



PADOVAN f -STATISTICAL CONVERGENCE AND A KOROVKIN-TYPE APPROXIMATION THEOREM

Prasanta Kumar Ray

Department of Mathematics, Sambalpur University, Burla, Odisha, India
 prasantamath@suniv.ac.in

Mrutyunjaya Meher

Department of Mathematics, Sambalpur University, Burla, Odisha, India
 bablu519@gmail.com

Received: 6/17/25, Accepted: 9/30/25, Published: 9/28/26

Abstract

This paper explores the concept of Padovan statistical convergence using modulus based density, called Padovan f -statistical convergence, which extends the classical notion of statistical convergence. The primary aim is to establish a Korovkin-type approximation theorem for functions of two variables within this framework. Further, a relationship between traditional statistical convergence and Padovan f -statistical convergence is established.

1. Introduction

Statistical convergence is a refinement of classical convergence concepts in analysis, where the emphasis lies on the statistical distribution of sequence elements rather than the pointwise convergence of each term. Unlike the traditional notion, which demands that all terms of a sequence converge to a specific limit, statistical convergence permits occasional deviations, provided that their frequency diminishes in the limit. Specifically, a sequence of real numbers (x_n) is said to be *statistically convergent* to a real number L if, for every $\epsilon > 0$, the density of terms satisfying $|x_k - L| \geq \epsilon$ tends to zero as $n \rightarrow \infty$. This is expressed mathematically as

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

The concept was independently introduced by Fast [6] and Steinhaus [15] in 1951 and has since been widely studied across various domains, including real analysis, functional analysis, and probability theory. Later, Mursaleen [13] extended that

idea to λ -statistical convergence, broadening the applicability of the convergence. Aizpuru et al. [1] further generalized this theory by introducing modulus-based densities. Building on this, Bhardwaj and Dhawan [2] formulated the concept of f -statistical convergence, which found applications in summability theory. Their work also established Korovkin-type approximation theorems using both f -statistical and f -lacunary statistical convergence methods [3].

Boyanov and Veselinov [4] established a Korovkin-type approximation theorem within the Banach space $C^*[0, \infty)$, employing the test functions $\{1, e^{-x}, e^{-2x}\}$. The space $C^*[0, \infty)$ is comprised of all real-valued continuous functions defined on the interval $[0, \infty)$ that possess a finite limit as $x \rightarrow \infty$, and convergence is measured using the supremum norm $\|\cdot\|_\infty$. The notion of difference sequence spaces was initially introduced by Kızmaz [12], who defined these in the form $X(\Delta) = \{x = (x_n) \in w : (x_k - x_{k-1}) \in X\}$, where w denotes the space of all real-valued sequences and X represents classical sequence spaces such as ℓ_∞ , c , or c_0 .

Building upon this, Kara [10] extended the framework by utilizing Fibonacci numbers, resulting in what are called *Fibonacci difference sequence spaces*. These are given as $X(F) = \{x = (x_n) \in w : Fx_n \in X\}$, where $F = (\tilde{f}_{nk})$ denotes an infinite matrix constructed from Fibonacci terms. In this context, f_n denotes the n^{th} Fibonacci number, and the matrix entries $\tilde{f}_{nk}$ are defined by:

$$\tilde{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n}, & \text{if } k = n - 1, \\ \frac{f_n}{f_{n+1}}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Accordingly, the transformation Fx_n , known as the F -transform of a sequence $x = (x_n)$, is described as:

$$Fx_n = \begin{cases} x_0, & \text{if } n = 0, \\ \frac{f_n}{f_{n+1}}x_n - \frac{f_{n+1}}{f_n}x_{n-1}, & \text{if } n \geq 1. \end{cases}$$

Kirişçi and Karaşa [11] further explored Fibonacci difference sequence spaces and introduced the concept of Fibonacci statistical convergence. Utilizing this convergence approach, they also established the second Korovkin-type theorem.

