



A SERIES INVOLVING A PRODUCT OF FOUR CONSECUTIVE HARMONIC NUMBERS

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Abstract

In correspondence with Goldbach, Euler began investigating series of the form $\sum_{k \geq 1} k^{-m} (1 + 2^{-n} + \cdots + k^{-n})$, which are known today as Euler sums. For the case where $n = 1$ and $m \geq 2$, Euler was able to obtain a closed form in terms of zeta values. We use elementary techniques in the spirit of Euler to evaluate the series $\sum_{k \geq 1} \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{k(k+1)(k+2)(k+3)}$, where $H_k := 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ is the k th harmonic number, in terms of zeta values. The closed form is a potential counterexample to a conjecture of Furdui and Sîntămărian. We relate this problem to conjectures regarding irrationality and $\mathbb{Q}$ -linear independence of zeta values.

1. Introduction

Euler originally investigated the series

$$\sum_{k=1}^{\infty} \frac{1}{k^n} \left(1 + \frac{1}{2^m} + \cdots + \frac{1}{k^m} \right), \quad (m \in \mathbb{N}, n \in \mathbb{N} \setminus \{1\})$$

in [7], where $\mathbb{N} := \{1, 2, \dots\}$. Euler and Goldbach were able to determine that

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3) \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{H_k}{k^3} = \frac{5}{4}\zeta(4) \quad (1)$$

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respectively [16, pp. 238–240]. Here, $H_k := 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ is the k th harmonic number and

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad (\Re(s) > 1) \quad (2)$$

denotes the classical Riemann zeta function. The sums Euler and Goldbach derived are specific cases of the formula

$$\sum_{k=1}^{\infty} \frac{H_k}{k^n} = \left(\frac{n}{2} + 1\right) \zeta(n+1) - \frac{1}{2} \sum_{k=1}^{n-2} \zeta(n-k) \zeta(k+1), \quad (n \in \mathbb{N} \setminus \{1\}). \quad (3)$$

The empty sum that arises at $n = 2$ is understood to be nil. We mention that Equation (3) is due to Euler [5, 18]. Numerous results pertaining to Euler sums can be found in [5, 9, 17, 14, 23, 26, 24, 22, 25]. Euler sums are related to the multiple zeta function, defined as

$$\zeta(s_1, \dots, s_r) = \sum_{n_1 > \dots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}},$$

which has been studied in more recent times for its significance in diverse areas of mathematics [4, 12, 27]. Note that when $s_1, \dots, s_r$ are positive integers with $s_1 > 1$, the result is an Euler sum.

In 1993, an undergraduate student at the University of Waterloo named Enrico Au-Yeung rediscovered and conjectured the following value for the following quadratic sum:

$$\sum_{k=1}^{\infty} \left(\frac{H_k}{k}\right)^2 = \frac{17}{4} \zeta(4). \quad (4)$$

David Borwein and Jonathan Borwein would eventually verify the result using techniques involving Fourier series and Parseval's theorem in [3]. A more elementary proof is provided by Vălean and Furdul in [20]. In honor of Au-Yeung, the series is called the quadratic series of Au-Yeung. The cubic version of Au-Yeung's series was computed in [19] to be

$$\sum_{k=1}^{\infty} \left(\frac{H_k}{k}\right)^3 = \frac{93}{16} \zeta(6) - \frac{5}{2} \zeta^2(3).$$

One would expect that the quartic analogue can also be expressed zeta values. However, it was determined in [21] that

$$\sum_{k=1}^{\infty} \left(\frac{H_k}{k}\right)^4 = \frac{13559}{144} \zeta(8) - 92 \zeta(3) \zeta(5) - 2 \zeta(2) \zeta^2(3) + 26 \mathbf{S}_{2,6}$$

where

$$\mathbf{S}_{m,n} := \sum_{k=1}^{\infty} \frac{1}{k^n} \left(1 + \frac{1}{2^m} + \cdots + \frac{1}{k^m}\right), \quad (m \in \mathbb{N}, n \in \mathbb{N} \setminus \{1\}),$$

which is an Euler sum not known to be reducible to zeta values when $m = 2$ and $n = 6$.

