



MATCHINGS IN MATROIDS OVER ABELIAN GROUPS, III

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Abstract

This paper develops matroidal analogues of classical results on matchings in abelian groups. By embedding matroid ground sets in an abelian group, we introduce base matchings between matroid bases, recover the group-theoretic setting in the uniform matroid case, and derive structural and combinatorial criteria for their existence. Our main focus is on paving matroids. We prove self-matchability for paving matroids, extend asymmetric matchability results using the hyperplane-nullity parameter, and show that stressed hyperplanes provide a natural route to matchability through relaxation.

– Dedicated to Melvyn Nathanson and Carl Pomerance on their 80th birthdays

1. Introduction**1.1. Overview**

In an abelian group G , a *matching* is a bijection $f: A \rightarrow B$ between finite subsets $A, B \subseteq G$ such that $a + f(a) \notin A$ for all $a \in A$. A matching is called *symmetric* if $A = B$, and *asymmetric* otherwise. If there exists a matching from A to B , then A is said to be *matched* or *matchable* to B . We say that G has the *matching property* if, for every pair of finite subsets $A, B \subseteq G$ with $|A| = |B|$ and $0 \notin B$, there exists such a matching. A geometric model for a class of bipartite graphs was introduced by Fan and Losonczy [12]. In that work, a special kind of matching, called an acyclic matching, was defined and shown to exist for a subclass of these graphs by geometric methods. The existence of acyclic matchings implies the nonvanishing of the determinant of certain weighted biadjacency matrices. This matching framework

was later applied to a question posed by E. K. Wakeford in 1920 [26], concerning which sets of monomials can be eliminated from a generic homogeneous polynomial by linear changes of variables. In a special case related to canonical forms of symmetric tensors, the authors of [12] reduced Wakeford's problem to the existence of acyclic matchings in $\mathbb{Z}^n$. This property was established for $\mathbb{Z}^n$ by Alon, Fan, Kleitman, and Losonczy [8], further developed in [1] and [5], and later completely characterized for all abelian groups in [4].

Matchings were extended to the abelian group setting in [19] and to non-abelian groups in [10], with further developments, particularly on enumerative aspects, presented in [15]. A linear formulation for field extensions was introduced in [11]. More recently, matchable sets and subspaces were characterized via Dyson transforms in [3]. A matroidal analogue was proposed in [7], and additional progress on matchable matroids appears in [6]. In particular, [6] and [7] investigated matchability for the following four classes of matroids.

- sparse paving matroids,
- transversal matroids,
- panhandle matroids,
- Schubert matroids.

In this paper, we continue this line of research by studying the matchability of paving matroids. Following the terminology introduced in [19], we present the definition of matchings in abelian groups. Throughout, we use the notation $[n] := \{1, \dots, n\}$ for any positive integer n . We also write $(G, +)$ for an abelian group with neutral element 0. Let $p(G)$ represent the smallest cardinality of a nontrivial (i.e., non-zero) finite subgroup of G . If G has no nontrivial finite subgroup, we make the convention $p(G) = \infty$. Characterizations of abelian groups satisfying the matching property, as well as necessary and sufficient conditions for the existence of symmetric matchings, were obtained by Losonczy.

Theorem 1 ([19]). *Let G be an abelian group, and let A be a nonempty finite subset of G . Then, there exists a matching from A to itself if and only if $0 \notin A$.*

Theorem 2 ([19]). *An abelian group G satisfies the matching property if and only if G is torsion-free or cyclic of prime order.*

Beyond these characterizations, sufficient conditions for matchings between sufficiently small finite subsets of abelian groups were obtained in [2].

Theorem 3 ([2]). *Let G be an abelian group, and let A and B be finite subsets of G with $0 \notin B$ and $|A| = |B| < p(G)$, where $p(G)$ represents the smallest cardinality of a nontrivial finite subgroup of G . Then, there exists a matching from A to B .*

While there are meaningful connections between the notion of matchings explored in this paper and the classical matching theory in graphs introduced by Philip Hall in 1935 [14], the two concepts are fundamentally distinct. In particular, in order to address a linear algebra problem posed by Wakeford [26], Fan [12] constructed a bipartite graph

$$\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$$

from two sets $A, B \subseteq \mathbb{Z}^n$. To each element $a \in A$ one associates a symbol x_a , and to each element $b \in B$ a symbol y_b . Let

$$X = \{x_a \mid a \in A\} \quad \text{and} \quad Y = \{y_b \mid b \in B\}.$$

The graph $\mathcal{G}$ is then defined to have vertex set

$$V(\mathcal{G}) = X \cup Y,$$

where an edge joins $x_a \in X$ to $y_b \in Y$ if and only if

$$a + b \notin A.$$

In graph theory, a *matching* is a set of edges with no shared endpoints, and a *perfect matching* is one that covers all vertices. In the bipartite graph with bipartition sets $V(\mathcal{G}) = X \cup Y$, indexed by A and B , perfect matchings correspond to bijections $f : A \rightarrow B$ such that $a + f(a) \notin A$ for every $a \in A$. Wakeford's problem [26] thus becomes equivalent to determining the existence of such a bijection, referred to as an *acyclic matching* from A to B .

It is worth noting that, in the classical graph-theoretic setting, a perfect matching merely requires that every vertex is incident to exactly one edge, without reference to a bijection $f : A \rightarrow B$ or to the additional condition $a + f(a) \notin A$. By contrast, the notion of matchings in abelian groups incorporates this extra structure in order to capture the linear-algebraic framework underlying Wakeford's problem.

1.2. Matching in Matroids

We begin with the definition of a matroid. The reader is referred to [21] for all undefined terms.

Definition 4. A *matroid* M is a pair $(E, \mathcal{I})$, where $E = E(M)$ is a finite *ground set* and $\mathcal{I}$ is a family of subsets of E , called *independent sets*, satisfying the following conditions. First, $\emptyset \in \mathcal{I}$. Second, if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$. Finally, if $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there exists $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$. The last condition is called the *augmentation property*.

Let $M = (E, \mathcal{I})$ be a matroid. The *rank* of a subset $X \subseteq E$ is given by

$$r_M(X) = r(X) = \max\{|X \cap I| : I \in \mathcal{I}\}.$$

The *rank* of a matroid M , denoted by $r(M)$, is defined to be $r_M(E(M))$. A set $X \subseteq E$ is called *dependent* if it is not independent. A maximal independent set is called a *basis*. It follows from the augmentation property that every two bases have the same size. A *matroid basis system* is a nonempty family $\mathcal{B} \subseteq 2^E$ of subsets, that satisfies the exchange axiom. For all distinct $B, B' \in \mathcal{B}$ and every $x \in B \setminus B'$, there exists $y \in B' \setminus B$ such that $(B \setminus \{x\}) \cup \{y\} \in \mathcal{B}$. It follows that $\mathcal{B}$ is the set of bases of a matroid; see Kung's survey [18]. A minimal dependent set is called a *circuit*. An element that forms a single-element circuit of M is called a *loop*. A matroid is called *loopless* if it does not have a loop.

