



THE MAXIMAL ORDER OF THE SHIFTED-PRIME DIVISOR FUNCTION

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Abstract

For each positive integer n , we denote by $\omega^*(n)$ the number of primes p such that $p - 1$ divides n . First introduced by Prachar in 1955, this function has interesting applications in primality testing and bears a strong connection with counting Carmichael numbers. Prachar showed that for a certain constant $c_0 > 0$,

$$\omega^*(n) > \exp\left(c_0 \frac{\log n}{(\log \log n)^2}\right)$$

for infinitely many n . This result was later improved by Adleman, Pomerance, and Rumely, who established an inequality of the same shape with $(\log \log n)^2$ replaced by $\log \log n$. Assuming the Generalized Riemann Hypothesis for Dirichlet L -functions (GRH), Prachar also proved the stronger inequality

$$\omega^*(n) > \exp\left(\left(\frac{1}{2} \log 2 + o(1)\right) \frac{\log n}{\log \log n}\right)$$

for infinitely many n . By refining the arguments of Prachar and of Adleman, Pomerance, and Rumely, we improve the factor “ $\frac{1}{2} \log 2 + o(1)$ ” above to $0.6736 \log 2$ unconditionally and to $\log\left(\frac{1+\sqrt{5}}{2}\right) + o(1)$ under GRH.

– *Dedicated to the 80th birthdays of Melvyn Nathanson and Carl Pomerance.*

1. Introduction

For each $n \in \mathbb{N}$, let $\omega^*(n)$ count the number of shifted primes $p - 1$ dividing n , i.e.,

$$\omega^*(n) := \sum_{p-1|n} 1.$$

Equivalently, $\omega^*(n)$ returns the number of positive divisors d of n such that $d + 1$ is prime. For instance, $\omega^*(2) = 2$, $\omega^*(4) = \omega^*(6) = \omega^*(8) = \omega^*(10) = 3$, and $\omega^*(12) = 5$. This function has found interesting applications in primality testing [1] and is closely related to counting Carmichael numbers [2]. It is easy to see that $\omega^*(n) \geq 1$ with equality if and only if n is odd. On the other hand, the function ω^* can attain fairly large values infinitely often. Indeed, Prachar, who initiated the study of ω^* in his influential work [17], showed that there is some absolute constant $c_1 > 0$ such that

$$\omega^*(n) > \exp \left(c_1 \frac{\log n}{(\log \log n)^2} \right) \quad (1)$$

for infinitely many n . Assuming the Generalized Riemann Hypothesis for Dirichlet L -functions (GRH), he was able to save a $\log \log n$ factor and provide a lower bound for c_1 . More precisely, he proved that GRH implies the inequality

$$\omega^*(n) > \exp \left(\left(\frac{1}{2} \log 2 + o(1) \right) \frac{\log n}{\log \log n} \right) \quad (2)$$

for infinitely many n .

Nearly three decades later, Adleman, Pomerance and Rumely [1] improved Prachar's unconditional lower bound (1) to

$$\omega^*(n) > \exp \left(c_2 \frac{\log n}{\log \log n} \right), \quad (3)$$

along a sequence of n tending to infinity, where $c_2 > 0$ is some absolute constant. This lower bound has the same shape as Prachar's GRH-conditional lower bound (2), up to the constant factor. They also conjectured that the constant $\frac{1}{2} \log 2$ in (2) can in fact be upgraded to $\log 2$. It is worth noting that the conjectured constant $\log 2$ would be best possible, revealing the true maximal order of ω^* , since $\omega^*(n) \leq \tau(n)$ for all $n \in \mathbb{N}$ and

$$\tau(n) \leq \exp \left((\log 2 + o(1)) \frac{\log n}{\log \log n} \right)$$

for all sufficiently large n [10, Theorem 317], where $\tau(n)$ denotes the number of positive divisors of n . Interestingly, this conjecture, if true, would imply that the Carmichael function $\lambda(n)$, which may be defined as the exponent of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$, satisfies

$$\liminf_{n \rightarrow \infty} \frac{\log \lambda(n)}{(\log \log n) \log \log \log n} = \frac{1}{\log 2},$$

an observation first made by Erdős, Pomerance, and Schmutz [5].

