



PARTITION-THEORETIC MODEL OF PRIME DISTRIBUTION, II

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Abstract

In recent work by Botkin, Dawsey, Hemmer, Just and the present author, a deterministic model of prime number distribution is developed based on properties of integer partitions that gives almost exact estimates for $\pi(n)$, the number of primes less than or equal to a positive integer n , up to $n = 10,000$. In this follow-up note, the author summarizes the ideas behind this partition-theoretic model of primes and formulates a computational model that is practically exact in its estimates of $\pi(n)$ up to $n = 100,000$.

1. Introduction

1.1. Partition Model of Prime Distribution

The sieve of Eratosthenes provides a method to identify prime numbers out of the larger set of positive integers $\mathbb{Z}^+$. In carrying out the algorithm, one gets the feeling that the primes $\mathbb{P}$ are positioned in the sequence of integers precisely where they need to be in order to generate the rest of $\mathbb{Z}^+$ by multiplication. Despite the sense of inevitability associated with the primes, many well-known theorems and conjectures related to prime numbers are very difficult (or seemingly impossible) to prove. As a way of sidestepping a direct attack on prime number phenomena, mathematicians often use a probabilistic approach based on the Prime Number Theorem.

We recall that if $\pi(x)$ denotes the number of prime numbers less than or equal to $x \in \mathbb{R}^+$, the classical Prime Number Theorem states that as $x \rightarrow \infty$,

$$\pi(x) \sim \frac{x}{\log x}, \quad (1)$$

where $\log x$ denotes the natural logarithm function. Thus the probability $\pi(n)/n$ that $n \in \mathbb{Z}^+$ is prime is asymptotically equal to $1/\log n$ as n increases. A random

model of prime distribution might assume the primes are modeled by a random sequence with probability $1/\log n$ of the n th term being prime, $n \geq 2$, or impose similar probabilistic measures on the prime gaps as in Cramér's model (see e.g. [13]).

In recent work [2], the author of the present paper and their collaborators develop a deterministic model of prime number distribution based on facts and observations about integer partitions. Inequalities that relate the partition norm [12] and super-norm [6] statistics impose constraints on the distribution of primes. Based on this, the authors of [2] deduce that the n th prime gap may be modeled by the relation

$$p_{n+1} - p_n = 2 \left\lceil \frac{d(n)}{2} \right\rceil, \quad (2)$$

where $d(n)$ is the *divisor function*, the number of positive divisors of $n \geq 1$. While this partition model of prime gaps is formulated based on somewhat heavy-handed simplifying assumptions, it yields a surprisingly accurate estimate of prime distribution; see Table 1. Below, we summarize the model of primes proposed in [2].

For $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the usual *ceiling function*. For $x \geq 0$, let $\lfloor x \rfloor$ denote the usual *floor function*; if $x < 0$, then we set $\lfloor x \rfloor := 0$.

Models 1, 2, and 2* (Partition models of prime numbers from [2]). The n th prime number p_n , $n \geq 1$, is modeled by setting $p_1 = 2$ and for $n \geq 2$ by using the formula

$$p_n = 1 + 2 \sum_{j=1}^{n-1} \left\lceil \frac{d(j)}{2} \right\rceil + \varepsilon(n), \quad (3)$$

where $d(j)$ is the divisor function and $\varepsilon(n)$ is an explicit error term that is negligible by comparison. The case $\varepsilon(n) := 0$ for all $n \geq 2$ is referred to in [2] as Model 1. The case

$$\varepsilon(n) := \lfloor \pi_2(p_{n-1}) - 2\gamma(n-1) \rfloor \quad (4)$$

is referred to as Model 2, where $\pi_2(k)$ is the number of semiprimes less than or equal to $k \geq 1$, and $\gamma = 0.5772\dots$ is the Euler–Mascheroni constant. For a third case, set $p_1 = 2$, $p_2 = 3$, and for $n \geq 3$ use an asymptotic approximation to the above error term,

$$\varepsilon(n) := \lfloor (n-1) \cdot (\log \log(n-1) - 2\gamma) \rfloor, \quad (5)$$

where $\log x$ denotes the natural logarithm, to yield a model variant referred to as Model 2*.

Model 1 predicts both the Prime Number Theorem and the Twin Prime Conjecture [2]. Here, we experiment with computational corrections to derive a new model variant.