Given a matroid $M = (E, \mathcal{I})$, the *dual matroid* $M^* = (E, \mathcal{I}^*)$ is defined so that the bases in M^* are exactly the complements of the bases in M . A matroid $M = (E, \mathcal{I})$ of rank n is said to be a *paving matroid* if every $(n - 1)$ -subset of E is independent. If M and M^* are both paving matroids, then M is called a *sparse paving matroid*. A *flat* in M is a set $F \subseteq E$ with the property that adjoining any new element to F strictly increases its rank. A flat of rank $r(M) - 1$ is called a *hyperplane*. A set $H \subseteq E(M)$ is called a *circuit-hyperplane* of M if it is at the same time a circuit and a hyperplane. For any $X \subseteq E$ the difference $|X| - r(X)$ is called the *nullity* of X and is denoted by $\text{null}(X)$. Let us denote the set of all hyperplanes of M by $\mathcal{H}_M$. Then, the *hyperplane nullity* of M is denoted by $\text{null}(\mathcal{H}_M)$ and defined as

$$\text{null}(\mathcal{H}_M) = \max\{\text{null}(H) : H \in \mathcal{H}_M\}.$$

Following [13], a hyperplane H of a matroid M is said to be *stressed* if every $(r(M) - 1)$ -subset of H is independent.

For integers m and n with $0 \leq n \leq m$, the *uniform matroid* $U_{n,m}$ is defined on a ground set E of size m such that a subset $A \subseteq E$ is independent if and only if $|A| \leq n$. Equivalently, the bases of $U_{n,m}$ are exactly the subsets of E of size n . Note that uniform matroids are a very special class of sparse paving matroids. A uniform matroid $U_{n,m}$ is called a *free matroid* if $m = n$.

If $X \subseteq E$, the *closure* of X is denoted by $\text{cl}(X)$ and defined as follows.

$$\text{cl}(X) = \{x \in E : r(X) = r(X \cup \{x\})\}.$$

The closure operator satisfies the following properties:

- For all subsets X of E , we have $X \subseteq \text{cl}(X)$;
- If X and Y are subsets of E with $X \subseteq Y$, then $\text{cl}(X) \subseteq \text{cl}(Y)$;
- If $r(X) = r(M) - 1$, then $\text{cl}(X)$ is a hyperplane of M .

Following [7] we introduce the definition of matchings in matroids over an abelian group. We may assume that all matroids are loopless (see Remark 5.1 in [7]). We say that $M = (E, \mathcal{I})$ is a *matroid over G* if E is a subset of G .

Definition 5. Let $(G, +)$ be an abelian group. Let M and N be two matroids over G with $r(M) = r(N) = n > 0$. Let

$$\mathcal{M} = \{a_1, \dots, a_n\} \quad \text{and} \quad \mathcal{N} = \{b_1, \dots, b_n\}$$

be ordered bases of M and N , respectively. We say that $\mathcal{M}$ is *matched* to $\mathcal{N}$ if

$$a_i + b_i \notin E(M) \quad \text{for all } i \in [n].$$

We say that M is *matched* to N if, for every basis $\mathcal{M}$ of M , there exists a basis $\mathcal{N}$ of N such that $\mathcal{M}$ is matched to $\mathcal{N}$.

Adopting the notation from Definition 5, suppose that $\mathcal{M}$ is matched to $\mathcal{N}$ as ordered matroid bases. Then, for each $1 \leq i \leq n$, we have $a_i + b_i \notin E(M)$. In particular, $a_i + b_i \notin \mathcal{M}$. Therefore, the map $a_i \mapsto b_i$ defines a matching, in the group-theoretic sense, from the subset $\mathcal{M}$ to the subset $\mathcal{N}$ of G . Thus, the definition of matchings in matroids is compatible with the classical notion of matchings in abelian groups.

The notion of matchings in matroids given in Definition 5 is consistent with the notion of matchability for vector spaces over field extensions introduced by Eliahou and Lecouvey [11]. Let $K \subseteq F$ be a field extension, and let $A, B \subseteq F$ be n -dimensional K -vector subspaces. Suppose $\mathcal{A} = \{a_1, \dots, a_n\}$ and $\mathcal{B} = \{b_1, \dots, b_n\}$ are ordered bases of A and B , respectively. Then, $\mathcal{A}$ is said to be *matched* to $\mathcal{B}$ if

$$a_i^{-1}A \cap B \subseteq \langle b_1, \dots, \hat{b}_i, \dots, b_n \rangle \quad \text{for every } i \in [n], \quad (1)$$

where $\langle b_1, \dots, \hat{b}_i, \dots, b_n \rangle$ denotes the K -vector subspace spanned by $\mathcal{B} \setminus \{b_i\}$. The space A is said to be *matched* to B if every basis $\mathcal{A}$ of A is matched to some basis $\mathcal{B}$ of B in this way.

Now let $G = F \setminus \{0\}$ be the multiplicative group of F , and define matroids over G by setting $M = (A \setminus \{0\}, \mathcal{I})$ and $N = (B \setminus \{0\}, \mathcal{I}')$, where $\mathcal{I}$ and $\mathcal{I}'$ are the families of linearly independent subsets of A and B , respectively. Then

$$r(M) = \dim_K A = n = \dim_K B = r(N).$$

Suppose the basis $\mathcal{A}$ is matched to $\mathcal{B}$ in the vector space setting. It follows from Equation (1) that $a_i b_i \notin A$ for all i , and hence A is also matched to B in the matroid-theoretic sense.

In this paper we broaden the scope of matroid matchability in three directions. First, we extend the symmetric paving case by proving that any paving matroid is matched to itself under suitable size conditions (Theorem 15). Second, we develop an asymmetric extension in which matchability is controlled by the hyperplane-nullity parameter t , generalizing the sparse paving setting (Theorem 21). Finally, we show that stressed hyperplanes provide a natural route to matchability in relaxations of paving matroids (Theorem 31).

The point of passing from sparse paving to paving matroids is not merely to enlarge the class of matroids for which matching results are known. In sparse paving matroids (with rank n), the only dependent n -sets are circuit-hyperplanes, so the obstruction is essentially binary. In other words, the failure of an n -subset to be a basis is controlled by a single type of obstruction, namely circuit-hyperplanes. By contrast, in paving matroids, hyperplanes may have larger nullity, and our results show that matchability is controlled by this richer geometric parameter. In this way, the paper shows that the key mechanism governing matroid matchings is the structure of hyperplanes, rather than near-uniformity alone.