The study of ω^* was recently revived by Murty and Murty [15]. For each $k \in \mathbb{N}$, we define the k th moment of ω^* by

$$M_k(x) := \frac{1}{x} \sum_{n \leq x} \omega^*(n)^k.$$

These moments encode useful information about the distribution of ω^* . A quick application of Mertens' theorem [10, Theorem 427] yields $M_1(x) = \log \log x + O(1)$. In the same paper [17], Prachar showed that $M_2(x) \ll (\log x)^2$. In their recent work [15], Murty and Murty [15] proved the estimate $(\log \log x)^3 \ll M_2(x) \ll \log x$ and conjectured the asymptotic formula $M_2(x) \sim C_2 \log x$ with some constant $C_2 > 0$. Two years later, Ding [4] achieved the order matching lower bound $M_2(x) \gg \log x$. More recently, Pomerance and the first author [6] established the estimate $M_3(x) \asymp (\log x)^4$ and made the more general conjecture that for each $k \geq 2$ there exists a constant $C_k > 0$ such that $M_k(x) \sim C_k (\log x)^{2^k - k - 1}$. Based on a heuristic argument, the first author [7] also conjectured that $C_2 = \zeta(2)^2 \zeta(3) / \zeta(6)$, where ζ is the Riemann zeta-function. Despite the fact that no asymptotic formula is currently known even for a single value of $k \geq 2$, quite recently Gabdullin [8] pinned down the correct order of magnitude for $M_k(x)$ by showing that $M_k(x) \asymp (\log x)^{2^k - k - 1}$ for every fixed $k \geq 2$.

The main objective of this paper is to furnish explicit values for the constant c_2 appearing in (3). Perhaps a bit surprisingly, we are able to obtain unconditionally a numerical value for c_2 which is slightly larger than $\frac{2}{3} \log 2$, hence surpassing the constant $\frac{1}{2} \log 2$ in Prachar's GRH-conditional lower bound (2). Under GRH, our argument yields a numerical value for c_2 which exceeds $(\log 2)^2$. The following theorem provides a precise statement of our results.

Theorem 1. *There exist infinitely many n such that*

$$\omega^*(n) > \exp \left(0.6736 \log 2 \cdot \frac{\log n}{\log \log n} \right). \quad (4)$$

Moreover, if GRH is true, then we have

$$\omega^*(n) > \exp \left(\left(\log \left(\frac{1 + \sqrt{5}}{2} \right) + o(1) \right) \frac{\log n}{\log \log n} \right) \quad (5)$$

for infinitely many n .

2. An Overview of Prachar's Argument

We first outline Prachar's simple proof of (2) upon which our proof of Theorem 1 is built. In what follows, the letter p always denotes a prime. We write $\pi(x)$ for the

number of primes $p \leq x$, and let $\pi(x; d, a)$ count the number of primes $p \leq x$ with $p \equiv a \pmod{d}$.

Suppose that x is sufficiently large. Let $\epsilon \in (0, 1)$ be arbitrary, and put

$$k := \prod_{p \leq (1-\epsilon) \log x} p. \quad (6)$$

Then $k = x^{1-\epsilon+o(1)}$ by the Prime Number Theorem. Under GRH we have [14, Corollary 13.8]

$$\pi(x; d, 1) = \frac{\text{Li}(x)}{\varphi(d)} + O(\sqrt{x} \log x) \gg \frac{x}{\varphi(d) \log x}$$

uniformly for all integers $1 \leq d \leq \sqrt{k}$, where φ is the Euler totient function, and

$$\text{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

For each $1 \leq d \leq \sqrt{k}$ dividing k , denote by A_d the number of pairs (m, p) , where $m \in \mathbb{N}$ and p is prime, such that $m, p \leq x$,

$$p \equiv 1 \pmod{d} \quad \text{and} \quad \gcd(m, k) = \frac{k}{d}. \quad (7)$$

Let A record the total number of pairs (m, p) with $m, p \leq x$ satisfying the congruence $m(p-1) \equiv 0 \pmod{k}$. Every pair counted by A_d , for some $d \mid k, d \leq \sqrt{k}$, is counted by A . Moreover, an individual pair is counted by A_d for at most one d . Since the number of choices for p is clearly $\pi(x; d, 1)$, and since the count of m is at least $\lfloor x/k \rfloor \varphi(d)$, we have

$$A_d \geq \pi(x; d, 1) \left\lfloor \frac{x}{k} \right\rfloor \varphi(d) \gg \frac{x^2}{k \log x},$$

from which it follows that

$$A \geq \sum_{\substack{d \leq \sqrt{k} \\ d \mid k}} A_d \gg \frac{x^2}{k \log x} \sum_{\substack{d \leq \sqrt{k} \\ d \mid k}} 1 \geq \frac{\tau(k)x^2}{2k \log x}.$$