Model 3 (Partition model of prime numbers with variable parameters). Let $r, t \in \mathbb{R}$ with $r \geq 0$, $0 \leq t \leq 1$. Let $\gamma = 0.5772\dots$ denote the Euler-Mascheroni constant and $M = 0.2614\dots$ denote the Meissel-Mertens constant. The n th prime number can be modeled by setting $p_1 = 2$, $p_2 = 3$, and for $n \geq 3$ by using Equation (3) with the error term

$$\varepsilon(n) = \varepsilon(n; r, t) := \lfloor (n + r - 1) \log \log(n + r - 1) - (n - 1)(2\gamma - Mt) \rfloor, \quad (6)$$

to yield the model variant we will refer to as Model 3.

Models 1, 2 and 2*, as well as Model 3 with $r = 0$, produce the sequence of values

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 27,$$

and do not produce the sequence of primes from there, but almost produce the correct sequence of prime gaps. See Table 1 to compare estimates for $\pi(n)$ from the different models.

n	$\pi(n)$	$n / \log n$	Model 1	Model 2	Model 2*	Model 3 $r = 6$ $t = 0.11$
10	4	4.34...	4	4	4	4
100	25	21.71...	27	26	27	25
1000	168	144.76...	184	168	171	168
10,000	1229	1085.73...	1352	1212	1233	1228
100,000	9592	8685.88...	10,602	9435	9618	9592
1,000,000	78,498	72,382.41...	86,739	77,322	78,740	78,575

Table 1: Comparing estimates for $\pi(n)$; Model 3 depends on real parameters r, t

1.2. Notation

We define the notation required to discuss these models. Let $\mathbb{Z}^+$ denote the positive integers. Let $\mathbb{P}$ denote the prime numbers. Let $p_i \in \mathbb{P}$ denote the i th prime number, viz. $p_1 = 2, p_2 = 3, p_3 = 5$, etc.; we define $p_0 := 1$, and refer to the subscript $i \in \mathbb{Z}^+$ of $p_i \in \mathbb{P}$ as the *index* of the prime number. For $n \in \mathbb{Z}^+$, we write its prime factorization as $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_i^{a_i} \cdots$, $a_i \in \mathbb{Z}_{\geq 0}$, where only finitely many primes p_i have nonzero multiplicity (number of occurrences); we omit p_i from the notation if $a_i = 0$. Define a *semiprime* to be a positive integer having exactly two prime factors including repetition, and define a *k-almost prime* to be a positive integer with $k \geq 2$ prime factors including repetition. For $x \in \mathbb{R}^+$, let $\pi(x)$ denote the prime counting function, the number of primes less than or equal to x , and let $\pi_k(x)$ denote the number of k -almost primes less than or equal to x .

Let $\mathcal{P}$ denote the set of integer partitions, unordered finite multisets of positive integers including the empty partition $\emptyset \in \mathcal{P}$ (see e.g. [1]). For a nonempty partition $\lambda \in \mathcal{P}$, we write $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_r)$, $\lambda_i \in \mathbb{Z}^+$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 1$. For $\lambda \in \mathcal{P}$, let $|\lambda| \geq 0$ denote the *size* (sum of parts), let $\ell(\lambda) = r$ be the *length* (number of parts), and let $m_i = m_i(\lambda) \geq 0$ be the *multiplicity* of $i \in \mathbb{Z}^+$ as a part of partition λ . Define a *non-unitary partition* to be a partition with no part equal to one.

Let $\gamma = 0.5772\dots$ denote the Euler-Mascheroni constant and let $M = 0.2614\dots$ denote the Meissel-Mertens constant.

A new, multiplicative partition statistic is introduced in [11]. The *norm* $N(\lambda)$ of partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_r)$ is the product of its parts:

$$N(\lambda) := \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_r = 1^{m_1} 2^{m_2} 3^{m_3} \cdots i^{m_i} \cdots \in \mathbb{Z}^+. \quad (7)$$

We define $N(\emptyset) := 1$ (it is an empty product); see [12] for more on the partition norm.¹

Furthermore, in [6] another multiplicative partition statistic is defined. The *supernorm* $\hat{N}$ of partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_r)$ is the product

$$\hat{N}(\lambda) := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_r} = 2^{m_1} 3^{m_2} 5^{m_3} \cdots p_i^{m_i} \cdots \in \mathbb{Z}^+, \quad (8)$$

where $p_i \in \mathbb{P}$ is the i th prime number, $i \geq 1$, and $m_j = m_j(\lambda) \geq 0$. We define $\hat{N}(\emptyset) := 1$; see [8, 9] for further reading about the partition supernorm.

