



CONSECUTIVE PRIMES AND IP SETS

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Abstract

For an infinite set $\mathcal{M} \subset \mathbb{N}$, let $\text{FS}(\mathcal{M})$ be the set of all nonzero finite sums of distinct numbers in $\mathcal{M}$. An *IP set* is any set of the form $\text{FS}(\mathcal{M})$. Let p_n denote the n^{th} smallest prime number. A *de Polignac number* is a natural number m such that $p_{n+1} - p_n = m$ holds for infinitely many n . Given an arbitrary IP set $\mathcal{A}$ and any integer $k \geq 1$, there is a sequence $a_1, a_2, \dots, a_k$ in $\mathcal{A}$ such that $\sum_{i \leq n \leq j} a_n$ is a de Polignac number for $1 \leq i \leq j \leq k$. In particular, the set **Pol** of all de Polignac numbers has an infinite intersection with every IP set.

1. Background

The present paper was inspired by work of Tao and Ziegler [11], who studied infinite restricted sumsets in the shifted primes and ascending chains of prime-producing tuples. Although our work also revolves around the study of sumsets and primes, we do not use any results from [11]. Instead, we rely on various theorems in the literature and one auxiliary result (Lemma 1).

We begin by briefly reviewing material associated with our main theorem and various results needed in the proof.

1.1. IP Sets

A classical result of Hindman [7] (answering a question posed by Graham and Rothschild [6]) relates to *finite sums sets*, i.e., those of the form

$$\text{FS}(\mathcal{M}) := \left\{ \sum_{m \in \mathcal{M}'} m : \emptyset \neq \mathcal{M}' \subset \mathcal{M}, |\mathcal{M}'| < \infty \right\}$$

for some infinite set $\mathcal{M}$ of natural numbers. Hindman showed that for any finite coloring of $\mathbb{N}$, there is an infinite set $\mathcal{M}$ such that $\text{FS}(\mathcal{M})$ is monochromatic.

In their treatise on topological dynamics and combinatorial number theory, Furstenberg and Weiss [4] coined the term *IP set* to mean any finite sums set $\mathbf{FS}(\mathcal{M})$. There are many interesting examples of IP sets. Bergelson and Rusza [3] gave a general method for constructing a variety of IP sets. For example, they showed that if $\mathcal{S}$ is the set of squarefree natural numbers and $s \in \mathbb{N}$, then $\mathcal{S} - s$ contains an IP set if and only if $s \in \mathcal{S}$.

1.2. De Polignac Numbers

Let p_n be the n^{th} smallest prime number. The notorious *twin prime conjecture* asserts the existence of infinitely many natural numbers n for which $p_{n+1} - p_n = 2$. This conjecture first appeared in the 1849 monograph of Alphonse de Polignac [9], who also speculated that for every even natural number $2m$, there are infinitely many prime gaps of size $2m$, i.e., that $p_{n+1} - p_n = 2m$ holds for infinitely many n . Let $\mathbf{Pol}$ denote the set of even integers $2m$ with this property. The elements of $\mathbf{Pol}$ are called *de Polignac numbers*, and de Polignac's hypothesis that $\mathbf{Pol} = 2\mathbb{N}$ is known as *Polignac's conjecture*.¹

The *bounded gap conjecture* (that is, the assertion that $\mathbf{Pol} \neq \emptyset$) was first proved in the celebrated 2013 paper of Zhang [12], who employed a modification of the Goldston-Pintz-Yıldırım sieve [5] to prove the existence of a de Polignac number not exceeding 7×10^7 . The **Polymath8** project sought to understand and improve Zhang's arguments and ultimately showed that $\mathbf{Pol} \cap [2, 4680] \neq \emptyset$. Later in 2013, Maynard [8] showed that for every $m \in \mathbb{N}$ one has²

$$\liminf_n (p_{n+m} - p_n) \ll m^3 e^{4m}.$$

His breakthrough paper also proved the existence of infinitely many prime gaps of size at most 600; in other words, it showed that $\mathbf{Pol} \cap [2, 600] \neq \emptyset$. Following Maynard's announcement, a new polymath project was launched (**Polymath8b**) and eventually achieved the result that $\mathbf{Pol} \cap [2, 246] \neq \emptyset$, currently the strongest known result in this direction.

