

SOME ARITHMETIC PROPERTIES OF OVERPARTITION K -TUPLES**Derrick M. Keister**

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Abstract

Recently, Lovejoy introduced the construct of overpartition pairs which are a natural generalization of overpartitions. Here we generalize that idea to overpartition k -tuples and prove several congruences related to them. We denote the number of overpartition k -tuples of a positive integer n by $\overline{p}_k(n)$ and prove, for example, that for all $n \geq 0$, $\overline{p}_{t-1}(tn + r) \equiv 0 \pmod{t}$ where t is prime and r is a quadratic nonresidue mod t .

1. Introduction

As defined by Corteel and Lovejoy [5], an *overpartition* of a positive integer n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a part may be overlined. For example, the overpartitions of the integer 3 are

$$3, \overline{3}, 2 + 1, \overline{2} + \overline{1}, \overline{2} + 1, 2 + \overline{1}, 1 + 1 + 1, \overline{1} + 1 + 1.$$

The number of overpartitions of a positive integer n is denoted by $\overline{p}(n)$, with $\overline{p}(0) = 1$ by definition. Thus $\overline{p}(3) = 8$ from the above example. As noted in Corteel and Lovejoy [5], the generating function for overpartitions is

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n}.$$

As the topic of overpartitions has already been examined rather thoroughly [3, 4, 5, 6, 7, 8, 10, 11], we look to new constructions. One such construction is that of an *overpartition pair* of a positive integer n , defined by Lovejoy [9] as a pair of overpartitions wherein the sum of all listed parts is n . For example, the overpartition pairs of 2 are

$$(2; \emptyset), (\overline{2}; \emptyset), (\emptyset; \overline{2}), (\emptyset; 2), (1 + 1; \emptyset), (\overline{1} + 1; \emptyset), (\emptyset; 1 + 1), (\emptyset; \overline{1} + 1), \\ (1; \overline{1}), (1; 1), (\overline{1}; \overline{1}), (\overline{1}; 1).$$

Lovejoy denoted the number of overpartition pairs of a positive integer n by $\overline{pp}(n)$, with $\overline{pp}(0) = 1$ by definition. Thus $\overline{pp}(2) = 12$ from the above example. Following lines similar to that for overpartitions, the generating function for overpartition pairs is

$$\sum_{n=0}^{\infty} \overline{pp}(n) q^n = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^2.$$

Several arithmetic properties of both overpartitions and their pairs have appeared in the literature. Since our interest here is primarily on congruence properties, there are a few theorems that are especially noteworthy. The first one is straightforward and proven intuitively.

Theorem 1. *For all $n > 0$, $\overline{p}(n) \equiv 0 \pmod{2}$.*

Next we have a theorem easily proven using results of Mahlburg [10].

Theorem 2. *For all $n > 0$,*

$$\overline{p}(n) \equiv \begin{cases} 2 \pmod{4} & \text{if } n \text{ is a square,} \\ 0 \pmod{4} & \text{otherwise.} \end{cases}$$

Several other congruences in arithmetic progressions were proven by Hirschhorn and Sellers. For example, the following were proven in [7].

Theorem 3. *For all $n \geq 0$,*

$$\begin{aligned} \overline{p}(5n+2) &\equiv 0 \pmod{4}, \\ \overline{p}(5n+3) &\equiv 0 \pmod{4}, \\ \overline{p}(4n+3) &\equiv 0 \pmod{8}, \\ \text{and } \overline{p}(8n+7) &\equiv 0 \pmod{64}. \end{aligned}$$

Also, Hirschhorn and Sellers [6] proved that $\overline{p}(n)$ satisfies congruences modulo non-powers of 2 by proving the following:

Theorem 4. *For all $n \geq 0$ and all $\alpha \geq 0$, $\overline{p}(9^\alpha(27n+18)) \equiv 0 \pmod{12}$.*

Finally, we note a theorem proven by Bringmann and Lovejoy [2]. This result provides much inspiration for the main result in the next section.