2. Improving the Computational Model of $\pi(n)$

2.1. Brief Summary of the Ideas Behind Models 1, 2, and 2*

In [2], Model 1 follows from two elementary observations and a subsequent simplifying assumption.

2.1.1. Elementary Observations

Observation 1. The supernorm induces a bijection between partitions and the prime factorizations of positive integers. The integers in prime gaps are the images of certain subsets of $\mathcal{P}$ under the supernorm map. Thus, as is noted in [2, Prop. 2.1], measuring the n th prime gap is equivalent to enumerating partitions that map into the interval $[p_n, p_{n+1})$ under the supernorm:

$$p_{n+1} - p_n = \# \left\{ \lambda \in \mathcal{P} : p_n \leq \hat{N}(\lambda) < p_{n+1} \right\}. \quad (9)$$

¹The partition norm is referred to as the “integer” of the partition and notated n_λ in [11].

Observation 2. Based on a mix of proof and computational evidence, it is observed in [2, Eqs. 1.9–1.10] that if λ is a non-unitary partition and $N(\lambda) \geq 5$, then

$$N(\lambda) < p_{N(\lambda)} \leq \widehat{N}(\lambda) \leq N(\lambda)^{\log 3 / \log 2}. \quad (10)$$

Noting $\frac{\log 3}{\log 2} = 1.5849\dots$, the supernorm maps λ to a “small” interval immediately above the prime $p_{N(\lambda)}$. Moreover, the fewer parts λ has, the closer we expect $\widehat{N}(\lambda)$ to be to $p_{N(\lambda)}$.

Observation 1 suggests that a reasonable way to estimate the size $p_{n+1} - p_n$ of the n th prime gap is to guess which partitions map into the interval under $\widehat{N}$, and enumerate them. Observation 2 suggests that a good place to start might be with partitions whose images under $\widehat{N}$ we already expect to be just above p_n , viz. non-unitary partitions of norm n with few parts. These partitions map to odd integers we anticipate to lie just above p_n on the number line. In [2], a somewhat drastic simplification is assumed based on this reasoning.

2.1.2. Simplifying Assumption

Following [2], assume for simplicity that all odd integers in the interval $[p_n, p_{n+1})$, $n \geq 2$, are the images of non-unitary partitions with norms equal to n , having one or two parts, under the supernorm $\widehat{N}$. That is, assume for simplicity

$$p_{n+1} - p_n = 2 \cdot \# \{ \lambda \in \mathcal{P} : N(\lambda) = n, m_1(\lambda) = 0, \ell(\lambda) = 1 \text{ or } 2 \}; \quad (11)$$

the factor of 2 on the right doubles the number of odd primes and semiprimes in the interval to count the even integers that follow them.

As noted in [2], assumption (11) is equivalent to writing

$$p_{n+1} - p_n = 2 \left\lceil \frac{d(n)}{2} \right\rceil. \quad (12)$$

Each length-two partition $\lambda = (\lambda_1, \lambda_2)$ such that $N(\lambda) = \lambda_1 \lambda_2 = n$ represents a pair of divisors λ_1 and λ_2 of n , and we associate the one-part partition (n) with the divisor pair n and 1, justifying (2) above. Then for $n \geq 2$, one obtains from (12) by a telescoping sum

$$p_n = p_2 + \sum_{k=2}^{n-1} (p_{k+1} - p_k) = 3 + 2 \sum_{k=2}^{n-1} \left\lceil \frac{d(k)}{2} \right\rceil = 1 + 2 \sum_{k=1}^{n-1} \left\lceil \frac{d(k)}{2} \right\rceil, \quad (13)$$

since $3 = 1 + 2 \lceil d(1)/2 \rceil$. This is the $\varepsilon(n) = 0$ case of (3) that defines Model 1 in [2].²

²Model 1 was initially proposed in the unpublished appendix of [5].

Remark 1. Note that $2 \lceil d(k)/2 \rceil = d(k)$ for all $k \neq m^2, m \in \mathbb{Z}^+$. If k is a perfect square then $2 \lceil d(k)/2 \rceil = d(k) + 1$. Then for $n \geq 2$, one may simplify computations by using

$$2 \sum_{k=1}^{n-1} \left\lceil \frac{d(k)}{2} \right\rceil = \sum_{k=1}^{n-1} d(k) + \lfloor \sqrt{n-1} \rfloor = \sum_{k=1}^{n-1} \left\lfloor \frac{n-1}{k} \right\rfloor + \lfloor \sqrt{n-1} \rfloor. \quad (14)$$

Since integers in prime gaps with more than two prime factors are not enumerated in Model 1, it is anticipated that the model should underestimate the prime gap sizes, and thus overestimate $\pi(n)$; Table 1 suggests this is indeed the case.