1.3. The Maynard-Tao Theorem

An ordered tuple $\mathcal{H}$ of distinct nonnegative integers is said to be *admissible* if it avoids at least one residue class mod p for every prime p . Following Tao and Ziegler [11], we say that a finite admissible tuple $\mathcal{H} = (h_1, \dots, h_k)$ is *prime-producing* if there are infinitely many $n \in \mathbb{N}$ such that $\{n + h_1, \dots, n + h_k\}$ are simultaneously

¹Another equally famous conjecture of Alphonse de Polignac is that every odd integer $k \geq 3$ can be expressed as the sum of a power of two and a prime number. Thus, in the literature, the term *de Polignac number* often refers to odd integers k of this form.

²The same result, albeit with a slightly weaker bound, had been independently proven by Tao around the same time; the latter work remains unpublished.

prime. The Dickson-Hardy-Littlewood conjecture asserts that every such tuple $\mathcal{H}$ is prime-producing. This conjecture remains one of the great unsolved problems in number theory, and the strongest unconditional result in this direction is the following theorem of Maynard [8] and Tao.

Theorem 1 (Maynard–Tao). *For every integer $m \geq 2$, there is a number k_m for which the following holds. If $(h_1, \dots, h_k)$ is admissible with $k \geq k_m$, then the set $\{n + h_1, \dots, n + h_k\}$ contains at least m primes for infinitely many $n \in \mathbb{N}$.*

Shortly after this theorem was announced, a version of the Maynard–Tao theorem producing *consecutive* primes in tuples was given by Banks, Freiberg and Turnage-Butterbaugh [1], who used the result to solve an old problem of Erdős. The following is a slightly weakened version of their theorem.

Theorem 2 (Banks–Freiberg–Turnage-Butterbaugh). *Fix an integer $m \geq 2$, and let k_m have the property stated in Theorem 1. If $(h_1, \dots, h_k)$ is admissible with $k \geq k_m$, then there is a set $\{h'_1, \dots, h'_m\} \subset \{h_1, \dots, h_k\}$ such that the set $\{n + h'_1, \dots, n + h'_m\}$ consists of m consecutive primes for infinitely many $n \in \mathbb{N}$.*

1.4. Ramsey’s Theorem

We recall the well-known result of Ramsey [10].

Theorem 3 (Ramsey). *For any $r, s \in \mathbb{N}$, there is a least natural number $R(r, s)$ such that every blue-red edge coloring of the complete graph on $R(r, s)$ vertices contains either a blue clique on r vertices or a red clique on s vertices.*

2. Statement of the Main Theorem

Our main result is the following.

Theorem 4. *Let $\mathcal{A}$ be an IP set. Then, for any $k \geq 1$, there is a sequence $a_1, a_2, \dots, a_k$ in $\mathcal{A}$ such that*

$$\sum_{i \leq n \leq j} a_n \in \mathbf{Pol} \quad \text{whenever} \quad 1 \leq i \leq j \leq k. \quad (1)$$

Moreover, the a_j ’s can be chosen so that all of the sums in (1) are distinct from one another.

The proof is straightforward and short, but it incorporates the deep results from number theory and additive combinatorics outlined in §1.

As a simple consequence, Theorem 4 implies that $\mathbf{Pol}$ contains infinitely many elements of any IP set (therefore, $\mathbf{Pol}$ is an IP^* set in the sense of Bergelson [2]), and we deduce the following corollaries by choosing appropriate IP sets.

Corollary 1. *For every $s \in \mathcal{S}$, there are infinitely many $m \in \mathcal{S} - s$ such that $p_{n+1} - p_n = m$ for infinitely many n .*

Corollary 2. *For every $c > 0$, there are infinitely many numbers m in the set*

$$\{m \in \mathbb{N} : p \mid m \text{ implies } p > c\}$$

with the property that $\frac{1}{2}(p_{n+1} - p_n) = m - 1$ for infinitely many n .

Corollary 3. *There are infinitely many $m \in \mathbb{N}$ that have only the digits 0 and 2 in their decimal expansion and satisfy $p_{n+1} - p_n = m$ for infinitely many n .*

3. Proof of the Main Theorem

We need the following lemma.

