} \times \frac{1}{\log \log x} \times \frac{\log \log x}{x} \gg \frac{1}{\log x}.$$

In the above we used that if $L := (u, \kappa u)$ is an interval where $\kappa > 1$ is a constant and u tends to infinity then

$$\sum_{\ell \in L} \frac{1}{\ell^2} \geq (1 + o(1)) \int_u^{\kappa u} \frac{dt}{t^2} = \left(1 - \frac{1}{\kappa}\right) \frac{1}{u} \gg_{\kappa} \frac{1}{u} \quad \text{as } u \rightarrow \infty.$$

This holds for us for I with $\kappa := 2$ and for the interval of even numbers in J with $\kappa := 11/10$, respectively.

This was only for $n \in [x, 2x]$. We now give x the values $2, 4, 8, \dots, 2^k$ obtaining that the number of such examples up to $T := 2^K$ should be

$$\gg \sum_{k=1}^K \frac{1}{\log(2^k)} \gg \sum_{k=1}^K \frac{1}{k} \gg \log K \gg \log \log T$$

as promised in Equation (1).

To conclude, we say a few words about the examples from Section 2. We ran a Mathematica code which did the following. Let $a, c \in [1, 10000]$ be such that $\gcd(a, c) = 1$. We put $n := ac$ and with $p := p_n$, we test if $b := (p - c)/a$ is an integer greater than or equal to 2 and if $q := cb + a$ is a prime p_m . If the pair (a, c) passes these tests, then we look for $b' \geq 2$ with $b' + 1 \mid n + m$, $b' - 1 \mid |n - m|$ such that n is the reversal of m in base b' . The code ran for about half a day on the first author's laptop producing the list of examples from Section 2.

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References

- [1] C. Pomerance and C. Spicer, Proof of the Sheldon conjecture, *Amer. Math. Monthly* **126** (8) (2019), 688–698.