The paper is organized as follows. Section 2 reviews preliminaries on paving and sparse paving matroids, including their characterization via partitions and hyperplane nullity, as well as additive tools such as Kneser’s theorem and results on critical pairs. Section 3 presents the main theorems. Subsection 3.1 treats the symmetric case, Subsection 3.2 develops the asymmetric theory incorporating hyperplane nullity, and Subsection 3.3 introduces stressed hyperplanes and relaxation, connecting paving matroids to uniform matroids and demonstrating how these ideas ensure broader matchability.

2. Preliminaries

2.1. Notions and Results on Paving Matroids

A d -partition of a finite set E is a collection $\mathcal{S}$ of subsets of E , each of cardinality at least d , such that the intersection of any two distinct members of $\mathcal{S}$ has cardinality at most $d - 1$. The collection $\mathcal{S} = \{E\}$ is a d -partition for every d ; we call it the *trivial d -partition*. The connection between paving matroids and d -partitions is given by the following result, whose proof can be found in [21].

Theorem 6 ([21]). *If M is a paving matroid of rank $n \geq 2$, then its hyperplanes form a non-trivial $(n - 1)$ -partition of $E(M)$. That is, the intersection of any two distinct hyperplanes in M is of size at most $n - 2$.*

The following result provides an alternative characterization of sparse paving matroids.

Theorem 7 ([21]). *A matroid M of rank n is a sparse paving matroid if and only if every n -subset of $E(M)$ is either a basis or a circuit-hyperplane.*

Returning to the nullity of hyperplanes, note that if M is a uniform matroid, then clearly $\text{null}(\mathcal{H}_M) = 0$. In other words, hyperplane nullity measures how far a matroid deviates from being uniform. In the more general case where $\text{null}(\mathcal{H}_M) \leq 1$, we obtain the following characterization, which follows directly from Theorem 7 and highlights the role of hyperplane nullity in identifying sparse paving matroids.

Lemma 8. *A paving matroid M is sparse paving if and only if $\text{null}(\mathcal{H}_M) \leq 1$.*

Proof. Suppose first that M is a sparse paving matroid. Let H be a hyperplane of M . Clearly $|H| \geq n - 1$. If $|H| = n - 1$, then H is independent and hence $\text{null}(H) = 0$. If $|H| > n - 1$, then H contains an n -subset H' . By Theorem 7, H' is itself a hyperplane, and thus $H = H'$, implying $|H| = n$. In this case, $\text{null}(H) = 1$. Therefore, in either case we have $\text{null}(\mathcal{H}_M) \leq 1$.

Conversely, assume that $\text{null}(\mathcal{H}_M) \leq 1$. Let A be an n -subset of $E(M)$ that is not a basis. We claim that A is a circuit-hyperplane. Since M is paving, A is a circuit with $r(A) = n - 1$. Consider the hyperplane $H = \text{cl}(A)$. Then $\text{null}(H) \leq 1$. We have

$$1 = |A| - r(A) \leq |H| - r(H) = \text{null}(H) \leq \text{null}(\mathcal{H}_M) \leq 1.$$

Hence, all inequalities are equalities, which forces $A = H$. Thus, A is a hyperplane. Since A is a circuit-hyperplane, by Theorem 7, M is sparse paving. $\square$

Corollary 9. *Let M be a paving matroid. Then M is uniform if and only if $\text{null}(\mathcal{H}_M) = 0$, and M is non-uniform and sparse paving if and only if $\text{null}(\mathcal{H}_M) = 1$.*

Proof. The result follows immediately from Lemma 8. $\square$

2.2. Abelian Additive Theory

The main additive number theory tools used in the proofs of our main results are Theorem 10 and Lemma 13. Let G be an abelian group and $A, B \subseteq G$. The sumset $A + B$ is $A + B = \{a + b : a \in A, b \in B\}$. We start with a result of Kneser providing a lower bound on the size of $A + B$.

Theorem 10 ([20, Theorem 4.3]). *For an abelian group G and finite nonempty subsets $A, B \subseteq G$, the sumset $A + B$ satisfies*

$$|A + B| \geq |A| + |B| - |H|,$$

where

$$H = \text{Stab}(A + B) = \{g \in G : (A + B) + g = A + B\}$$

is the stabilizer of $A + B$.

Let G be an abelian group, let $x, a \in G$, and let k be a positive integer. A progression of length k with difference x and initial term a is a subset of G of the form $\{a, a + x, a + 2x, \dots, a + (k - 1)x\}$. We say that a set $A \subseteq G$ is a *semi-progression* if $A \setminus \{a\}$ is a progression for some $a \in A$. In the following lemma due to Kemperman, a certain family of critical pairs is characterized. Note that a pair (A, B) of finite subsets of a group G is called *critical* if $|G| > |A + B| = |A| + |B| - 1$.

Lemma 11 ([17]). *Let (A, B) be a critical pair of a finite abelian group G with $|A| > 1$, $|B| > 1$, and $|A| + |B| - 1 \leq p(G) - 2$, where $p(G)$ represents the smallest cardinality of a nontrivial finite subgroup of G . Then, A and B are progressions with the same difference.*

We next recall a classical theorem of Kemperman [16]. It provides a lower bound for a sumset containing an element with a unique representation.

Theorem 12 ([16]). *Let A and B be nonempty finite subsets of a group G . Assume there exists an element $c \in A + B$ appearing exactly once as a sum $c = a + b$ with $a \in A, b \in B$. Then*

$$|A + B| \geq |A| + |B| - 1.$$

The following lemma is an immediate consequence of Theorem 12.

Lemma 13. *Let A and B be nonempty finite subsets of a group G such that A , B , and $A + B$ are all contained in a subset X of $G \setminus \{0\}$. Then*

$$|X| \geq |A| + |B| + 1.$$

Proof. Define $A_0 = A \cup \{0\}$ and $B_0 = B \cup \{0\}$. Then, $0 \in A_0 + B_0$ and appears exactly once as a sum in $A_0 + B_0$. That is, if $0 = a + b$ with $a \in A$ and $b \in B$, since $0 \notin A + B$, then $a = 0$ or $b = 0$, and hence $a = b = 0$. Applying Theorem 12 to A_0 and B_0 we have

$$|A_0 + B_0| \geq |A_0| + |B_0| - 1.$$

Set $X = (A + B) \cup A \cup B$. Since $|A_0| = |A| + 1$, $|B_0| = |B| + 1$, and $A_0 + B_0 = (A + B) \cup A \cup B$, we have

$$|X| = |A_0 + B_0| \geq |A| + |B| + 1,$$

as claimed. □

3. Main Results

Throughout, all matroids are assumed to be loopless. In this section, we investigate matchability within paving matroids. Although paving matroids possess a seemingly restrictive structure, Crapo and Rota [9] conjectured, and it is now widely believed, that asymptotically almost all matroids are paving. While this conjecture remains unresolved, it is supported by strong asymptotic estimates [22, 23]. What is certain, however, is that paving matroids constitute a large and significant class of matroids.