Furdui and Sîntămărian considered a different type of series involving products of harmonic numbers; rather than raising $\frac{H_k}{k}$ to an integer power, they considered series involving the consecutive product $\frac{H_k H_{k+1} \cdots H_{k+q}}{k(k+1) \cdots (k+q)}$ for some integer $q \geq 1$. They were able to compute the following series in closed form:

$$\sum_{k=1}^{\infty} \frac{H_k H_{k+1}}{k(k+1)} = \zeta(2) + 2\zeta(3) \quad (5)$$

and

$$\sum_{k=1}^{\infty} \frac{H_k H_{k+1} H_{k+2}}{k(k+1)(k+2)} = -\frac{1}{2}\zeta(2) + \frac{5}{4}\zeta(3) + \frac{5}{8}\zeta(4). \quad (6)$$

A generalization of Equation (5) is provided in [10]:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{H_k H_{k+\ell}}{k(k+\ell)} &= \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell-1}{i} \frac{1}{(i+1)^3} \left(\frac{3}{i+1} - \frac{2}{\ell} \right) \\ &\quad + \frac{\pi^2}{6} \frac{H_{\ell}}{\ell} + \frac{2\zeta(3)}{\ell} - \frac{1}{\ell} \sum_{j=1}^{\ell} \frac{H_j}{j^2}, \quad (\ell \geq 1). \end{aligned}$$

In [16, p. 119], Equation (5) is computed and Equation (6) is left as an open problem. In [11], Equation (6) is computed. In the very same article, Furdui and Sîntămărian make the following conjecture.

Conjecture 1. If $q \geq 1$ is an integer, then

$$\sum_{k=1}^{\infty} \frac{H_k H_{k+1} \cdots H_{k+q}}{k(k+1) \cdots (k+q)} = \alpha_{q,2}\zeta(2) + \alpha_{q,3}\zeta(3) + \cdots + \alpha_{q,q+2}\zeta(q+2)$$

where $\alpha_{q,2}, \alpha_{q,3}, \dots, \alpha_{q,q+2}$ are rationals.

We note that Conjecture 1 possibly fails when $q = 3$, as the series might not be expressible as a linear combination of zeta values $\zeta(2), \zeta(3), \zeta(4), \zeta(5)$ over the rationals. To illustrate this, we use elementary methods to directly show that

$$\sum_{k=1}^{\infty} \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{k(k+1)(k+2)(k+3)} = -\frac{4}{9}\zeta(2) - \frac{1}{6}\zeta(2)\zeta(3) - \frac{7}{24}\zeta(3) + \frac{191}{144}\zeta(4) + \frac{1}{2}\zeta(5).$$

2. Proof of the Main Result

We require some lemmas. The proof of the following lemma relies on repeatedly applying partial fraction decomposition and the well-known relation $H_{k+q} = H_{k+p} + \frac{1}{k+p+1} + \frac{1}{k+p+2} + \cdots + \frac{1}{k+q}$ where $q > p \geq 0$ are integers.

Lemma 1. *Let*

$$\mathcal{H}_k := \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{k(k+1)(k+2)(k+3)}.$$

Then we have

$$\begin{aligned} \mathcal{H}_k = & \frac{11}{24(k+1)^2} - \frac{131}{72(k+2)^2} + \frac{11}{12(k+3)^2} - \frac{5}{9(k+1)^3} + \frac{109}{72(k+2)^3} \\ & - \frac{41}{36(k+3)^3} + \frac{1}{6(k+1)^4} - \frac{5}{12(k+2)^4} + \frac{11}{36(k+3)^4} + \frac{11H_k}{24k} \\ & - \frac{41H_{k+1}}{18(k+1)} + \frac{197H_{k+2}}{72(k+2)} - \frac{11H_{k+3}}{12(k+3)} + \frac{73H_k^2}{72k} - \frac{11H_{k+1}^2}{3(k+1)} \\ & + \frac{97H_{k+2}^2}{24(k+2)} - \frac{25H_{k+3}^2}{18(k+3)} + \frac{13H_k^3}{18k} - \frac{13H_{k+1}^3}{6(k+1)} + \frac{13H_{k+2}^3}{6(k+2)} - \frac{13H_{k+3}^3}{18(k+3)} \\ & + \frac{H_k^4}{6k} - \frac{H_{k+1}^4}{2(k+1)} + \frac{H_{k+2}^4}{2(k+2)} - \frac{H_{k+3}^4}{6(k+3)} + \frac{9H_{k+1}}{4(k+1)^2} - \frac{29H_{k+2}}{6(k+2)^2} \\ & + \frac{91H_{k+3}}{36(k+3)^2} - \frac{2H_{k+1}}{3(k+1)^3} + \frac{13H_{k+2}}{6(k+2)^3} - \frac{4H_{k+3}}{3(k+3)^3} + \frac{H_{k+3}}{6(k+3)^4} \\ & + \frac{9H_{k+1}^2}{4(k+1)^2} - \frac{15H_{k+2}^2}{4(k+2)^2} + \frac{7H_{k+3}^2}{4(k+3)^2} + \frac{H_{k+2}^2}{2(k+2)^3} - \frac{H_{k+3}^2}{2(k+3)^3} \\ & + \frac{H_{k+1}^3}{2(k+1)^2} - \frac{H_{k+2}^3}{(k+2)^2} + \frac{H_{k+3}^3}{2(k+3)^2}. \end{aligned}$$