In [2], Model 2 is derived from Model 1 by making a simplistic estimate of the missing contribution to prime gaps of partitions having three parts and norms equal to n . The $\pi_2(p_{n-1})$ term in (4) arises in this estimate; the $2\gamma(n-1)$ term is subtracted for computational reasons and significantly improves the model's estimates. See [2] for details.

Model 2* is then produced in [2] by substituting the following estimate, derived from a formula of Landau [10], into Model 2 to yield the estimated correction term $\varepsilon(n)$ in (5):

$$\pi_2(p_{n-1}) \sim (n-1) \log \log(n-1) \quad (15)$$

as $n \rightarrow \infty$. The authors of [2] make a comparison of Models 2 and 2*: "Since it involves simpler computations and is evidently more accurate in its estimates of $\pi(n)$ as n increases, we consider Model 2* to be a better model of primes."

In the following section, we focus on refining the error term (15).

2.2. Refining the Estimate of $\pi_2(p_{n-1})$ in the Error Term (Model 3)

In [2], the authors note that as n increases, "the missing contribution of partitions (and factorizations) of lengths greater than three creeps up, causing the [Model 2 and 2*] estimates to become farther from exact as n increases." It is reasonable to expect that estimating the contributions of partitions with more than three parts may improve the models' accuracy; Model 2 was formulated for the same reason, but related to partitions of lengths greater than two. That the Model 2 correction term (4) and subsequent Model 2* correction (5) were suggested by the premises of Model 1, and do produce improved estimates of prime distribution, strengthens the plausibility of the model's premises to some extent.

However, as the number of parts increases, the premise is compromised that partitions of norm n with few parts should map immediately above p_n . To eliminate this concern, let us focus instead on modifying the asymptotic estimate (15) to produce a better fit.

2.2.1. Interpolating Between Models 2 and 2*

As discussed in [2], inspection of Table 1 suggests that from some point on, the Model 2 estimate for $\pi(n)$ is strictly less than $\pi(n)$ itself while the Model 2* estimate is strictly greater than $\pi(n)$; this observation is recorded in [2, Conjecture 4.1]. If these inequalities were true, they would imply that Model 2 accurately captures the behavior of $\pi(n)$ with smaller error than the error in the Landau estimate (15) for $\pi_2(p_{n-1})$. (In [2], the authors note this conjecture relies on limited data.)

It is proved in [2] based on a formula from [4] that, for M the Meissel-Mertens constant,

$$\pi_2(p_n) - n \log \log n \sim Mn \quad (16)$$

as $n \rightarrow \infty$. Then a better approximation to (4) than the one provided by (5) is given by

$$\varepsilon(n) := \lfloor (n-1) \cdot (\log \log(n-1) - 2\gamma + M) \rfloor. \quad (17)$$

One can effectively interpolate between Model 2 and Model 2* by introducing a parameter $t \in [0, 1]$ into (17), so that $t = 0$ gives Model 2* and $t = 1$ approximates Model 2:

$$\varepsilon(n) = \varepsilon(n; t) := \lfloor (n-1) \cdot (\log \log(n-1) - 2\gamma + Mt) \rfloor. \quad (18)$$

By varying t , perhaps one can locate a “sweet spot” between the estimates of Model 2 and Model 2* to produce a better model of $\pi(n)$. See Table 2 for examples.

2.2.2. Introducing a Second Parameter to Produce a Better Fit

Another variable parameter can be introduced into the asymptotic estimate $\pi_2(p_n) \sim n \log \log n$ in a somewhat ad hoc manner, to further fine-tune estimates for $\pi(n)$. In [7] it is noted that for fixed $r \geq 0$, it is also the case that $(n+r) \log \log(n+r) \sim n \log \log n$ as $n \rightarrow \infty$.³ Computational experiments reported in [7] show that replacing (5) in Model 2* with

$$\varepsilon(n) = \varepsilon(n; r) := \lfloor (n+r-1) \log \log(n+r-1) - 2\gamma(n-1) \rfloor \quad (19)$$

and varying r may improve Model 2* estimates of $\pi(n)$ at small n , with a decreasing relative effect as $n \rightarrow \infty$ since $(n+r) \log \log(n+r) \sim n \log \log n$. See Table 3 for examples.