Note that matchability in the group setting can be viewed as a special case of matchability in the matroid setting. Let A and B be two finite nonempty subsets

of an abelian group G , each of cardinality n , and assume $0 \notin B$. Define a uniform matroid $M \cong U_{n,n}$ on the ground set A , and similarly define $N \cong U_{n,n}$ on the ground set B . Then, A is matched to B in the group-theoretic sense if and only if M is matched to N in the matroid-theoretic sense. Note that uniform matroids form a very narrow subclass of all matroids, and $U_{n,n}$ represents a particularly special case.

The importance of paving matroids is not only that they extend sparse paving matroids, but that they make the structural source of matchability more visible. Since the classical group-theoretic notion of matching is recovered in the uniform case, paving matroid matchability extends the original problem of matchings within groups. Passing to all paving matroids, therefore, helps separate features of matchability that depend on near-uniformity from those that persist in a richer dependence structure. Our results show that, in paving matroids, matchability is governed by geometric features such as hyperplane intersections, hyperplane nullity, and stressed hyperplanes. This both explains why the theory extends beyond the sparse paving case and places the classical group case within a broader structural framework. Since paving matroids can also be related back to uniform matroids through relaxation (Theorem 31), the group-theoretic setting appears not as an isolated phenomenon, but as the first instance of a more general theory of matchings.

3.1. Matching Paving Matroids over Abelian Groups (Symmetric Case)

Throughout, for an abelian group G , we let $p(G)$ denote the smallest cardinality of a nontrivial finite subgroup of G . The following result was proved in [7] for matchings in sparse paving matroids under the symmetric case, providing a matroidal version of Theorem 1.

Theorem 14 ([7]). *Let M be a sparse paving matroid over G . Then, M is matched to itself if and only if $0 \notin E(M)$.*

We now extend this to the paving matroid analogue of Theorem 14.

Theorem 15. *Let M be a paving matroid over G . Then, M is matched to itself if and only if $0 \notin E(M)$.*

Note that, for any pair of matroids M, N , the conditions $0 \notin E(N)$ and $r(M) = r(N)$ are necessary for M to be matched to N (see [7, Proposition 1.7]). Thus, the “only if” implication in Theorem 15 is immediate. We now establish the reverse implication.

Proof of Theorem 15. We assume that M is not a free matroid (the case when M is free is addressed in Remark 16). Since $0 \notin E(M)$, it follows from Theorem 1 that $E(M)$ is matched to itself in the group sense. That is, there exists a bijection $f : E(M) \rightarrow E(M)$ with $a + f(a) \notin E(M)$ for all $a \in E(M)$.

Let $\mathcal{M} = \{a_1, \dots, a_n\}$ be a basis of M . Set $f(a_i) := b_i$ and define $\mathcal{N} = \{b_1, \dots, b_n\}$. If $\mathcal{N}$ is a basis of M , then $\mathcal{M}$ is matched to $\mathcal{N}$ in the matroid sense and we are done. Otherwise, suppose that $\mathcal{N}$ is not a basis of M .

We claim that there exist $x \in E(M) \setminus \mathcal{N}$ and $i \in [n]$ such that $a_i + x \notin E(M)$. Let us suppose not. Then

$$\mathcal{M} + (E(M) \setminus \mathcal{N}) \subseteq E(M).$$

Since the three sets $\mathcal{M}$, $E(M) \setminus \mathcal{N}$, and $\mathcal{M} + (E(M) \setminus \mathcal{N})$ are contained in $E(M)$ and $0 \notin E(M)$, Lemma 13 yields

$$|E(M)| \geq |\mathcal{M}| + |E(M) \setminus \mathcal{N}| + 1 = |E(M)| + 1,$$

which is a contradiction. This completes the proof of the claim.

Now let x and i be as given as above. Without loss of generality, assume $i = n$ and define

$$\mathcal{P} = \{b_1, \dots, b_{n-1}, x\}.$$

We claim that $\mathcal{P}$ is a basis of M . Suppose, to the contrary, that $\mathcal{P}$ is not a basis of M . Since M is paving, we must have

$$r(\mathcal{P}) = n - 1.$$

Consider the hyperplanes $H_1 = \text{cl}(\mathcal{N})$ and $H_2 = \text{cl}(\mathcal{P})$. By Theorem 6, it follows that

$$|H_1 \cap H_2| \leq n - 2.$$

However, it is clear that $|H_1 \cap H_2| \geq n - 1$, since $b_i \in H_1 \cap H_2$ for all $i \in [n - 1]$. This contradiction proves that $\mathcal{P}$ is indeed a basis.

Finally, observe that the basis $\mathcal{M}$ is matched to the basis $\mathcal{P}$. Hence, M is matched to itself, completing the proof. $\square$

Remark 16. In the proof of Theorem 15, if M is a free matroid, then it has a unique basis, namely $E(M)$. Since $0 \notin E(M)$, Theorem 1 guarantees that $E(M)$ is matched to itself in the group sense. Consequently, the basis $E(M)$ is also matched to itself in the matroid sense. Hence, M is matched to itself.

Remark 17. The significance of Theorem 15 is that it shows the classical self-matching criterion from abelian groups (Theorem 1) is not confined to sets or to uniform matroids. Even in a paving matroid, where substantial dependence may occur, the only obstruction to symmetric self-matchability remains the trivial additive obstruction $0 \in E(M)$. Thus, the theorem identifies the paving condition as a structural setting in which matroidal dependence does not create new symmetric matching obstructions. In particular, it shows that the source of self-matchability lies not merely in near-uniformity, but in the hyperplane geometry of paving matroids.

Example 18. We construct a paving matroid over $\mathbb{Z}$ whose ground set does not contain 0, and then show that it is matched to itself by Theorem 15. To illustrate a case not covered by Theorem 14, let M be the rank-3 matroid on $E(M) = [5]$ whose bases are

$$\mathcal{B}(M) = \{\{i, j, 5\} \mid i, j \in [4], i \neq j\}.$$

The 3-circuits are

$$\mathcal{C}_3(M) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}.$$

Thus, M is paving but not sparse paving, since $\{1, 2, 3\}$ is a 3-subset that is neither a basis nor a hyperplane. Consider the group matching $f : [5] \rightarrow [5]$ given by

$$f(1) = 5, \quad f(2) = 4, \quad f(3) = 3, \quad f(4) = 2, \quad f(5) = 1.$$

For the bases $\{1, 2, 5\}$, $\{1, 3, 5\}$, and $\{1, 4, 5\}$, the images $\{5, 4, 1\}$, $\{5, 3, 1\}$, and $\{5, 2, 1\}$ are already bases of M , respectively. For the remaining bases, the replacement step in the proof of Theorem 15 gives

$$\{2, 3, 5\} \mapsto \{4, 3, 5\}, \quad \{2, 4, 5\} \mapsto \{5, 2, 1\}, \quad \{3, 4, 5\} \mapsto \{3, 5, 1\}.$$

Each displayed target is a basis of M , and each pairing satisfies the required condition $a_i + b_i \notin E(M)$. Hence, every basis of M is matched to a basis of M .