Proof. Our goal is to expand $\mathcal{H}_k$ to express it as a rational linear combination of terms of the form

$$\frac{H_{k+m}^\ell}{(k+m)^n}, \quad (\ell, m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}).$$

We begin by applying partial fraction decomposition:

$$\begin{aligned} \mathcal{H}_k &= H_k H_{k+1} H_{k+2} H_{k+3} \left(\frac{1}{6k} - \frac{1}{2(k+1)} + \frac{1}{2(k+2)} - \frac{1}{6(k+3)} \right) \\ &= \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{6k} - \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{2(k+1)} + \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{2(k+2)} \\ &\quad - \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{6(k+3)}. \end{aligned}$$

We direct our attention to $\frac{H_k H_{k+1} H_{k+2} H_{k+3}}{6k}$:

$$\begin{aligned} & \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{6k} \\ &= \frac{H_k \left(H_k + \frac{1}{k+1}\right) \left(H_k + \frac{1}{k+1} + \frac{1}{k+2}\right) \left(H_k + \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3}\right)}{6k} \\ &= \frac{H_k^4}{6k} + \frac{H_k^3}{2k(k+1)} + \frac{H_k^3}{3k(k+2)} + \frac{H_k^3}{6k(k+3)} + \frac{H_k^2}{2k(k+1)^2} + \frac{H_k^2}{6k(k+2)^2} \\ & \quad + \frac{2H_k^2}{3k(k+1)(k+2)} + \frac{H_k^2}{3k(k+1)(k+3)} + \frac{H_k^2}{6k(k+2)(k+3)} \\ & \quad + \frac{H_k}{6k(k+1)^3} + \frac{H_k}{6k(k+1)(k+2)^2} + \frac{H_k}{3k(k+1)^2(k+2)} \\ & \quad + \frac{H_k}{6k(k+1)^2(k+3)} + \frac{H_k}{6k(k+1)(k+2)(k+3)}. \end{aligned}$$

We again apply partial fraction decomposition to every single term:

$$\begin{aligned} \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{6k} &= \frac{11H_k}{24k} - \frac{11H_k}{24(k+1)} + \frac{H_k}{24(k+2)} - \frac{H_k}{24(k+3)} \\ & \quad - \frac{7H_k}{12(k+1)^2} + \frac{H_k}{12(k+2)^2} - \frac{H_k}{6(k+1)^3} + \frac{73H_k^2}{72k} \\ & \quad - \frac{4H_k^2}{3(k+1)} + \frac{5H_k^2}{24(k+2)} + \frac{H_k^2}{9(k+3)} - \frac{H_k^2}{2(k+1)^2} \\ & \quad - \frac{H_k^2}{12(k+2)^2} + \frac{13H_k^3}{18k} - \frac{H_k^3}{2(k+1)} - \frac{H_k^3}{6(k+2)} \\ & \quad - \frac{H_k^3}{18(k+3)} + \frac{H_k^4}{6k}. \end{aligned}$$

If there is a term of the form $\frac{H_{k+p}^\ell}{(k+q)^n}$ where $p = q$ then we are done. If $p > q$, we substitute $H_{k+p} = H_{k+q} + \frac{1}{k+q+1} + \frac{1}{k+q+2} + \cdots + \frac{1}{k+p}$. If $p < q$, we substitute $H_{k+p} = H_{k+q} - \frac{1}{k+q} - \frac{1}{k+q-1} - \cdots - \frac{1}{k+p+1}$. After this substitution, expand the new expression and apply partial fraction decomposition. The result may again contain some terms where $p \neq q$, in which case we repeat our substitution of H_{k+p} . We achieve our goal by iterating this process. We apply the same principles to $\frac{H_k H_{k+1} H_{k+2} H_{k+3}}{2(k+1)}$, $\frac{H_k H_{k+1} H_{k+2} H_{k+3}}{2(k+2)}$, and $\frac{H_k H_{k+1} H_{k+2} H_{k+3}}{6(k+3)}$, eventually giving us the desired result. The tedious calculation details are omitted for the sake of brevity. $\square$

We proceed with the following lemma.