On the other hand, each pair (m, p) counted by A yields a positive integer $n = m(p-1) \leq x^2$ divisible by k . Thus,

$$A \leq \sum_{\substack{n \leq x^2 \\ k \mid n}} \#\{(m, p): m, p \leq x \text{ and } n = m(p-1)\}.$$

¹In fact, Prachar [17] took

$$k = \prod_{p \leq (1/2-\epsilon) \log x} p.$$

However, this appears to be a misstep: This choice of k only yields the smaller constant $\frac{1}{4} \log 2$ rather than $\frac{1}{2} \log 2$ asserted in [17] and (2).

Combining the above upper and lower bounds for A , we deduce that there exists some large $n \leq x^2$ which is divisible by k and admits

$$\gg \frac{\tau(k)x^2/(k \log x)}{x^2/k} = \frac{\tau(k)}{\log x} = \frac{2^{\pi((1-\epsilon) \log x)}}{\log x} > \exp\left((\log 2 - 2\epsilon) \frac{\log x}{\log \log x}\right)$$

representations of the form $n = m(p-1)$. For this particular n , we have

$$\omega^*(n) \geq \exp\left((\log 2 - 2\epsilon) \frac{\log x}{\log \log x}\right) > \exp\left(\left(\frac{1}{2} \log 2 - 2\epsilon\right) \frac{\log n}{\log \log n}\right),$$

as desired.

In the argument outlined above, the representations $n = m(p-1)$ counted by A_d all have $m, p \leq x$ and $d \leq \sqrt{k}$, with k defined by (6). In the next section, we shall prove Theorem 1 by adjusting the choices for k, d, m, p . We make use of insights from the theory of anatomy of integers to maximize the total number of representations $n = m(p-1)$ counted by A .

3. Refining Prachar's Argument: Proof of Theorem 1

We start by proving the GRH-conditional inequality (5).

Proof of (5) of Theorem 1. Suppose that x is sufficiently large. Put

$$\epsilon := (\log \log x)^{-1/2} \quad \text{and} \quad u := \frac{3 + \sqrt{5}}{4}, \tag{8}$$

and set

$$\begin{aligned} k &:= \prod_{p \leq (u-\epsilon) \log x} p \\ &= x^{u-\epsilon} \exp\left(O\left(\frac{\log x}{\log \log x}\right)\right). \end{aligned}$$

We will show momentarily that there is a set $\mathcal{D}$ of divisors of k , where each $d \in \mathcal{D}$ obeys the estimate

$$d = x^{\frac{1}{2}-\epsilon} \exp\left(O\left(\frac{\log x}{(\log \log x)^2}\right)\right), \tag{9}$$

and where, with $X := x^{\frac{1}{2}+u}$,

$$\#\mathcal{D} \geq \exp\left(\left(\log\left(\frac{1+\sqrt{5}}{2}\right) + o(1)\right) \frac{\log X}{\log \log X}\right)^2. \tag{10}$$

²Mikhail Gabdullin has kindly pointed out to us that the optimality of this lower bound for $\#\mathcal{D}$ can be seen to follow from [9, Theorem 2].

Let us see now how this claim implies (5).

For each $d \in \mathcal{D}$, let A_d denote the number of pairs (m, p) with $m \leq y_d := x^u/d$ and $p \leq x$, satisfying the same conditions described in (7). By inclusion-exclusion, the number of choices for m is

$$\sum_{\substack{m' \leq y_d/(k/d) \\ (m', d)=1}} 1 = \frac{\varphi(d)}{d} \cdot \frac{y_d}{k/d} + O(\tau(d)) \gg \frac{\varphi(d)}{d} \cdot \frac{x^u}{k},$$

since $\tau(d) \leq \exp\left(O\left(\frac{\log x}{\log \log x}\right)\right)$, and

$$\frac{\varphi(d)}{d} \cdot \frac{y_d}{k/d} = \frac{\varphi(d)}{d} \cdot \frac{x^u}{k} \gg \exp\left(\frac{\log x}{2(\log \log x)^{1/2}}\right).$$

Our estimate (9) for d , along with the choice of ϵ in (8), implies that $d \leq x^{1/2}/(\log x)^3$ (say), so that

$$\pi(x; d, 1) = \frac{\text{Li}(x)}{\varphi(d)} + O(\sqrt{x} \log x) \gg \frac{x}{\varphi(d) \log x}.$$