Example 19. In this example we give a non-paving matroid over an abelian group for which the construction used in the proof of Theorem 15 breaks down. Let $G = \mathbb{Z}$ and let

$$E = \{1, 3, 7, 9\}.$$

Define a matroid M of rank 3 on E by declaring its bases to be exactly

$$\mathcal{B}(M) = \{\{1, 7, 9\}, \{3, 7, 9\}\}.$$

It is immediate that this is a matroid. The collection $\mathcal{B}(M)$ is nonempty, and the basis exchange axiom is trivially satisfied since the only two bases differ by exchanging 1 and 3. Since the 2-subset $\{1, 3\}$ is contained in no basis, it is dependent. Hence, M is not paving. Now define a matching $f : E \rightarrow E$ in the group setting by

$$f(1) = 9, \quad f(3) = 1, \quad f(7) = 3, \quad f(9) = 7.$$

Consider the basis

$$\mathcal{M} = \{3, 7, 9\}.$$

Following the construction in the proof of Theorem 15, set

$$\mathcal{N} = f(\mathcal{M}) = \{1, 3, 7\}.$$

Then, $\mathcal{N}$ is not a basis of M , since $\mathcal{N} \notin \mathcal{B}(M)$. Now take $a_3 = 9$ and $x = 9 \in E \setminus \mathcal{N}$. Since

$$9 + 9 = 18 \notin E,$$

the analogue of the claim in the proof of Theorem 15 holds. Replacing the corresponding element 7 of $\mathcal{N}$ by $x = 9$, we obtain

$$\mathcal{P} = \{1, 3, 9\}.$$

Again, $\mathcal{P}$ is not a basis of M , since $\mathcal{P} \notin \mathcal{B}(M)$. Therefore, the replacement step used in the proof of Theorem 15 may fail for non-paving matroids. This shows that the paving assumption is needed in order to conclude that the modified set is again a basis.

3.2. Matching Paving Matroids over Abelian Groups (Asymmetric Case)

The following result about the matchability of sparse paving matroids in the asymmetric situation was proved in [7].

Theorem 20 ([7]). *Let M be a matroid over G and let N be a sparse paving matroid over G , both of rank n , with $0 \notin E(N)$. Assume further that one of the following conditions holds:*

- (i) $|E(M)| < \min |E(N)| - 1, p(G)$;
- (ii) *The set $E(M)$ is not a progression, the group G is finite, $|E(M)| = |E(N)| - 1$, and $|E(N)| < p(G)$;*
- (iii) *The set $E(M)$ is neither a progression nor a semi-progression, the group G is finite, and $|E(M)| = |E(N)| < p(G)$;*
- (iv) $|E(M)| < |E(N)| - n - 1$.

Then M is matched to N .

We now extend this result to general paving matroids by incorporating the hyperplane-nullity parameter. In particular, we generalize the sparse paving case ($t \leq 1$) to arbitrary paving matroids, with explicit dependence on the parameter t .

Theorem 21. *Let M be a matroid over G and let N be a paving matroid over G , both of rank n , with $0 \notin E(N)$. Let $t = \text{null}(\mathcal{H}_N)$, where $0 < t \leq n$. Assume further that one of the following conditions holds:*

- (i) $|E(M)| < \min |E(N)| - 2t + 1, p(G)$;
- (ii) *The set $E(M)$ is not a progression, the group G is finite, $|E(M)| = |E(N)| - 2t + 1$, and $|E(N)| < p(G)$;*

- (iii) The set $E(M)$ is neither a progression nor a semi-progression, the group G is finite, $|E(M)| = |E(N)| - 2t + 2$, and $|E(N)| < p(G)$;
- (iv) $|E(M)| < |E(N)| - n - 1$.

Then M is matched to N .

Proof. We begin by addressing the following three special cases.

Case a: M is a free matroid. Then $\mathcal{M} = E(M)$ is the unique basis of M . Let $\mathcal{N}$ be a basis of N . By Theorem 3, $\mathcal{M}$ is matched to $\mathcal{N}$ in the group-theoretic sense; that is, there exists a bijection $f : \mathcal{M} \rightarrow \mathcal{N}$ such that $a + f(a) \notin \mathcal{M}$ for every $a \in \mathcal{M}$. It follows that $\mathcal{M}$ is also matched to $\mathcal{N}$ in the matroid sense. Hence, M is matched to N .

Case b: $t = 1$. By Lemma 8, N is a sparse paving matroid. Therefore, Theorem 20 implies that M is matched to N .

Case c: $n = 1$. Let $\mathcal{M} = \{a\}$ be a basis of M . Since $|E(M)| < |E(N)|$, there exists $b \in E(N)$ such that $a + b \notin E(M)$. Set $\mathcal{N} = \{b\}$. Since N is loopless, $\mathcal{N}$ is a basis of N . Thus, $\mathcal{M}$ is matched to $\mathcal{N}$, and consequently M is matched to N .

We now assume that M is not a free matroid, that $t > 1$, and that $n > 1$. We first treat conditions (i), (ii), and (iii), postponing condition (iv) until later. The common feature of conditions (i), (ii), and (iii) is that they imply $|E(M)| < p(G)$. Choose a subset $A \subseteq E(N)$ such that $|A| = |E(M)|$. Since $|A| = |E(M)| < p(G)$, Theorem 3 guarantees the existence of a matching $f : E(M) \rightarrow A$ in the group-theoretic sense. Let $\mathcal{M} = \{a_1, \dots, a_n\}$ be a basis of M , and define $b_i := f(a_i)$ for all $i \in [n]$. Set $\mathcal{N} = \{b_1, \dots, b_n\} \subseteq E(N)$. Since N is a paving matroid of rank n , the set $\mathcal{N}$ is either a basis or a circuit. If $\mathcal{N}$ is a basis, then $\mathcal{M}$ is matched to $\mathcal{N}$ in the matroid sense, and hence M is matched to N , completing the proof in this case. If instead $\mathcal{N}$ is a circuit, we proceed by proving the claim that for every integer s with $s \in [t]$, there exist distinct elements $x_1, \dots, x_s \in E(N) \setminus \mathcal{N}$ and distinct indices $i_1, \dots, i_s \in [n]$ such that

$$a_{i_j} + x_j \notin E(M) \quad \text{for every } j \in [s].$$

In particular, the conclusion holds for $s = t$. We proceed by induction on s . For the base case $s = 1$, we show that there exist $x \in E(N) \setminus \mathcal{N}$ and $a \in \mathcal{M}$ such that $a + x \notin E(M)$. Suppose, for contradiction, that no such pair (a, x) exists. Then

$$\mathcal{M} + (E(N) \setminus \mathcal{N}) \subseteq E(M).$$

Applying Theorem 10 to the sets $\mathcal{M}$ and $E(N) \setminus \mathcal{N}$, we obtain a subgroup $H \leq G$

such that

$$|\mathcal{M} + (E(N) \setminus \mathcal{N})| \geq |\mathcal{M}| + |E(N) \setminus \mathcal{N}| - |H|, \quad (2)$$

$$\mathcal{M} + (E(N) \setminus \mathcal{N}) + H = \mathcal{M} + (E(N) \setminus \mathcal{N}). \quad (3)$$