Lemma 2. *The following equalities hold:*

$$\begin{aligned} \text{(a)} \quad \mathcal{S}_1 &:= \sum_{k=1}^{\infty} \left(\frac{11}{2(k+1)^2} - \frac{131}{6(k+2)^2} + \frac{11}{(k+3)^2} \right) = \frac{491}{72} - \frac{16}{3}\zeta(2), \\ \text{(b)} \quad \mathcal{S}_2 &:= \sum_{k=1}^{\infty} \left(-\frac{5}{(k+1)^3} + \frac{109}{8(k+2)^3} - \frac{41}{4(k+3)^3} \right) = \frac{2735}{1728} - \frac{13}{8}\zeta(3), \\ \text{(c)} \quad \mathcal{S}_3 &:= \sum_{k=1}^{\infty} \left(\frac{1}{(k+1)^4} - \frac{5}{2(k+2)^4} + \frac{11}{6(k+3)^4} \right) = -\frac{611}{1944} + \frac{1}{3}\zeta(4). \end{aligned}$$

Proof. (a) By re-indexing, we have

$$\begin{aligned} \mathcal{S}_1 &= \frac{11}{2} \sum_{k=2}^{\infty} \frac{1}{k^2} - \frac{131}{6} \sum_{k=3}^{\infty} \frac{1}{k^2} + 11 \sum_{k=4}^{\infty} \frac{1}{k^2} \\ &= \frac{11}{2} \left(-\frac{1}{1^2} + \sum_{k=1}^{\infty} \frac{1}{k^2} \right) - \frac{131}{6} \left(-\frac{1}{1^2} - \frac{1}{2^2} + \sum_{k=1}^{\infty} \frac{1}{k^2} \right) \\ &\quad + 11 \left(-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} + \sum_{k=1}^{\infty} \frac{1}{k^2} \right) \\ &= \frac{491}{72} - \frac{16}{3} \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &= \frac{491}{72} - \frac{16}{3} \zeta(2). \end{aligned}$$

The proofs of (b) and (c) are very similar to that of (a), and are thus omitted for the sake of brevity. $\square$

Lemma 3. *The following equalities hold:*

$$\begin{aligned} \text{(a)} \quad \mathcal{S}_4 &:= \sum_{k=1}^{\infty} \left(\frac{11H_k}{4k} - \frac{41H_{k+1}}{3(k+1)} + \frac{197H_{k+2}}{12(k+2)} - \frac{11H_{k+3}}{2(k+3)} \right) = -\frac{299}{144}, \\ \text{(b)} \quad \mathcal{S}_5 &:= \sum_{k=1}^{\infty} \left(\frac{73H_k^2}{24k} - \frac{11H_{k+1}^2}{k+1} + \frac{97H_{k+2}^2}{8(k+2)} - \frac{25H_{k+3}^2}{6(k+3)} \right) = -\frac{6445}{5184}, \\ \text{(c)} \quad \mathcal{S}_6 &:= \sum_{k=1}^{\infty} \left(\frac{H_k^3}{3k} - \frac{H_{k+1}^3}{k+1} + \frac{H_{k+2}^3}{k+2} - \frac{H_{k+3}^3}{3(k+3)} \right) = -\frac{26}{243}, \\ \text{(d)} \quad \mathcal{S}_7 &:= \sum_{k=1}^{\infty} \left(\frac{H_k^4}{3k} - \frac{H_{k+1}^4}{k+1} + \frac{H_{k+2}^4}{k+2} - \frac{H_{k+3}^4}{3(k+3)} \right) = -\frac{577}{5832}. \end{aligned}$$