It follows that for all $d \in \mathcal{D}$,

$$A_d \gg \pi(x; d, 1) \frac{\varphi(d)}{d} \cdot \frac{x^u}{k} \gg \frac{x^{u+1}}{kd \log x} = \frac{x^{u+\frac{1}{2}+\epsilon}}{k \log x} \exp\left(O\left(\frac{\log x}{(\log \log x)^2}\right)\right). \quad (11)$$

Therefore,

$$\sum_{d \in \mathcal{D}} A_d \gg \#\mathcal{D} \frac{x^{u+\frac{1}{2}+\epsilon}}{k \log x} \exp\left(O\left(\frac{\log x}{(\log \log x)^2}\right)\right). \quad (12)$$

On the other hand, if (m, p) is counted by some A_d , then $m(p-1)$ is a multiple of k and

$$m(p-1) \leq \left(\max_{d \in \mathcal{D}} y_d\right) x = x^{u+\frac{1}{2}+\epsilon} \exp\left(O\left(\frac{\log x}{(\log \log x)^2}\right)\right). \quad (13)$$

Reasoning as in §2, we conclude upon comparing (12) and (13) that there is a large multiple n of k , not exceeding the final expression in (13), with

$$\omega^*(n) \geq \frac{\#\mathcal{D}}{\log x} \exp\left(O\left(\frac{\log x}{(\log \log x)^2}\right)\right).$$

Substituting the lower bound (10) for $\#\mathcal{D}$ gives

$$\begin{aligned} \omega^*(n) &\geq \exp\left(\left(\log\left(\frac{1+\sqrt{5}}{2}\right) + o(1)\right) \frac{\log X}{\log \log X}\right) \\ &\geq \exp\left(\left(\log\left(\frac{1+\sqrt{5}}{2}\right) + o(1)\right) \frac{\log n}{\log \log n}\right), \end{aligned}$$

where we use in the second line that $n \leq X^{1+o(1)}$.

To show the existence of the set $\mathcal{D}$, we employ the probabilistic method. Let

$$L = (u - \epsilon) \log x \quad \text{and} \quad R = \pi(L).$$

Furthermore, set

$$\rho = \frac{\frac{1}{2} - \epsilon}{u - \epsilon}.$$

We introduce independent and identically distributed Bernoulli random variables v_r , for each prime $r \leq L$, where every v_r takes the value 1 with probability ρ . Then

$$d := \prod_{r \leq L} r^{v_r}$$

is a random divisor of k . We proceed to study the distribution of d .

It is straightforward to compute the expectation and variance of the random variable $\log d = \sum_{r \leq L} v_r \log r$: We have

$$\mathbb{E}[\log d] = \sum_{r \leq L} \mathbb{E}[v_r \log r] = \rho \sum_{r \leq L} \log r = \left(\frac{1}{2} - \epsilon\right) \log x + O(L/(\log L)^3),$$

$$\mathbb{V}[\log d] = \sum_{r \leq L} \mathbb{V}[v_r \log r] = \rho(1 - \rho) \sum_{r \leq L} (\log r)^2 \ll \sum_{r \leq L} (\log r)^2 \ll L \log L.$$

This latter estimate implies, via Chebyshev's inequality, that $|\log d - \mathbb{E}[\log d]| > L^{2/3}$ with probability $O(L^{-1/3} \log L) = o(1)$. Thus, with probability $1 + o(1)$,

$$\begin{aligned} \left| \log d - \left(\frac{1}{2} - \epsilon\right) \log x \right| &\leq |\log d - \mathbb{E}[\log d]| + \left| \mathbb{E}[\log d] - \left(\frac{1}{2} - \epsilon\right) \log x \right| \\ &\leq L^{2/3} + L/(\log L)^2 < 2L/(\log L)^2. \end{aligned} \quad (14)$$

Let $\mathcal{D}$ be the set of divisors d of k for which $|\log d - (\frac{1}{2} - \epsilon) \log x| < 2L/(\log L)^2$. For each $d \in \mathcal{D}$, the desired estimate (9) holds, and we have just seen that $d \in \mathcal{D}$ with probability $1 + o(1)$, as $x \rightarrow \infty$. We proceed to translate this probability bound into a lower bound on $\#\mathcal{D}$. In fact, we will obtain the claimed lower bound (10) for a certain convenient subset of $\mathcal{D}$.