Since $\mathcal{M} + (E(N) \setminus \mathcal{N}) \subseteq E(M)$, it follows from Equation (3) that

$$|H| \leq |\mathcal{M} + (E(N) \setminus \mathcal{N}) + H| = |\mathcal{M} + (E(N) \setminus \mathcal{N})| \leq |E(M)| < p(G),$$

so $H = \{0\}$. Substituting this into Equation (2) yields

$$\begin{aligned} |E(M)| &\geq |\mathcal{M}| + |E(N) \setminus \mathcal{N}| - 1 \\ &= n + (|E(N)| - n) - 1 \\ &= |E(N)| - 1 \\ &> |E(N)| - 2t + 1. \end{aligned} \quad (4)$$

We now consider statements (i), (ii), and (iii) separately. In (i), the condition

$$|E(M)| < \min\{|E(N)| - 2t + 1, p(G)\}$$

directly contradicts Equation (4). In (ii), the condition $|E(M)| = |E(N)| - 2t + 1$ directly contradicts Equation (4). In (iii), G is finite. Combining the assumption $|E(M)| = |E(N)| - 2t + 2$ with Equation (4), we obtain

$$|E(M)| \geq |\mathcal{M} + (E(N) \setminus \mathcal{N})| \geq |\mathcal{M}| + |E(N) \setminus \mathcal{N}| - 1 \geq |E(N)| - 2t + 2 = |E(M)|.$$

Therefore, we have

$$|\mathcal{M}| + |E(N) \setminus \mathcal{N}| - 1 = |\mathcal{M} + (E(N) \setminus \mathcal{N})|.$$

The pair $(\mathcal{M}, E(N) \setminus \mathcal{N})$ is critical and satisfies the conditions of Theorem 11 because:

- since $n > 1$, we have $|\mathcal{M}| > 1$;
- we have $|E(N) \setminus \mathcal{N}| = |E(N)| - n > 1$;
- we have $|\mathcal{M}| + |E(N) \setminus \mathcal{N}| - 1 = |E(N)| - 1 \leq p(G) - 2$.

Hence, both sets are progressions with the same common difference. Since

$$E(M) = \mathcal{M} + (E(N) \setminus \mathcal{N}),$$

it follows that $E(M)$ is also a progression, a contradiction.

Assume that distinct elements $x_1, \dots, x_{s-1} \in E(N) \setminus \mathcal{N}$ and distinct indices $i_1, \dots, i_{s-1} \in [n]$ have been chosen such that

$$a_{i_j} + x_j \notin E(M) \quad \text{for every } j \in [s-1].$$

Claim 22. There exists $x_s \in E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})$ and $i_s \in [n] \setminus \{i_1, \dots, i_{s-1}\}$ such that $a_{i_s} + x_s \notin E(M)$.

Suppose, for contradiction, that no such pair exists. Then

$$(E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})) + (\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}) \subseteq E(M). \quad (5)$$

Applying Theorem 10 to the sets

$$\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\} \quad \text{and} \quad E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\}),$$

we obtain a subgroup $H \leq G$ such that

$$\begin{aligned} & \left| (\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}) + (E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})) \right| \\ & \geq |\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}| + |E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})| - |H|, \\ & (\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}) + (E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})) + H \\ & = (\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}) + (E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})). \end{aligned} \quad (6)$$

The same argument used in the base case shows that $H = \{0\}$. Combining this with Equations (5) and (6), we conclude that

$$\begin{aligned} |E(M)| & \geq |(\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}) + (E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\}))| \\ & \geq |\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}| + |E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})| - 1 \\ & = |E(N)| - 2s + 1 \geq |E(N)| - 2t + 1. \end{aligned} \quad (7)$$

We now split the argument into three cases.

Case 1: Statement (i) holds. Then the condition

$$|E(M)| < \min\{|E(N)| - 2t + 1, p(G)\}$$

contradicts Equation (7).

Case 2: Statement (ii) holds. Then we have $|E(M)| = |E(N)| - 2t + 1$. From Equation (7) we conclude that

$$\begin{aligned} & \left| (\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}) + (E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})) \right| \\ & = |\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}| + |E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})| - 1. \end{aligned}$$

Therefore, the pair

$$(\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}, E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\}))$$

is a critical pair satisfying the conditions of Theorem 11, since first of all we have $|\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}| = n - (s - 1) \geq n - (t - 1) > 1$. Furthermore, since M is not a free matroid and $t > 1$, we have

$$\begin{aligned} |E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})| &= |E(N)| - (n + s - 1) \\ &\geq |E(N)| - (n + t - 1) \\ &= |E(M)| + 2t - 1 - (n + t - 1) \\ &> t > 1. \end{aligned}$$

Finally, we have

$$\begin{aligned} |\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}| + |E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})| - 1 \\ = |E(M)| \\ \leq |E(N)| - 1 \\ \leq p(G) - 2. \end{aligned}$$

By Theorem 11, both sets are progressions with the same common difference. Thus, we obtain

$$E(M) = (\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}) + (E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})),$$

so $E(M)$ is itself a progression, contradicting the assumption that $E(M)$ is not a progression.

Case 3: Statement (iii) holds. Then we have $|E(M)| = |E(N)| - 2t + 2$. From Equation (7) it follows that one of the following must hold:

$$|\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}| + |E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})| - 1 = |E(M)|,$$

or

$$|\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}| + |E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})| - 1 = |E(M)| - 1.$$

We now consider two subcases.

Subcase a: $|\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}| + |E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})| - 1 = |E(M)|$. Then, as in (ii), the pair

$$(\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}, E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\}))$$

is critical and satisfies the conditions of Theorem 11. Hence, both $\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}$ and $E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})$ are progressions with the same common difference. Since

$$E(M) = (\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}) + (E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})),$$

it follows that $E(M)$ is itself a progression, a contradiction.

Subcase b: $|\mathcal{M} \setminus \{a_{i_1}, \dots, a_{i_{s-1}}\}| + |E(N) \setminus (\mathcal{N} \cup \{x_1, \dots, x_{s-1}\})| - 1 = |E(M)| - 1$. Then $|E(M)| = |E(N)|$. On the other hand, we also know that $|E(M)| = |E(N)| - 2t + 2$, which forces $t = 1$, a contradiction.

Thus, the proof of the Claim 22 is complete. Now let $x_1, \dots, x_t$ and $i_1, \dots, i_t$ be obtained as above. Without loss of generality, we may assume that $i_j = j$ for every $j \in [t]$. That is, $a_j + x_j \notin E(M)$ for all $j \in [t]$. Set $X = \{x_1, \dots, x_t\}$ and $\mathcal{P} = X \cup \mathcal{N}$.