It is essential to highlight that an unbounded *modulus function* f is a mapping from $[0, \infty)$ to $[0, \infty)$ that satisfies specific properties: $f(x) = 0$ if and only if $x = 0$, and it adheres to the inequality $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$. Moreover, f is both increasing and right-continuous at 0. Building upon the aforementioned theoretical developments, we extend the notion of statistical convergence by incorporating Padovan numbers in conjunction with an unbounded modulus function. This leads us to introduce a new variant, termed Padovan f -statistical convergence.

Within this framework, we establish key foundational properties and utilize them to derive a Korovkin-type approximation theorem analogous to the one presented by Boyanov and Veselinov [4].

To proceed, we recall several essential definitions that are pivotal to our subsequent analysis.

Definition 1 ([14]). A matrix $A = (a_{mn})$ is said to be *regular* if it satisfies the following three conditions:

1. There exists a constant $K > 0$ such that for all m , $\sum_{n=1}^{\infty} |a_{mn}| \leq K$.
2. For each fixed n , $\lim_{m \rightarrow \infty} a_{mn} = 0$.
3. $\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} a_{mn} = 1$.

Definition 2 ([7]). A sequence (x_n) is called *statistically bounded* if there exists a real number $M > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \in \mathbb{N} : |x_k| > M\}| = 0.$$

Remark 1. Every statistically convergent sequence is necessarily statistically bounded.

Definition 3 ([1]). Let f be an unbounded modulus function. The f -density of a subset $K \subset \mathbb{N}$, denoted by $D^f(K)$, is defined as

$$D^f(K) = \lim_{n \rightarrow \infty} \frac{1}{f(n)} f(|\{k \leq n : k \in K\}|),$$

provided the limit exists.

Definition 4 ([2]). Let f be an unbounded modulus function and let X be a normed linear space. A sequence $(x_n) \subset X$ is said to be f -statistically convergent to $L \in X$ if, for every $\epsilon > 0$,

$$D^f(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}) = 0.$$

2. Padovan Statistical Convergence

Kirişci and Karaısa [11] were the pioneers in introducing the notion of Fibonacci statistical convergence. According to their definition, a sequence (x_n) is said to be *Fibonacci statistically convergent* to a limit L if, for every $\epsilon > 0$, the following condition holds:

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |Fx_k - L| \geq \epsilon\}| = 0,$$

where Fx_k denotes the transformation of the sequence under the Fibonacci matrix. This concept was further employed by the authors to prove a Korovkin-type approximation theorem in the setting of Fibonacci statistical convergence.

Motivated by this approach, we introduce an infinite regular matrix $Q = (\tilde{Q}_{nk})$, structured as follows:

$$\tilde{Q}_{nk} = \begin{cases} \frac{Q_{k+3}}{Q_k}, & \text{if } k = n - 1, \\ -\frac{Q_{k+1}}{Q_k}, & \text{if } k = n, \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

where Q_n denotes the n^{th} Padovan number. The transformation of a sequence (x_n) under the matrix Q , denoted by (Qx_n) , is defined by:

$$Qx_n = \frac{Q_{n+3}}{Q_n}x_{n-1} - \frac{Q_{n+1}}{Q_n}x_n, \quad \text{for } n \geq 1.$$

Using this matrix transformation, we define the concept of Padovan statistical convergence as follows.

Definition 5. A sequence (x_n) is said to be *Padovan statistically convergent* to a limit L if for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |Qx_k - L| \geq \epsilon\}| = 0.$$

For the notational convenience, we write the above limit as $Q\text{-st} \lim_{n \rightarrow \infty} x_n = L$.

Example 1. The sequence $x = (x_n)$ defined by

$$x_n = \begin{cases} n^2 + 1 & \text{if } n = k^2 + 1, \ k \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

is Padovan statistically convergent to zero since

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |Qx_k| \geq \epsilon\}| = 0.$$

Theorem 1. Let $x = (x_n)$ be a sequence for which there exists a Padovan statistically convergent sequence $y = (y_n)$ such that $Qx_k = Qy_k$ for almost all k . Then, the sequence $x = (x_n)$ is also Padovan statistically convergent.