2.2.3. Combining the Above Modifications

We combine the modifications introduced to $\varepsilon(n)$ in (18) and (19) to yield the new model that was stated in Section 1.1, as follows.

³The author thanks Eli DeWitt who, as an undergraduate research student at Michigan Technological University, performed computational experiments that informed this study (see [7]).

n	$\pi(n)$	$t = 0$	$t = 0.1$	$t = 0.5$	$t = 1$
10	4	4	4	4	4
100	25	27	27	26	26
1000	168	171	170	168	165
10,000	1229	1233	1230	1216	1200
100,000	9592	9618	9595	9510	9404
1,000,000	78,498	78,740	78,591	78,000	77,274

Table 2: Estimates for $\pi(n)$ when varying t in Equation (18)

2.2.4. Statement and Discussion of Model 3

Model 3 is the statement that the prime numbers $p_1, p_2, p_3, \dots$, can be modeled by the sequence having initial values $p_1 = 2$, $p_2 = 3$, and for $n \geq 3$ having the values

$$p_n = 1 + 2 \sum_{k=1}^{n-1} \left\lceil \frac{d(k)}{2} \right\rceil + \lfloor (n+r-1) \log \log(n+r-1) - (n-1)(2\gamma - Mt) \rfloor,$$

with $r, t \in \mathbb{R}$, $r \geq 0$, $0 \leq t \leq 1$, as was noted in Section 1.1 above. See Table 1 to compare the Model 3 estimates for $\pi(n)$ to estimates from other models.

Setting $r = t = 0$, Model 3 reduces to Model 2* from [2]. If $r = 0$, then computations show that Model 3 outputs the same initial values 2, 3, 5, 7, 11, 13, 17, 19, 23, 27, as the previous variants of the model. However, as r increases above zero, the terms of the modeled p_n sequence become spaced farther apart and Model 3 does not model the sequence of primes at all, while still providing a reasonably good model of $\pi(n)$ that remains asymptotic to $n/\log n$.

Here is an example of this phenomenon for a particular selection of the values r, t . Note in Table 2 that the $t = 0.1$ column provides quite a good fit to $\pi(n)$; also, setting $r = 6$ is noted to yield a good fit in [7]. One finds that setting $r = 6$, $t = 0.11$, in Model 3 produces an accurate estimate of $\pi(n)$ up to at least $n = 1,000,000$, as Table 1 displays. But setting $r = 6$, $t = 0.11$, in Model 3 outputs the sequence of values 2, 3, 8, 10, 14, 16, 21, 23, This integer sequence has the wrong values from the third term on, including even integers that are unmistakably composite,⁴ yet is almost equinumerous with the primes up to one million.

⁴In [2] it is noted that replacing the floor function $\lfloor x \rfloor$ in the correction terms of these models with a variant $\lfloor x \rfloor^*$ such that $\lfloor x \rfloor^* = \lfloor x \rfloor$ if $\lfloor x \rfloor$ is even, and $\lfloor x \rfloor^* = \lfloor x \rfloor - 1$ if $\lfloor x \rfloor$ is odd, will produce a sequence of only odd numbers without affecting estimates for $\pi(n)$.

3. Closing Remarks

The models formulated in [2] and here, as well as recent works like [3, 6], demonstrate that partition theory – the theory of addition – encodes information about prime numbers. To what extent this insight is useful to the theory of prime distribution, compared to classical approaches like random models, is yet to be seen. In [3], prime numbers are proved to be intrinsically connected to the coefficients of certain quasimodular forms via special partition-theoretic functions; this beautiful application is seemingly unconnected to the topic of prime distribution. In [2], while a partition model of primes and of the Prime Number Theorem is established, the authors test new predictions from Model 1 with mixed findings and, in successful tests, with results only weakly supporting predictions. Perhaps the prime number formulas produced by these models are as accurate as they are due to different reasons from those given in [2]. Yet because these partition models for $\pi(n)$ are accurate at small n and provide discrete models of the Prime Number Theorem as n increases, one wonders: do they capture something essential about prime distribution?

n	$\pi(n)$	$r = 0$	$r = 1$	$r = 6$	$r = 10$	$r = 20$
10	4	4	4	4	2	2
100	25	27	27	25	24	20
1000	168	171	170	169	168	166
10,000	1229	1233	1232	1232	1232	1229
100,000	9592	9618	9618	9616	9615	9614
1,000,000	78,498	78,740	78,740	78,740	78,738	78,736

Table 3: Estimates for $\pi(n)$ when varying r in (19), reproduced from [7]

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