Claim 23. $r(\mathcal{P}) = n$.

Assume, for contradiction, that $r(\mathcal{P}) \neq n$. Since N is a paving matroid and $|\mathcal{P}| > n$, it follows that $r(\mathcal{P}) = n - 1$. Let $H = \text{cl}(\mathcal{P})$. Then, H is a hyperplane, and we have

$$t = \text{null}(\mathcal{H}_N) \geq \text{null}(H) = |H| - (n - 1) \geq |\mathcal{P}| - (n - 1) = (n + t) - (n - 1) = t + 1,$$

a contradiction. Hence, $r(\mathcal{P}) = n$, completing the proof of Claim 23. We now define $\mathcal{N}_0 = \mathcal{N}$ and consider the following n -subsets of $E(N)$.

$$\mathcal{N}_i = \{x_1, \dots, x_i, b_{i+1}, \dots, b_n\}, \quad \text{for all } i \in [t].$$

We claim that $\mathcal{N}_i$ is a basis of N for some $i \in [t]$. Suppose, for contradiction, that every $\mathcal{N}_i$ is not a basis. Since N is a paving matroid, this means $r(\mathcal{N}_i) = n - 1$. Set $H_i = \text{cl}(\mathcal{N}_i)$ for $0 \leq i \leq t$. Clearly, each H_i is a hyperplane of N . On the other hand,

$$|H_i \cap H_{i+1}| \geq |\mathcal{N}_i \cap \mathcal{N}_{i+1}| = n - 1.$$

Invoking Theorem 6, we conclude that $H_i = H_{i+1}$ for all $0 \leq i \leq t - 1$. Thus, we have

$$\mathcal{P} \subseteq \bigcup_{i=0}^t H_i = H_0,$$

implying $r(\mathcal{P}) \leq r(H_0) = n - 1$. This contradicts the fact that $r(\mathcal{P}) = n$. Hence, $\mathcal{N}_i$ is a basis of N for some $i \in [t]$. This completes the proof of the claim. Now assume that $i \in [t]$ is obtained as above. Then, $\mathcal{M}$ is matched to $\mathcal{N}_i$ via the assignment

$$a_j \mapsto x_j \quad \text{for } 1 \leq j \leq i, \quad a_k \mapsto b_k \quad \text{for } i + 1 \leq k \leq n.$$

Therefore, M is matched to N . This proves Theorem 21 under assumptions (i), (ii), and (iii).

It remains to consider statement (iv). Let $\mathcal{M} = \{a_1, \dots, a_n\}$ be a basis for M . We show that $\mathcal{M}$ is matched to a basis of N . We claim that there exist n distinct elements $b_1, b_2, \dots, b_n \in E(N)$ such that

$$a_i + b_i \notin E(M) \quad \text{for all } i \in [n].$$

We first choose $b_1 \in E(N)$ so that $a_1 + b_1 \notin E(M)$. Such b_1 exists since $|E(M)| < |E(N)|$. Next, choose $b_2 \in E(N) \setminus \{b_1\}$ such that $a_2 + b_2 \notin E(M)$. Such b_2 exists as $|E(M)| < |E(N)| - 1$. We proceed by induction on n . Suppose $b_1, \dots, b_j$ have been chosen. We show that there exists $b_{j+1} \in E(N) \setminus \{b_1, \dots, b_j\}$ satisfying $a_{j+1} + b_{j+1} \notin E(M)$. Assume to the contrary that no such b_{j+1} exists. Then

$$a_{j+1} + (E(N) \setminus \{b_1, \dots, b_j\}) \subseteq E(M),$$

which implies

$$|E(N) \setminus \{b_1, \dots, b_j\}| \leq |E(M)|.$$

Thus, we have

$$|E(M)| \geq |E(N)| - j \geq |E(N)| - n,$$

a contradiction. This proves the claim. Let $\mathcal{N} = \{b_1, \dots, b_n\}$, where $b_1, \dots, b_n$ are as above. Since N is a paving matroid, $\mathcal{N}$ is either a basis of N or a circuit of rank $n - 1$. If $\mathcal{N}$ is a basis, then $\mathcal{M}$ is matched to $\mathcal{N}$ and we are done. Suppose $\mathcal{N}$ is a circuit of rank $n - 1$. We claim that there exist $x \in E(N) \setminus \mathcal{N}$ and $i \in [n]$ such that $a_i + x \notin E(M)$. Assume to the contrary that no such x and a_i exist. Then

$$\mathcal{M} + (E(N) \setminus \{b_1, \dots, b_n\}) \subseteq E(M),$$

which implies

$$|E(N)| - n = |E(N) \setminus \{b_1, \dots, b_n\}| \leq |E(M)|.$$

This contradicts the assumption $|E(M)| < |E(N)| - n - 1$. The proof of the claim is complete. Now, choose $i \in [n]$ and $x \in E(N) \setminus \mathcal{N}$ as above. Consider the subset

$$\mathcal{P} = \{b_1, \dots, b_{i-1}, x, b_{i+1}, \dots, b_n\} \subseteq E(N).$$

In a similar manner as in the proof of Theorem 21(i)-(iii), one may argue that $\mathcal{P}$ is a basis of N . Evidently, $\mathcal{M}$ is matched to $\mathcal{P}$. Hence, M is matched to N , completing the proof. $\square$

Remark 24. It is worth noting that, although the conditions in Theorem 21(i)-(iii), such as $E(M)$ not being a semi-progression may seem unused in the proof of Theorem 21, they are in fact applied in the special case $t = 1$. In this instance, we invoke Theorem 20, where this condition plays a role.

Remark 25. Note that Theorem 21 does not address the case $t = 0$, since this forces N to be a uniform matroid (see Corollary 9). Matchability in uniform matroids is straightforward and has been completely characterized in [7, Proposition 2.9].

Remark 26. Note that in the asymmetric problem of the sparse paving case, dependence is controlled by circuit-hyperplanes, whereas in the general paving case, the size of the obstruction is measured by hyperplane nullity. Theorem 21 shows that

the existence of matchings is governed not merely by the size gap $|E(N)| - |E(M)|$, but also by the extent to which N deviates from uniformity, as quantified by the hyperplane nullity parameter t .