The mean and variance of $\Omega(d) = \sum_{r \leq L} v_r$ satisfy $\mathbb{E}[\Omega(d)] = \rho R$, $\mathbb{V}[\Omega(d)] \ll R$. So if we let $\mathcal{E}$ denote the set of $d \mid k$ with $|\Omega(d) - \rho R| > R^{2/3}$, then $\mathbb{P}(d \in \mathcal{E}) = o(1)$ by another application of Chebyshev's inequality. We put $\mathcal{D}' := \mathcal{D} \setminus \mathcal{E}$ and observe that

$$\mathbb{P}(d \in \mathcal{D}') \geq \mathbb{P}(d \in \mathcal{D}) - \mathbb{P}(d \in \mathcal{E}) = 1 + o(1).$$

If $d \in \mathcal{D}'$, then $v_r = 1$ for $\rho R + O(R^{2/3})$ primes $r \leq L$, while $v_r = 0$ for $(1 - \rho)R + O(R^{2/3})$ primes $r \leq L$. Hence, each $d \in \mathcal{D}'$ carries a probability mass of

$$\rho^{\rho R} (1 - \rho)^{(1 - \rho)R} \exp(O(R^{2/3})).$$

In order for the probability masses corresponding to $d \in \mathcal{D}'$ to sum to $1 + o(1)$, it must be that

$$\#\mathcal{D} \geq \#\mathcal{D}' \geq \rho^{-\rho R} (1 - \rho)^{-(1-\rho)R} \exp(O(R^{2/3})). \quad (15)$$

Since $R = (1 + o(1))u \frac{\log x}{\log \log x}$ while $\rho = \frac{1}{2u} + o(1)$, we have

$$\rho^{-\rho R} (1 - \rho)^{-(1-\rho)R} \exp(O(R^{2/3})) = \exp((C + o(1)) \log x / \log \log x), \quad (16)$$

where

$$\begin{aligned} C &= \frac{1}{2} \log(2u) + \left(u - \frac{1}{2}\right) \log \frac{2u}{2u-1} \\ &= \left(u + \frac{1}{2}\right) \log \frac{2u}{2u-1} + \left(\log \sqrt{2u} - \log \frac{2u}{2u-1}\right) \\ &= \left(u + \frac{1}{2}\right) \log \frac{1 + \sqrt{5}}{2}, \end{aligned}$$

noting for the last line that $\frac{2u}{2u-1} = \frac{1+\sqrt{5}}{2} = \sqrt{2u}$. Finally, since $X = x^{\frac{1}{2}+u}$,

$$\left(u + \frac{1}{2}\right) \frac{\log x}{\log \log x} = (1 + o(1)) \frac{\log X}{\log \log X}. \quad (17)$$

The lower bound (10) on $\#\mathcal{D}$ follows from (15), (16) and (17). $\square$

The unconditional inequality (4) follows in a similar fashion, using the following result of Harman (see [12, Theorem 1.2]) as a proxy for the GRH.

Proposition 1 ([12]). *There is an absolute constant $\delta > 0$ making the following true. For each $\eta > 0$, there are constants $K \geq 2$ and $c > 0$ such that the following holds. Suppose*

$$K < d < x^{0.4736}, \quad \text{and} \quad p \mid d \Rightarrow p < d^\delta.$$

Furthermore, assume that for every $f \mid d$ and primitive character $\chi \bmod f$,

$$L(s, \chi) \neq 0 \quad \text{for} \quad \operatorname{Re}(s) > 1 - \frac{1}{(\log d)^{3/4}}, \quad |\operatorname{Im}(s)| \leq \exp(\eta(\log d)^{3/4}). \quad (18)$$

Then for every a with $\gcd(a, d) = 1$, we have

$$\pi(x; d, a) \geq \frac{cx}{\phi(d) \log x}.$$

Proof of (4) of Theorem 1. Put

$$\epsilon := (\log \log x)^{-1/2} \quad \text{and} \quad \theta := 0.4736,$$

and define $u \approx 1.2694$ to be the unique point at which the function

$$\begin{aligned} f_\theta(t) &:= \frac{1}{t+1-\theta} \left(\theta \log \frac{t}{\theta} - (t-\theta) \log \left(1 - \frac{\theta}{t} \right) \right) \\ &= \frac{t \log t - (t-\theta) \log(t-\theta) - \theta \log \theta}{t+1-\theta} \end{aligned} \quad (19)$$

attains its global maximum $f_\theta(u) \approx 0.4669$ on $[\theta, \infty)$.