Proof. Assume that $Qx_k = Qy_k$ for all k except on a set of natural density zero, and that $Q\text{-st} \lim y_k = L$. Then, for any $\epsilon > 0$ and for all n , the following inclusion holds:

$$\{k \leq n : |Qx_k - L| \geq \epsilon\} \subset \{k \leq n : Qx_k \neq Qy_k\} \cup \{k \leq n : |Qy_k - L| \geq \epsilon\}.$$

Taking the natural density on both sides, we obtain:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |Qx_k - L| \geq \epsilon\}| &\leq \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : Qx_k \neq Qy_k\}| \\ &+ \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |Qy_k - L| \geq \epsilon\}|. \end{aligned}$$

Since both sets on the right-hand side have natural density zero, it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |Qx_k - L| \geq \epsilon\}| = 0.$$

Hence, $x = (x_n)$ is Padovan statistically convergent to L . This completes the proof. $\square$

The following is a new result that establishes a relationship between statistical convergence and Padovan statistical convergence.

Theorem 2. *Every statistically convergent sequence is Padovan statistically convergent.*

Proof. Suppose a sequence (x_n) statistically converges to a limit L . According to Theorem 1 in [8], this can be written as $\lim_{n \in \mathbb{N}/K} x_n = L$, where K is a subset of natural numbers with zero natural density. The matrix defined in (1) is regular, allowing us to state $\lim_{n \in \mathbb{N}/K} Qx_n = L$, where the set K has zero natural density. Consequently,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |Qx_k - L| \geq \epsilon\}| = 0.$$

This concludes the proof. $\square$

Remark 2. The converse of the above theorem is not true. For example, the sequence $\{1, 2, 3, 4.5, 7.5, 12.5, 20, \dots\}$ obtained from $x_n = \frac{Q_{n+3}}{Q_{n+1}} x_{n-1}$, $n \geq 1$ and $x_0 = 1$ is not statistically convergent, but it is Padovan statistically convergent to zero.

3. Padovan f -Statistical Convergence

The purpose of this section is to present the concept of Padovan f -statistical convergence, derive associated results, and establish a Korovkin-type theorem within this context.

Definition 6. Let f be an unbounded modulus function. A sequence (x_n) is said to be convergent to L in *Padovan f -statistical* sense, denoted as Q_f -st $\lim_n x_n = L$, if for all $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{f(n)} f(|\{k \leq n : |Qx_k - L| \geq \epsilon\}|) = 0.$$

It is important to observe that every Padovan f -statistically convergent sequence is also Padovan statistically convergent. However, the converse may not necessarily hold. The following theorem confirms this statement.

Theorem 3. *Every sequence that converges in the Padovan f -statistical sense also converges in the Padovan statistical sense.*

Proof. Suppose that a sequence (x_n) converges to L in the Padovan f -statistical sense. Then, for a given $p \in \mathbb{N}$, there exists n such that $p \geq n$, and we have

$$\frac{1}{f(n)} f(|\{k \leq n : |Qx_k - L| \geq \epsilon\}|) \leq \frac{1}{p},$$

where f is an unbounded modulus function. By the subadditive property of f , we write

$$f(|\{k \leq n : |Qx_k - L| \geq \epsilon\}|) \leq \frac{1}{p} \cdot pf\left(\frac{n}{p}\right).$$

As the function f is increasing, we have

$$|\{k \leq n : |Qx_k - L| \geq \epsilon\}| \leq \frac{n}{p}.$$

Now, $p \rightarrow \infty$ when $n \rightarrow \infty$, so we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |Qx_k - L| \geq \epsilon\}| = 0.$$

This ends the proof. $\square$

Definition 7. A sequence (x_n) is said to be a *Padovan f -statistical Cauchy* sequence if for all $\epsilon > 0$ there exists a number $N(\epsilon)$ such that

$$\frac{1}{f(n)} f(|\{k \leq n : |Qx_k - Qx_N| \geq \epsilon\}|) = 0.$$

Theorem 4. *Every Padovan f -statistically convergent sequence is also a Padovan f -statistically Cauchy sequence.*