Lemma 1. *Every infinite set $\mathcal{B} \subset \mathbb{N}$ contains a strictly increasing sequence $(b_n)_{n \in \mathbb{N}}$ for which the tuple $(b_1, b_2, \dots)$ is admissible.*

Proof. Put $b_0 := 0$ and $\mathcal{B}_0 := \mathcal{B}$. Using induction, we construct a sequence $(b_n)_{n \in \mathbb{N}}$ contained in $\mathcal{B}$ and a family of sets $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ satisfying

$$b_n > b_{n-1}, \quad \mathcal{B}_n = \mathcal{B}_{n-1} \cap (p_n \mathbb{Z} + b_n) \quad \text{and} \quad |\mathcal{B}_n| = \infty. \quad (2)$$

Indeed, let $n \geq 1$, and suppose b_j and $\mathcal{B}_j$ have been determined for all $j < n$. Since $\mathcal{B}_{n-1}$ is infinite, the pigeonhole principle shows that $\mathcal{B}_n := \mathcal{B}_{n-1} \cap (p_n \mathbb{Z} + h)$ is infinite for at least one integer $h \in [0, p_n)$. Replacing h by any number $b_n \in \mathcal{B}_n$ larger than b_{n-1} , we obtain (2), thereby completing the induction.

To finish the proof, we show that $(b_1, b_2, \dots)$ is admissible. Indeed, let p_n be an arbitrary prime, and observe that $\mathcal{B}_m \subset \mathcal{B}_n \subset p_n \mathbb{Z} + b_n$ for all integers $m \geq n$; consequently,

$$|\{b_1, b_2, \dots\} \bmod p_n| = |\{b_1, \dots, b_n\} \bmod p_n| \leq n < p_n,$$

and thus $(b_1, b_2, \dots)$ avoids at least one residue class mod p_n . $\square$

Proof of Theorem 4. As $\mathcal{A}$ is an IP set, $\mathcal{A} = \text{FS}(\mathcal{M})$ for some infinite set $\mathcal{M} \subset \mathbb{N}$. We order the elements $\{m_1, m_2, \dots\}$ of $\mathcal{M}$ with $m_1 < m_2 < \dots$ and denote by $\mathcal{B}$ the set of all partial sums of the form $\sum_{k \leq \ell} m_k$. According to Lemma 1, the set $\mathcal{B}$ contains a strictly increasing sequence $(b_n)_{n \in \mathbb{N}}$ for which the tuple $(b_1, b_2, \dots)$ is admissible. Note that, for any pair (i, j) with $1 \leq i < j$, there are numbers $\ell'_i < \ell'_j$ such that

$$b_j - b_i = \sum_{\ell'_i < k \leq \ell'_j} m_k \in \mathcal{A}.$$

Next, we extract from $(b_n)_{n \in \mathbb{N}}$ a lacunary subsequence $(v_n)_{n \in \mathbb{N}}$ such that $v_{n+1}/v_n > 10$ (say) for each n ; this ensures that the differences $v_j - v_i$ are all distinct from one another. The resulting tuple $(v_1, v_2, \dots)$ is also admissible. As before, for any pair (i, j) with $1 \leq i < j$, there are numbers $\ell_i < \ell_j$ such that

$$v_j - v_i = \sum_{\ell_i < k \leq \ell_j} m_k \in \mathcal{A}. \quad (3)$$

Now, in the notation of Theorems 2 and 3, we set

$$r := L + 1, \quad s := k_2, \quad \text{and} \quad N := R(r, s).$$

We now view the finite collection $\{v_1, \dots, v_N\}$ as the vertex set $\mathcal{V}$ of a complete graph $\mathcal{G}$ with edge set $\mathcal{E} := \{e_{i,j} : 1 \leq i < j \leq N\}$; each edge $e_{i,j}$ connects vertex v_i to vertex v_j . Define a blue–red coloring of the edges via the map

$$\mathcal{C} : \mathcal{E} \rightarrow \{\text{blue}, \text{red}\}, \quad \mathcal{C}(e_{i,j}) := \begin{cases} \text{blue} & \text{if } v_j - v_i \in \mathbf{Pol}, \\ \text{red} & \text{if } v_j - v_i \notin \mathbf{Pol}. \end{cases}$$

By Theorem 3, the graph $\mathcal{G}$ contains either a blue clique on $r = L + 1$ vertices or a red clique on $s = k_2$ vertices.