Below, we describe how Proposition 1 can be used to find a positive integer k dividing $\prod_{p \leq (u-\epsilon) \log x} p$, along with a set $\mathcal{D}$ of divisors of k , where each $d \in \mathcal{D}$ has the property that

$$\pi(x; d, 1) \gg \frac{x}{\phi(d) \log x}. \quad (20)$$

Furthermore, $\mathcal{D}$ will be selected in such a way that each $d \in \mathcal{D}$ obeys the estimate

$$d = x^{\theta-\epsilon} \exp \left(O \left(\frac{\log x}{(\log \log x)^2} \right) \right), \quad (21)$$

and such that

$$\#\mathcal{D} \geq \exp \left((C' + o(1)) \frac{\log x}{\log \log x} \right) = \exp \left((f_\theta(u) + o(1)) \frac{\log Y}{\log \log Y} \right), \quad (22)$$

with

$$Y := x^{u+1-\theta}, \quad \text{and} \quad C' := \theta \log \frac{u}{\theta} - (u-\theta) \log \left(1 - \frac{\theta}{u} \right) = (u+1-\theta)f_\theta(u).$$

After k and $\mathcal{D}$ have been located, the rest of the argument can be carried out as before. For each $d \in \mathcal{D}$, let A_d denote the number of pairs (m, p) with $m \leq y_d = x^u/d$ and $p \leq x$, satisfying the conditions in (7). Then we have

$$A_d \gg \pi(x; d, 1) \frac{\varphi(d)}{d} \cdot \frac{x^u}{k} \gg \frac{x^{u+1}}{kd \log x} = \frac{x^{u+1-\theta+\epsilon}}{k \log x} \exp \left(O \left(\frac{\log x}{(\log \log x)^2} \right) \right),$$

and

$$\sum_{d \in \mathcal{D}} A_d \gg \frac{x^{u+1-\theta+\epsilon}}{k \log x} \exp \left((f_\theta(u) + o(1)) \frac{\log Y}{\log \log Y} \right).$$

Each pair (m, p) counted by some A_d corresponds to a multiple $m(p-1)$ of k for which

$$m(p-1) \leq x^{u+1-\theta+\epsilon} \exp \left(O \left(\frac{\log x}{(\log \log x)^2} \right) \right).$$

Comparing the last two displays, we conclude that there is an $n \leq x^{u+1-\theta+o(1)}$ with

$$\omega^*(n) \geq \exp \left((f_\theta(u) + o(1)) \frac{\log Y}{\log \log Y} \right) \geq \exp \left((f_\theta(u) + o(1)) \frac{\log n}{\log \log n} \right).$$

As $f_\theta(u)/\log 2 = 0.67365\dots > 0.6736$, the estimate (4) follows.

To produce k and $\mathcal{D}$, we borrow ideas and results from [11, pp. 647–648]. Let

$$W := \left(\frac{2}{5} \log x\right)^{3/4}.$$

As on p. 647 of [11], for some absolute constant $\eta > 0$, there is at most one primitive character $\chi_1 \bmod f_1$ of conductor

$$f_1 < V := \exp(\eta(\log x)^{3/4})$$

for which $L(s, \chi_1)$ has a zero ρ with

$$\operatorname{Re}(\rho) > 1 - \frac{1}{W}, \quad |\operatorname{Im}(\rho)| \leq V. \quad (23)$$

(This follows from the results on exceptional zeros appearing on pp. 93–95 of [3].) We will apply Proposition 1 with this η . Note that if $x^{0.4} < d < x^\theta$, in order for (18) to fail, $L(s, \chi)$ must have a zero ρ belonging to the region (23).

If the primitive character $\chi_1 \bmod f_1$ of the last paragraph exists, we let p_1 be a prime factor of f_1 . Otherwise, we let $p_1 = 1$. Let $L = (u - \epsilon) \log x$ and $R = \pi(L)$ (as before), and set

$$\rho = \frac{\theta - \epsilon}{u - \epsilon}.$$

We take

$$k = \prod_{\substack{p \leq L \\ p \neq p_1}} p = x^{u-\epsilon} \exp\left(O\left(\frac{\log x}{\log \log x}\right)\right),$$

and we let $d = \prod_{r|k} r^{v_r}$, where the v_r are independent and identically distributed Bernoulli random variables with each $\mathbb{P}(v_r = 1) = \rho$.