Proof. Let (x_n) be a Padovan f -statistically convergent sequence with limit L . Given any $\epsilon > 0$, choose an index $N = N(\epsilon)$ such that $|Qx_N - L| < \frac{\epsilon}{2}$. Then, for each $n \in \mathbb{N}$, we observe the inclusion:

$$\{k \leq n : |Qx_k - Qx_N| \geq \epsilon\} \subset \{k \leq n : |Qx_k - L| \geq \frac{\epsilon}{2}\} \cup \{k = N \leq n : |Qx_N - L| \geq \frac{\epsilon}{2}\}.$$

Since the modulus function f is increasing and subadditive, we have:

$$\begin{aligned} \frac{1}{f(n)} f(|\{k \leq n : |Qx_k - Qx_N| \geq \epsilon\}|) &\leq \frac{1}{f(n)} f(|\{k \leq n : |Qx_k - L| \geq \frac{\epsilon}{2}\}|) \\ &\quad + \frac{1}{f(n)} f(|\{k \leq n : |Qx_N - L| \geq \frac{\epsilon}{2}\}|). \end{aligned}$$

Since (x_n) is Padovan f -statistically convergent to L , both terms on the right-hand side tend to zero as $n \rightarrow \infty$. Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{f(n)} f(|\{k \leq n : |Qx_k - Qx_N| \geq \epsilon\}|) = 0.$$

Therefore, (x_n) is a Padovan f -statistically Cauchy sequence, which completes the proof. $\square$

Korovkin's first and second theorems were initially formulated in a statistical setting by Gadjiev and Orhan [9], and Duman [5], respectively. Boyanov and Veselinov [4] later established Korovkin-type theorems using the test functions $\{1, e^{-x}, e^{-2x}\}$ within the Banach space $C^*[0, \infty)$. Further, Bhardwaj and Dhawan [2] extended these results to the f -statistical framework.

The following theorem is a Korovkin-type theorem of Boyanov and Veselinov [4], which we have proved in the Padovan f -statistical sense.

Theorem 5. *Let f be an unbounded modulus function and (L_n) be the sequence of positive linear operators from $C^{*2}[0, \infty)$ to itself. Then, we have*

$$Q_f - \lim_n \|L_n(g, x, y) - g(x, y)\|_\infty = 0 \quad (2)$$

if and only if

$$Q_f - \lim_n \|L_n(f_i, x, y) - f_i\|_\infty = 0, \quad i = 0, 1, 2, 3, \quad (3)$$

where $f_0(x, y) = 1$, $f_1(x, y) = e^{-x}$, $f_2(x, y) = e^{-y}$, and $f_3(x, y) = e^{-2(x+y)}$.

Proof. If (2) holds, then (3) is a direct consequence of (2) for some particular values of $g(x, y)$. Conversely, suppose that (3) holds. Let $g(x, y) \in C^*[0, \infty)$. Then, there exists $M \in \mathbb{R}$ such that

$$|g(t, u) - g(x, y)| \leq 2M, \quad \text{for } t, u, x, y \in [0, \infty), \quad (4)$$

and it can be easily demonstrated that for every $\epsilon > 0$ there exists a $\delta > 0$, such that

$$\begin{aligned} & |g(t, u) - g(x, y)| < \epsilon \quad \text{whenever} \\ & \sqrt{(e^{-t} - e^{-x})^2 + (e^{-u} - e^{-y})^2} < \delta, \quad t, x \in [0, \infty). \end{aligned} \quad (5)$$

From (4) and (5), we have

$$\begin{aligned} & -\epsilon - \frac{2M}{\delta^2} ((e^{-t} - e^{-x})^2 + (e^{-u} - e^{-y})^2) \\ & < g(t) - g(x) \\ & < \epsilon + \frac{2M}{\delta^2} ((e^{-t} - e^{-x})^2 + (e^{-u} - e^{-y})^2). \end{aligned}$$