Example 27. We define two matroids M and N over the cyclic group $\mathbb{Z}_{13}$. Let $E(M) = \{\bar{2}, \bar{4}, \bar{6}, \bar{11}\}$ and define matroid M of rank 3 by its set of bases

$$\mathcal{B}(M) = \binom{E(M)}{3} \setminus \{\{\bar{2}, \bar{4}, \bar{6}\}\}.$$

Equivalently, the unique 3-circuit is $\{\bar{2}, \bar{4}, \bar{6}\}$, and the bases are $\{\bar{2}, \bar{4}, \bar{11}\}$, $\{\bar{2}, \bar{6}, \bar{11}\}$, and $\{\bar{4}, \bar{6}, \bar{11}\}$. Let $E(N) = \{\bar{1}, \dots, \bar{9}\}$ and fix $H = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\} \subseteq E(N)$. Define a rank-3 matroid N by

$$\mathcal{B}(N) = \binom{E(N)}{3} \setminus \binom{H}{3},$$

i.e., the only 3-subsets that are not bases are the four triples inside H . Then, N is a paving matroid that is not sparse paving. Its largest hyperplane is H (size 4), so its hyperplane nullity is $|H| - r(H) = 4 - 2 = 2$. Thus, M and N satisfy Condition (1) of Theorem 21 and M must be matched to N . We now explicitly show that M is matched to N by pairing the three bases of M with bases of N as follows.

$$\{\bar{2}, \bar{4}, \bar{11}\} \mapsto \{\bar{1}, \bar{3}, \bar{7}\}, \quad \{\bar{2}, \bar{6}, \bar{11}\} \mapsto \{\bar{1}, \bar{4}, \bar{9}\},$$

and

$$\{\bar{4}, \bar{6}, \bar{11}\} \mapsto \{\bar{1}, \bar{6}, \bar{9}\}.$$

Each target set is a basis of N , and the displayed pairings satisfy the required additive avoidance condition.

3.3. Matchable Bases through Stressed Hyperplanes

We now consider an alternative route to matchability through stressed hyperplanes. By relaxing such a hyperplane, certain dependent (n)-subsets become bases while the resulting matroid remains paving. This allows a basis to be matched either directly to the target matroid or to a suitable relaxation of it. We begin with the following characterization of paving matroids in terms of their hyperplanes.

Proposition 28 ([13]). *A matroid is paving if and only if all its hyperplanes are stressed.*

Definition 29. Let M be a matroid of rank $n = r(M)$ with bases $\mathcal{B}$, and let $S \subseteq E(M)$. Denote by $\binom{S}{n}$ the family of n -subsets of S . Suppose that no element of $\binom{S}{n}$ is a basis of M . We say that S can be relaxed if

$$\text{Rel}_S(\mathcal{B}) := \mathcal{B} \cup \binom{S}{n}$$

is again a matroid basis system. In this case, the resulting matroid is called the *relaxation of M at S* , and is denoted by $\text{Rel}_S(M)$.

Proposition 30 ([13]). *If H is a stressed hyperplane of M , then H can be relaxed.*

As a consequence, given an n -rank paving matroid M on m elements, one can obtain the uniform matroid $U_{n,m}$ by repeatedly relaxing its stressed hyperplanes. Indeed, [13] shows that every relaxation of a paving matroid is again paving, and that relaxing all stressed hyperplanes of M yields $U_{n,m}$.

Theorem 31. *Let M be a matroid and N a paving matroid over an abelian group G , both of rank n , with $0 \notin E(N)$ and*

$$|E(M)| \leq \min\{p(G) - 1, |E(N)|\}.$$

If $\mathcal{M}$ is a basis of M , then $\mathcal{M}$ is either matched to a basis of N , or to a basis of $\text{Rel}_S(N)$ for some $S \subseteq E(N)$.

Proof. Let $A \subseteq E(N)$ with $|A| = |E(M)|$. By Theorem 3, $E(M)$ is matched to A in the group sense; equivalently, there exists a matching $f : E(M) \rightarrow A$. Let $\mathcal{M} = \{a_1, \dots, a_n\}$ and set $b_i = f(a_i)$ for $i \in [n]$, and $\mathcal{N} = \{b_1, \dots, b_n\}$. If $\mathcal{N}$ is a basis of N , then $\mathcal{M}$ is matched to $\mathcal{N}$ in the matroid sense and we are done. Otherwise, set $H = \text{cl}(\mathcal{N})$. Since N is paving, H is a hyperplane of N , and by Proposition 28 it is stressed. Hence, by Proposition 30, H can be relaxed. By definition of relaxation, $\mathcal{N}$ is a basis of $\text{Rel}_H(N)$ (equivalently, $\mathcal{N} \in \text{Rel}_H(\mathcal{B}_N)$, where $\mathcal{B}_N$ is the basis system of N). Consequently, $\mathcal{M}$ is matched to $\mathcal{N}$ in the matroid sense in $\text{Rel}_H(N)$, as required. $\square$

Remark 32. When matchings between matroid bases are not available, stressed hyperplanes show that compatibility can often be recovered by relaxing independence constraints. A stressed hyperplane may be viewed as a minimal obstruction to uniformity. Relaxing such hyperplanes, turning certain dependent sets into bases, gradually transforms a paving matroid toward a uniform matroid while preserving structure. Theorem 31 shows that this process offers an alternative route to matchability. Even if bases of M cannot be matched to those of N directly, they may be matched to bases of a relaxation $\text{Rel}_S(N)$. Thus, matchability is not a fragile property tied to exact independence, but a stable phenomenon that persists under natural modifications. Relaxation therefore links the restrictive setting of paving matroids with the uniform case, where matching problems are straightforward.

3.4. Concluding Remarks and Future Directions

Let M be a matroid of rank n and let $k \in [n]$. We say that M is *k -paving* if every circuit of M has size $> n - k$. For $k = 1$ this recovers the usual paving condition,

where all circuits have size at least n . Clearly, the following hierarchy holds.

$$\begin{aligned} \text{paving} &= 1\text{-paving} \subseteq 2\text{-paving} \subseteq \cdots \\ &\subseteq n\text{-paving} = \{\text{all matroids of rank } n\}. \end{aligned}$$

The notion of k -paving matroids was first introduced in [24], where a characterization was given; see also [25] for more recent developments. We believe that some of the techniques developed here for the matchability of paving matroids could be adapted to investigate matchability in the broader class of k -paving matroids for $k > 1$. In particular, one may ask whether the dependence on t in Theorem 21 extends in a clean way. More precisely, one may ask whether the bound $|E(M)| < |E(N)| - 2t + 1$ for paving matroids admits a natural analogue of the form

$$|E(M)| < |E(N)| - c(k, t) + 1,$$

for some explicit function $c(k, t)$. Further work in this direction appears promising.

For a paving matroid N , one may obtain the uniform matroid $U_{n,m}$ by successively relaxing its stressed hyperplanes, forming a hierarchy of relaxations

$$N \longrightarrow \text{Rel}_{H_1}(N) \longrightarrow \cdots \longrightarrow U_{n,m}.$$

Assuming the conditions of Theorem 31, what is the minimal number of steps in this hierarchy needed to guarantee that M can be matched to some consecutive relaxations of N ? Clearly, this process must terminate, since in the worst-case scenario M is eventually matched to $U_{n,m}$. Is a single relaxation always sufficient, or can certain pairs (M, N) require multiple relaxations within the hierarchy?

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