We claim that the latter possibility is untenable, i.e., no such red clique exists. Indeed, suppose on the contrary that there is a complete subgraph $\mathcal{G}^b \subset \mathcal{G}$ determined by vertices $\mathcal{V}^b := \{h_1, \dots, h_{k_2}\}$ in $\mathcal{V}$ such that $\mathcal{G}^b$ is monochromatic and colored red by the map $\mathcal{C}$. By Theorem 2, there is a set $\{h'_1, h'_2\} \subset \mathcal{V}^b$ such that $n + h'_1$ and $n + h'_2$ are consecutive primes for infinitely many $n \in \mathbb{N}$; in other words, $|h'_2 - h'_1| \in \mathbf{Pol}$. But this is not possible since h'_1 and h'_2 are distinct vertices in $\mathcal{V}^b$ and hence the corresponding edge is colored red by $\mathcal{C}$.

Consequently, within the graph $\mathcal{G}$ there is a blue clique on $r = L + 1$ vertices, i.e., there is a complete subgraph $\mathcal{G}^\# \subset \mathcal{G}$ determined by vertices $\mathcal{V}^\# := \{h_1, \dots, h_{L+1}\}$ in $\mathcal{V}$ such that $\mathcal{G}^\#$ is monochromatic and colored blue by $\mathcal{C}$. Ordering $\mathcal{V}^\#$ so that $h_1 < \dots < h_{L+1}$, we have

$$\{h_j - h_i : 1 \leq i < j \leq m + 1\} \subset \mathbf{Pol}$$

since the edge determined by h_i and h_j is colored blue by $\mathcal{C}$ for $1 \leq i < j \leq m + 1$. By construction, all of the differences $h_j - h_i$ are distinct from one another. Since h_i and h_j are distinct vertices in $\mathcal{V}$, we have $h_{n+1} - h_n \in \mathcal{A}$ for each $n \in [1, L]$ in view of (3). Defining $a_n := h_{n+1} - h_n$, we obtain a sequence $a_1, \dots, a_L$ with the properties stated in the theorem, since

$$\sum_{i \leq n \leq j} a_n = \sum_{i \leq n \leq j} (h_{n+1} - h_n) = h_{j+1} - h_i \in \mathbf{Pol}.$$

This completes the proof of Theorem 4. $\square$

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References

- [1] W. D. Banks, T. Freiberg, and C. L. R. Turnage-Butterbaugh, Consecutive primes in tuples, *Acta Arith.* **167** (2015), no. 3, 261–266.
- [2] V. Bergelson, Ultrafilters, IP sets, dynamics, and combinatorial number theory, in *Ultrafilters Across Mathematics*, 23–47, Contemp. Math., 530, Amer. Math. Soc., Providence, RI, 2010.
- [3] V. Bergelson and I. Z. Ruzsa, Squarefree numbers, IP sets and ergodic theory, in *Paul Erdős and His Mathematics, I (Budapest, 1999)*, 147–160, Bolyai Soc. Math. Stud., 11, János Bolyai Math. Soc., Budapest.
- [4] H. Furstenberg and B. Weiss, Topological dynamics and combinatorial number theory, *J. Analyse Math.* **34** (1978), 61–85 (1979).
- [5] D. A. Goldston, J. Pintz, and C. Y. Yıldırım, Primes in tuples. I, *Ann. of Math. (2)* **170** (2009), no. 2, 819–862.
- [6] R. L. Graham and B. L. Rothschild, Ramsey’s theorem for n -parameter sets, *Trans. Amer. Math. Soc.* **159** (1971), 257–292.
- [7] N. B. Hindman, Finite sums from sequences within cells of a partition of N , *J. Combinatorial Theory Ser. A* **17** (1974), 1–11.
- [8] J. A. Maynard, Small gaps between primes, *Ann. of Math. (2)* **181** (2015), no. 1, 383–413.
- [9] A. de Polignac, Recherches nouvelles sur les nombres premiers, *Comptes Rendus des Séances de l’Académie des Sciences* (1849).
- [10] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc. (2)* **30** (1929), no. 4, 264–286.
- [11] T. C. Tao and T. D. Ziegler, Infinite partial sumsets in the primes, *J. Anal. Math.* **151** (2023), no. 1, 375–389.
- [12] Y. T. Zhang, Bounded gaps between primes, *Ann. of Math. (2)* **179** (2014), no. 3, 1121–1174.