As L_n is linear and monotonic, it follows that

$$\begin{aligned} & -\epsilon L_n(1, x) - \frac{2M}{\delta^2} L_n((e^{-t} - e^{-x})^2 + (e^{-u} - e^{-y})^2, x) \\ & < L_n(g, x) - g(x) L_n(1, x) \\ & < \epsilon L_n(1, x) + \frac{2M}{\delta^2} L_n((e^{-t} - e^{-x})^2 + (e^{-u} - e^{-y})^2, x). \end{aligned} \quad (6)$$

On the other hand,

$$L_n(g, x) - g(x) = [L_n(g, x) - g(x) L_n(1, x)] + g(x) [L_n(1, x) - 1]. \quad (7)$$

Using Inequality (6) in (7), we have

$$\begin{aligned} |L_n(g; x, y) - g(x, y)| & \leq \epsilon |L_n(1; x, y) - 1| \\ & \quad + \frac{2\|f\|_B}{\delta^2} \left\{ |L_n(e^{-2(t+u)}; x, y) - e^{-2(x+y)}| \right. \\ & \quad + (e^{-2x} + e^{-2y}) |L_n(1; x, y) - 1| \\ & \quad + 2e^{-x} |L_n(e^{-t}; x, y) - e^{-x}| \\ & \quad \left. + 2e^{-y} |L_n(e^{-u}; x, y) - e^{-y}| \right\} \\ & \quad + |g(x, y)| |L_n(1; x, y) - 1| \\ & \quad + \epsilon \\ & \leq \left(\epsilon + \frac{2M}{\delta^2} (e^{-2x} + e^{-2y}) + g(x) \right) [L_n(1, x) - 1] \\ & \quad + \frac{2M}{\delta^2} [L_n(e^{-2(t+u)}, x, y) - e^{-2(x+y)}] \\ & \quad - \frac{4M}{\delta^2} e^{-x} [L_n(e^{-t}, x, y) - e^{-x}] \\ & \quad - \frac{4M}{\delta^2} e^{-y} [L_n(e^{-u}, x, y) - e^{-y}] + \epsilon. \end{aligned}$$

Choosing $S = \epsilon + \frac{4M}{\delta^2} + M$, we then obtain

$$\begin{aligned} \|L_n(g, x, y) - g(x, y)\|_\infty & \leq S \left\{ \|L_n(1, x, y) - 1\|_\infty + \|L_n(e^{-2(t+u)}, x, y) - e^{-2(x+y)}\|_\infty \right. \\ & \quad \left. + \|L_n(e^{-t}, x, y) - e^{-x}\|_\infty + \|L_n(e^{-u}, x, y) - e^{-y}\|_\infty \right\} + \epsilon. \end{aligned}$$

Now replacing $L_n(\cdot, x, y)$ with $U_n = QL_n(\cdot, x, y)$, for $\epsilon' > 0$, we have

$$\begin{aligned} P &= \left\{ k \in \mathbb{N} : \|U_k(1, x, y) - 1\|_\infty + \|U_k(e^{-2(t+u)}, x, y) - e^{-2(x+y)}\|_\infty \right. \\ &\quad \left. + \|U_k(e^{-t}, x, y) - e^{-x}\|_\infty + \|U_k(e^{-u}, x, y) - e^{-y}\|_\infty \geq \frac{\epsilon' - \epsilon}{S} \right\}, \\ P_1 &= \left\{ k \in \mathbb{N} : \|U_k(1, x, y) - 1\|_\infty \geq \frac{\epsilon' - \epsilon}{4S} \right\}, \\ P_2 &= \left\{ k \in \mathbb{N} : \|U_k(e^{-t}, x, y) - e^{-x}\|_\infty \geq \frac{\epsilon' - \epsilon}{4S} \right\}, \\ P_3 &= \left\{ k \in \mathbb{N} : \|U_k(e^{-u}, x, y) - e^{-y}\|_\infty \geq \frac{\epsilon' - \epsilon}{4S} \right\}, \\ P_4 &= \left\{ k \in \mathbb{N} : \|U_k(e^{-2(t+u)}, x, y) - e^{-2(x+y)}\|_\infty \geq \frac{\epsilon' - \epsilon}{4S} \right\}. \end{aligned}$$