By our earlier arguments, a random d satisfies both

$$|\log d - (\theta - \epsilon) \log x| < L^{2/3} \quad (24)$$

and

$$|\Omega(d) - \rho R| < R^{2/3} \quad (25)$$

with probability $1 + o(1)$. (The possibly missing prime p_1 has a negligible effect.) Let $\mathcal{D}_0$ be the set of divisors d of k satisfying (24) and (25). We proceed to remove from $\mathcal{D}_0$ those d for which there is a primitive character $\chi \bmod f$, $f \mid d$, where (18) fails. Since $\gcd(d, p_1) = 1$, in these cases we necessarily have $f \geq V$. Furthermore, $L(s, \chi)$ has a zero in the region (23).

Zero density estimates (e.g., [13, Theorem 1] suffices here) show that there are no more than $\exp(O((\log x)^{1/4}))$ primitive characters $\chi \bmod f$, $f \leq x$, having a zero

in the region (23). (Compare with pp. 647–648 of [11].) From that set of characters, throw away those of conductors smaller than V , and collect their remaining conductors in a set $\mathcal{F}$. Then each d to be removed from $\mathcal{D}_0$ is divisible by some $f \in \mathcal{F}$.

We fix $f \in \mathcal{F}$ and examine the probability that $f \mid d$. If $f \nmid k$, then $\mathbb{P}(f \mid d) = 0$. Otherwise, $\mathbb{P}(f \mid d) = \rho^{\omega(f)}$, where $\omega(f)$ denotes the number of distinct prime factors of f . Since

$$V \leq f \leq (u \log x)^{\omega(f)},$$

we have

$$\omega(f) \geq \frac{\log V}{\log(u \log x)} = \frac{\eta(\log x)^{3/4}}{\log(u \log x)}.$$

Thus (for large x),

$$\mathbb{P}(f \mid d) = \rho^{\omega(f)} < \exp(-(\log x)^{7/10}).$$

Hence,

$$\mathbb{P}(f \mid d \text{ for some } f \in \mathcal{F}) \leq \#\mathcal{F} \exp(-(\log x)^{7/10}) = o(1),$$

recalling for the last equality that $\#\mathcal{F} \leq \exp(O((\log x)^{1/4}))$.

Therefore, after removing all $d \in \mathcal{D}_0$ divisible by an $f \in \mathcal{F}$, we are left with a set $\mathcal{D}$ of divisors of k for which $\mathbb{P}(d \in \mathcal{D}) = 1 + o(1)$. We see from (24) that each $d \in \mathcal{D}$ satisfies (21). Furthermore, invoking (25) and repeating the argument leading to (10), we arrive at (22). Finally, Proposition 1 furnishes the desired lower bound (20) on $\pi(x; d, 1)$ for all $d \in \mathcal{D}$. $\square$

4. Concluding Remarks

We have seen that a key, common ingredient in Prachar's argument and the proof of Theorem 1 is an inequality of the form

$$\pi(x; d, 1) \gg \frac{x}{\phi(d) \log x} \tag{26}$$

for $d \mid k$, where k is essentially the product of all primes $p \leq \theta \log x$ with some $\theta \in (0, 1)$. The proof of Theorem 1 reveals that if (26) holds for some fixed $\theta \in (0, 1)$, then we have

$$\omega^*(n) \geq \exp \left(\left(\max_{t \geq \theta} f_\theta(t) + o(1) \right) \frac{\log n}{\log \log n} \right)$$

for infinitely many n , where $f_\theta(t)$ is defined as in (19). In particular, the conjecture of Adleman, Pomerance and Rumely [1] mentioned in the introduction, that

$$\omega^*(n) \geq \exp \left((\log 2 + o(1)) \frac{\log n}{\log \log n} \right) \tag{27}$$

for infinitely many n , would follow if (26) holds for any fixed $\theta \in (0, 1)$ (since, for instance, $\max_{t \geq \theta} f_\theta(t) \geq f_\theta(2\theta) = \frac{2\theta}{\theta+1} \log 2$). As [2, Theorem 2.1] shows, the lower bound (26) is intimately related to zero densities for Dirichlet L -functions. An implication of this is that our inequality (5) still holds under the Density Hypothesis which is weaker than GRH.