Proof. (a) We take advantage of telescoping:

$$\begin{aligned}\mathcal{S}_4 &= \frac{11}{4} \sum_{k=1}^{\infty} \left(\frac{H_k}{k} - \frac{H_{k+1}}{k+1} \right) - \frac{131}{12} \sum_{k=1}^{\infty} \left(\frac{H_{k+1}}{k+1} - \frac{H_{k+2}}{k+2} \right) \\ &\quad + \frac{11}{2} \sum_{k=1}^{\infty} \left(\frac{H_{k+2}}{k+2} - \frac{H_{k+3}}{k+3} \right) \\ &= \frac{11}{4} \cdot \frac{H_1}{1} - \frac{131}{12} \cdot \frac{H_2}{2} + \frac{11}{2} \cdot \frac{H_3}{3} \\ &= -\frac{299}{144}.\end{aligned}$$

(b) We again take advantage of telescoping:

$$\begin{aligned}\mathcal{S}_5 &= \frac{73}{24} \sum_{k=1}^{\infty} \left(\frac{H_k^2}{k} - \frac{H_{k+1}^2}{k+1} \right) - \frac{191}{24} \sum_{k=1}^{\infty} \left(\frac{H_{k+1}^2}{k+1} - \frac{H_{k+2}^2}{k+2} \right) \\ &\quad + \frac{25}{6} \sum_{k=1}^{\infty} \left(\frac{H_{k+2}^2}{k+2} - \frac{H_{k+3}^2}{k+3} \right) \\ &= \frac{73}{24} \cdot \frac{H_1^2}{1} - \frac{191}{24} \cdot \frac{H_2^2}{2} + \frac{25}{6} \cdot \frac{H_3^2}{3} \\ &= -\frac{6445}{5184}.\end{aligned}$$

(c) Telescoping in $\mathcal{S}_6$ is more clear:

$$\begin{aligned}\mathcal{S}_6 &= \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{H_k^3}{k} - \frac{H_{k+3}^3}{k+3} \right) - \sum_{k=1}^{\infty} \left(\frac{H_{k+1}^3}{k+1} - \frac{H_{k+2}^3}{k+2} \right) \\ &= \frac{1}{3} \left(\frac{H_1^3}{1} + \frac{H_2^3}{2} + \frac{H_3^3}{3} \right) - \frac{H_2^3}{2} \\ &= -\frac{26}{243}.\end{aligned}$$

(d) The calculations for $\mathcal{S}_7$ is nearly identical to that of $\mathcal{S}_6$.

□

Lemma 4. *The following equalities hold:*

$$\begin{aligned}
 (a) \quad \mathcal{S}_8 &:= \sum_{k=1}^{\infty} \left(\frac{9H_{k+1}}{2(k+1)^2} - \frac{29H_{k+2}}{3(k+2)^2} + \frac{91H_{k+3}}{18(k+3)^2} \right) = \frac{3151}{3888} - \frac{2}{9}\zeta(3), \\
 (b) \quad \mathcal{S}_9 &:= \sum_{k=1}^{\infty} \left(-\frac{2H_{k+1}}{(k+1)^3} + \frac{13H_{k+2}}{2(k+2)^3} - \frac{4H_{k+3}}{(k+3)^3} \right) = -\frac{1807}{2592} + \frac{5}{8}\zeta(4), \\
 (c) \quad \mathcal{S}_{10} &:= \sum_{k=1}^{\infty} \frac{H_{k+3}}{(k+3)^4} = -\frac{8681}{7776} - \zeta(2)\zeta(3) + 3\zeta(5), \\
 (d) \quad \mathcal{S}_{11} &:= \sum_{k=1}^{\infty} \left(\frac{9H_{k+1}^2}{(k+1)^2} - \frac{15H_{k+2}^2}{(k+2)^2} + \frac{7H_{k+3}^2}{(k+3)^2} \right) = \frac{287}{324} + \frac{17}{4}\zeta(4).
 \end{aligned}$$