It is easy to verify that $P \subset P_1 \cup P_2 \cup P_3 \cup P_4$. By utilizing the subadditive and monotonic properties of the modulus function f and multiplying both sides by $\frac{1}{f(n)}$, we can express it as follows:

$$\begin{aligned} &\frac{1}{f(n)} f(|\{k \leq n : \|U_k(g, x, y) - g(x, y)\|_\infty \geq \epsilon'\}|) \\ &\leq \frac{1}{f(n)} f\left(\left|\left\{k \leq n : \|U_k(1, x, y) - 1\|_\infty \geq \frac{\epsilon' - \epsilon}{4S}\right\}\right|\right) \\ &\quad + \frac{1}{f(n)} f\left(\left|\left\{k \leq n : \|U_k(e^{-t}, x, y) - e^{-x}\|_\infty \geq \frac{\epsilon' - \epsilon}{4S}\right\}\right|\right) \\ &\quad + \frac{1}{f(n)} f\left(\left|\left\{k \leq n : \|U_k(e^{-u}, x, y) - e^{-y}\|_\infty \geq \frac{\epsilon' - \epsilon}{4S}\right\}\right|\right) \\ &\quad + \frac{1}{f(n)} f\left(\left|\left\{k \leq n : \|U_k(e^{-2(t+u)}, x, y) - e^{-2(x+y)}\|_\infty \geq \frac{\epsilon' - \epsilon}{4S}\right\}\right|\right). \end{aligned}$$

Using (3), we have $Q_f - \lim_n \|L_n(g, x, y) - g(x, y)\|_\infty = 0$. This completes the proof. $\square$

References

- [1] A. Aizpuru, M. C. Listán-García, and F. Rambla-Barreno, Density by moduli and statistical convergence, *Quaest. Math.* **37** (4) (2014), 525–530.
- [2] V. K. Bhardwaj and S. Dhawan, Korovkin type approximation theorem via f-statistical convergence, *J. Math. Anal.* **9** (2) (2018), 99–117.
- [3] V. K. Bhardwaj and S. Dhawan, Application of f-lacunary statistical convergence to approximation theorems, *J. Inequal. Appl.* **2018** (1) (2018), 1–25.

- [4] B. D. Boyanov and V. M. Veselinov, A note on the approximation of functions in an infinite interval by linear positive operators, *Bull. Math. Soc. Sci. Math. R. S. Roumanie* **14** (1) (1970), 9–13.
- [5] O. Duman, Statistical approximation for periodic functions, *Demonstr. Math.* **36** (4) (2003), 873–878.
- [6] H. Fast, Sur la convergence statistique, *Colloq. Math.* **2** (1951), 241–244.
- [7] J. Fridy and C. Orhan, Statistical limit superior and limit inferior, *Proc. Amer. Math. Soc.* **125** (12) (1997), 3625–3631.
- [8] J. A. Fridy, On statistical convergence, *Analysis* **5** (4) (1985), 301–314.
- [9] A. D. Gadjiev and C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.* **32** (2002), 129–138.
- [10] E. E. Kara, Some topological and geometrical properties of new Banach sequence spaces, *J. Inequal. Appl.* **2013** (1) (2013), 1–15.
- [11] M. Kirisci and A. Kara, Fibonacci statistical convergence and Korovkin type approximation theorems, *J. Inequal. Appl.* **2017** (229) (2017), 1–12.
- [12] H. Kizmaz, On certain sequence spaces, *Canad. Math. Bull.* **24** (2) (1981), 169–176.
- [13] M. Mursaleen, λ -statistical convergence, *Math. Slovaca* **50** (1) (2000), 111–115.
- [14] G. M. Petersen, *Regular Matrix Transformations*, McGraw–Hill, London, 1966.
- [15] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.* **2** (1951), 73–74.