Note that in Prachar's argument, if we sum A_d over all divisors d of k instead, we would have

$$\sum_{d|k} A_d \gg \frac{x}{k} \sum_{d|k} \phi(d) \pi(x; d, 1) = \frac{x}{k} \sum_{p \leq x} \sum_{\substack{d|k \\ d|p-1}} \phi(d) = \frac{x}{k} \sum_{p \leq x} \gcd(p-1, k).$$

The last sum hints at the connection between the maximal order of ω^* and the distribution of smooth shifted primes $p-1$. Given $y \geq 1$, we say that $n \in \mathbb{N}$ is y -smooth if $P^+(n) \leq y$, where $P^+(n)$ denotes the greatest prime factor of n , with the convention that $P^+(1) = 1$. In other words, y -smooth numbers are precisely those integers with no prime factors exceeding y . For $x \geq y \geq 1$, we define the counting functions

$$\begin{aligned} \Psi(x, y) &:= \#\{n \leq x : P^+(n) \leq y\}, \\ \pi(x, y) &:= \#\{p \leq x : P^+(p-1) \leq y\}. \end{aligned}$$

In contrast to $\Psi(x, y)$ whose asymptotic behavior is rather well-understood (see for instance [18, Chapter III.5]), the function $\pi(x, y)$ has remained elusive. Nevertheless, it is widely believed that smooth shifted primes have the same asymptotic density relative to shifted primes as smooth integers do relative to integers. Indeed, Pomerance [16] has conjectured that if $x \geq y \geq 1$, then

$$\frac{\pi(x, y)}{\pi(x)} \sim \frac{\Psi(x, y)}{x} \quad (28)$$

as $y \rightarrow \infty$. We conclude our paper with a demonstration that the Adleman–Pomerance–Rumely conjecture (27) is an easy consequence of Pomerance's conjecture (28).

Assume (28). Fix $v > 0$ and set $y = v \log x$ with x sufficiently large. The number of pairs (m, p) , with $m, p \leq x$, $P^+(m) \leq y$, and $P^+(p-1) \leq y$, is $\Psi(x, y)\pi(x, y)$. Since each $n = (m-1)p \leq x^2$ is y -smooth, and since the number of y -smooth numbers up to x^2 is precisely given by $\Psi(x^2, y)$, we deduce from (28) that there is some large y -smooth number $n \leq x^2$ with at least

$$\frac{\Psi(x, y)\pi(x, y)}{\Psi(x^2, y)} \sim \frac{\Psi(x, y)^2}{\Psi(x^2, y)} \cdot \frac{\pi(x)}{x} \sim \frac{\Psi(x, y)^2}{\Psi(x^2, y)} \cdot \frac{1}{\log x} \quad (29)$$

representations as $n = m(p - 1)$. By [18, Theorem III.5.2], we have

$$\begin{aligned}\log \Psi(x, y) &= \left(1 + O\left(\frac{1}{\log \log x}\right)\right) \frac{\log x}{\log y} \int_0^1 \log\left(1 + \frac{y}{t \log x}\right) dt \\ &= \left(1 + O\left(\frac{1}{\log \log x}\right)\right) \frac{\log x}{\log \log x} \int_0^1 \log\left(1 + \frac{v}{t}\right) dt,\end{aligned}$$

and analogously,

$$\log \Psi(x^2, y) = \left(1 + O\left(\frac{1}{\log \log x}\right)\right) \frac{2 \log x}{\log \log x} \int_0^1 \log\left(1 + \frac{v}{2t}\right) dt.$$

Since $\log(1 + z) \geq z/(1 + z)$ for all $z > -1$, it follows that

$$\begin{aligned}\log \frac{\Psi(x, y)^2}{\Psi(x^2, y)} &= \left(\log 2 + \int_0^1 \log\left(1 - \frac{t}{2t + v}\right) dt + O\left(\frac{1}{\log \log x}\right)\right) \frac{2 \log x}{\log \log x} \\ &\geq \left(\log 2 - \int_0^1 \frac{t}{t + v} dt + O\left(\frac{1}{\log \log x}\right)\right) \frac{2 \log x}{\log \log x} \\ &\geq \left(\log 2 - \frac{1}{1 + v} + O\left(\frac{1}{\log \log x}\right)\right) \frac{2 \log x}{\log \log x}.\end{aligned}$$

Inserting this in (29), we find that this particular n has at least

$$\begin{aligned}\frac{\Psi(x, y)\pi(x, y)}{\Psi(x^2, y)} &\gg \exp\left(\left(\log 2 - \frac{1}{1 + v} + O\left(\frac{1}{\log \log x}\right)\right) \frac{2 \log x}{\log \log x}\right) \\ &\geq \exp\left(\left(\log 2 - \frac{1}{v}\right) \frac{\log n}{\log \log n}\right)\end{aligned}$$

representations as $n = m(p - 1)$. Since $v > 0$ is arbitrary, this verifies our claim that (27) follows from (28).

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