Proof. (a) By re-indexing, we have

$$\begin{aligned}
 \mathcal{S}_8 &= \frac{9}{2} \sum_{k=2}^{\infty} \frac{H_k}{k^2} - \frac{29}{3} \sum_{k=3}^{\infty} \frac{H_k}{k^2} + \frac{91}{18} \sum_{k=4}^{\infty} \frac{H_k}{k^2} \\
 &= \frac{9}{2} \left(-\frac{H_1}{1^2} + \sum_{k=1}^{\infty} \frac{H_k}{k^2} \right) - \frac{29}{3} \left(-\frac{H_1}{1^2} - \frac{H_2}{2^2} + \sum_{k=1}^{\infty} \frac{H_k}{k^2} \right) \\
 &\quad + \frac{91}{18} \left(-\frac{H_1}{1^2} - \frac{H_2}{2^2} - \frac{H_3}{3^2} + \sum_{k=1}^{\infty} \frac{H_k}{k^2} \right) \\
 &= \frac{3151}{3888} - \frac{1}{9} \sum_{k=1}^{\infty} \frac{H_k}{k^2} \\
 &\stackrel{(1),(3)}{=} \frac{3151}{3888} - \frac{2}{9}\zeta(3).
 \end{aligned}$$

(b) The calculations for $\mathcal{S}_9$ are very similar to that of $\mathcal{S}_8$.

(c) Re-index $\mathcal{S}_{10}$ and use Equation (3).

(d) The calculations for $\mathcal{S}_{11}$ are very similar to that of $\mathcal{S}_8$. However, instead of applying the formula for Euler sums (3), we plug in the quadratic series of Au-Yeung (4). $\square$

Lemma 5. *The following equalities hold:*

$$\begin{aligned}
 (a) \quad \mathcal{S}_{12} &:= \sum_{k=1}^{\infty} \left(\frac{H_{k+2}^2}{(k+2)^3} - \frac{H_{k+3}^2}{(k+3)^3} \right) = \frac{121}{972}, \\
 (b) \quad \mathcal{S}_{13} &:= \sum_{k=1}^{\infty} \left(\frac{H_{k+1}^3}{2(k+1)^2} - \frac{H_{k+2}^3}{(k+2)^2} + \frac{H_{k+3}^3}{2(k+3)^2} \right) = \frac{1237}{15552}.
 \end{aligned}$$

Proof. (a) The series telescopes.

(b) We have

$$\mathcal{S}_{13} = \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{H_{k+1}^3}{(k+1)^2} - \frac{H_{k+2}^3}{(k+2)^2} \right) - \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{H_{k+2}^3}{(k+2)^2} - \frac{H_{k+3}^3}{(k+3)^2} \right).$$

The two remaining series telescope. $\square$

We are ready to prove the main result.

Theorem 1. *The following equality holds:*

$$\sum_{k=1}^{\infty} \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{k(k+1)(k+2)(k+3)} = -\frac{4}{9}\zeta(2) - \frac{1}{6}\zeta(2)\zeta(3) - \frac{7}{24}\zeta(3) + \frac{191}{144}\zeta(4) + \frac{1}{2}\zeta(5).$$

Proof. Let

$$\mathfrak{S} := \sum_{k=1}^{\infty} \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{k(k+1)(k+2)(k+3)}.$$

Recall $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{13}$ defined in Lemmas 2, 3, 4, and 5. Using Lemma 1, we expand $\mathfrak{S}$:

$$\begin{aligned} \mathfrak{S} &= \frac{1}{12}\mathcal{S}_1 + \frac{1}{9}\mathcal{S}_2 + \frac{1}{6}\mathcal{S}_3 + \frac{1}{6}\mathcal{S}_4 + \frac{1}{3}\mathcal{S}_5 + \frac{13}{6}\mathcal{S}_6 + \frac{1}{2}\mathcal{S}_7 + \frac{1}{2}\mathcal{S}_8 + \frac{1}{3}\mathcal{S}_9 + \frac{1}{6}\mathcal{S}_{10} \\ &\quad + \frac{1}{4}\mathcal{S}_{11} + \frac{1}{2}\mathcal{S}_{12} + \mathcal{S}_{13}. \end{aligned}$$

Apply Lemmas 2, 3, 4, and 5; this concludes the proof of the theorem. $\square$

The closed form in Theorem 1 is a potential counterexample to Conjecture 1. Using Theorem 1, express the series as

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{H_k H_{k+1} H_{k+2} H_{k+3}}{k(k+1)(k+2)(k+3)} &= -\left(\frac{4}{9} + \frac{1}{6}\zeta(3)\right)\zeta(2) - \frac{7}{24}\zeta(3) + \frac{191}{144}\zeta(4) + \frac{1}{2}\zeta(5) \\ &= -\frac{4}{9}\zeta(2) - \left(\frac{1}{6}\zeta(2) + \frac{7}{24}\right)\zeta(3) + \frac{191}{144}\zeta(4) + \frac{1}{2}\zeta(5). \end{aligned} \tag{7}$$

However, this does not show that the series satisfies Conjecture 1. Since $\zeta(2)$ and Apéry's constant, $\zeta(3)$, are irrational [1], we deduce that $-\left(\frac{4}{9} + \frac{1}{6}\zeta(3)\right)\zeta(2)$ and $-\left(\frac{1}{6}\zeta(2) + \frac{7}{24}\right)\zeta(3)$ are irrational. Thus, we cannot conclude Equation (7) shows that our series satisfies Conjecture 1.

Observe that our series satisfies Conjecture 1 if and only if $\zeta(2)\zeta(3) \in \text{span}_{\mathbb{Q}}(\mathcal{A})$ where $\mathcal{A} := \{\zeta(2), \zeta(3), \zeta(4), \zeta(5)\}$. A possible way this could happen is if $\zeta(n)$ is a rational multiple of π^n for integers $n \geq 2$. Euler himself noticed through his calculations that for integers $n \geq 1$, $\zeta(2n)$ is a rational multiple of π^{2n} [6], which

prompted a search for a way to express $\zeta(2n+1)$ as a rational multiple of π^{2n+1} . If $\zeta(3)$ and $\zeta(5)$ are rational multiples of π^3 and π^5 respectively, then $\zeta(2)\zeta(3)$ would be a rational multiple of π^5 and therefore a rational multiple of $\zeta(5)$. However, there is still no known representation for $\zeta(2n+1)$ for integers $n \geq 1$, even for $\zeta(3)$ [2, pp. 266-267]. In particular, Euler conjectured that $\zeta(3) = N\pi^3$ where N is an expression involving $\ln(2)$ [13, p. 12].

In contemporary literature, mathematicians are more open to the possibility that $\frac{\zeta(3)}{\pi^3}$ is irrational [8, p. 41]. Gun, Murty, and Rath give a conditional proof of the irrationality of $\frac{\zeta(2n+1)}{\pi^{2n+1}}$ for integers $n \geq 1$ in [15, p. 1341]; the proof is contingent on the following conjecture of Chowla and Milnor.

Conjecture 2 (Chowla–Milnor). Let $\zeta(s, a) = \sum_{k \geq 0} \frac{1}{(k+a)^s}$ denote the classical Hurwitz zeta function. Then the set

$$\mathfrak{L} := \{\zeta(q, m/n) : 1 \leq m \leq n, \gcd(m, n) = 1\}, \quad (q > 1, n > 2)$$

is linearly independent over $\mathbb{Q}$.

In [15], Gun et al. perform a slight modification on $\mathfrak{L}$ by restricting $m < n$ and define the following Chowla–Milnor space:

$$V_q(n) := \text{span}_{\mathbb{Q}}(\{\zeta(q, m/n) : 1 \leq m < n, \gcd(m, n) = 1\}).$$

The dimension of $V_q(n)$ as a $\mathbb{Q}$ -linear space can shed light on problems regarding irrationality by expressing the Hurwitz zeta function in terms of the Riemann zeta function. It is also worth noting that the Chowla–Milnor conjecture has implications in multiple zeta values. Again in [15], Gun et al. give an important proposition arising from the Chowla–Milnor space.

Proposition 1. *The Chowla–Milnor conjecture for $V_q(4)$ is equivalent to the irrationality of $\frac{\zeta(2n+1)}{\pi^{2n+1}}$ for integers $n \geq 1$.*

In relation to Theorem 1, the falsehood of the Chowla–Milnor conjecture allows for the rationality of certain values of $\frac{\zeta(2n+1)}{\pi^{2n+1}}$, which relates to Conjecture 1 of Furdui and Sîntămărian. On the other hand, proving the Chowla–Milnor conjecture would only eliminate one possible way of expressing $\zeta(2)\zeta(3)$ as a rational multiple of $\zeta(5)$. More work must be done to establish a stronger implication between these related conjectures. Nevertheless, one may conjecture that $\zeta(2)\zeta(3) \notin \text{span}_{\mathbb{Q}}(\mathcal{A})$; however, verifying this is difficult.

Perhaps we can learn more by computing the closed form for

$$\sum_{k=1}^{\infty} \frac{H_k H_{k+1} \cdots H_{k+q}}{k(k+1) \cdots (k+q)}, \quad (q \in \mathbb{N}).$$

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