Theorem 5. *For all $n \geq 0$, $\overline{pp}(3n+2) \equiv 0 \pmod{3}$.*

We now introduce a generalization of overpartition pairs. An *overpartition k -tuple* of a positive integer n is a k -tuple of overpartitions wherein all listed parts sum to n . We denote the number of overpartition k -tuples of n by $\overline{p}_k(n)$, with $\overline{p}_k(0) = 1$ by

definition. Consequently, the number of overpartition pairs of n is denoted as $\bar{p}_2(n)$. The generating function for $\bar{p}_k(n)$ is easily seen to be

$$\sum_{n \geq 0} \bar{p}_k(n) q^n = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^k.$$

The aim of this note is to prove several congruence properties for families of overpartition k -tuples. In the process, we will prove several natural generalizations of results quoted above.

2. Results for Overpartition k -Tuples

Our first theorem of this section provides a natural generalization of Bringmann and Lovejoy's Theorem 5 above. Moreover, the proof technique is extremely elementary, making this a very satisfying result.

Theorem 6. *For all $n \geq 0$, $\bar{p}_{t-1}(tn+r) \equiv 0 \pmod{t}$, where t is an odd prime and r is a quadratic nonresidue mod t .*

Remarks. First, note that the $t = 3$ case of this theorem is exactly Theorem 5. Secondly, note that, for each odd prime t , this theorem provides $\frac{t-1}{2}$ congruence properties for $\bar{p}_{t-1}(n)$.

Proof. Consider the following generating function manipulations:

$$\begin{aligned} \sum_{n=0}^{\infty} \bar{p}_{t-1}(n) q^n &= \prod_{i=1}^{\infty} \left(\frac{1+q^i}{1-q^i} \right)^{t-1} \\ &= \left[\prod_{i=1}^{\infty} \frac{1+q^i}{1-q^i} \right]^{t-1} \\ &= \left[\prod_{i=1}^{\infty} \frac{1+q^i}{1-q^i} \right]^t \left[\prod_{i=1}^{\infty} \frac{1-q^i}{1+q^i} \right] \\ &\equiv \left[\prod_{i=1}^{\infty} \frac{1+q^{ti}}{1-q^{ti}} \right] \left[\prod_{i=1}^{\infty} \frac{1-q^i}{1+q^i} \right] \pmod{t} \text{ since } t \text{ is prime} \\ &= \sum_{m=0}^{\infty} \bar{p}(m) q^{tm} \left[\prod_{i=1}^{\infty} \frac{1-q^i}{1+q^i} \right] \\ &= \sum_{m=0}^{\infty} \bar{p}(m) q^{tm} \sum_{s=-\infty}^{\infty} (-1)^s q^{s^2} \text{ thanks to Gauss [1, Cor. 2.10].} \end{aligned}$$

But note that $tn + r$ can never be represented as $tm + s^2$ for some integers m and s if r is a quadratic nonresidue mod t . This implies that $\bar{p}_{t-1}(tn + r) \equiv 0 \pmod{t}$ for all $n \geq 0$. $\square$

The next theorem is a broad generalization of Theorem 1. It is found with proof in [12], but is included here for the sake of completeness. We require a brief technical lemma.

Lemma 7. *Let m be a nonnegative integer. For all $1 \leq n \leq 2^m$,*

$$\binom{2^m}{n} 2^n \equiv 0 \pmod{2^{m+1}}.$$

Proof. Let $\text{ord}_2(N)$ be the exponent of the highest power of 2 dividing N . Thus, for example, $\text{ord}_2(8) = 3$ while $\text{ord}_2(80) = 4$. To prove Lemma 7, we need to prove that

$$\text{ord}_2 \left(\binom{2^m}{n} 2^n \right) \geq m + 1. \quad (1)$$

Note that

$$\begin{aligned} \text{ord}_2 \left(\binom{2^m}{n} 2^n \right) &= \text{ord}_2 \left(\frac{2^m(2^m - 1)(2^m - 2) \cdots (2^m - (n - 1))}{n!} \cdot 2^n \right) \\ &\geq \text{ord}_2 \left(\frac{2^{m+n}}{n!} \right) \\ &= m + n - \text{ord}_2(n!) \\ &= m + n - \left(\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \cdots \right) \end{aligned}$$

where $\lfloor x \rfloor$ is the floor function of x .

Now assume $n = c_0 2^0 + c_1 2^1 + \cdots + c_t 2^t$ where each $c_i \in \{0, 1\}$. Then

$$\begin{aligned} \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \cdots &= c_1 2^0 + c_2 2^1 + \cdots + c_t 2^{t-1} \\ &\quad + c_2 2^0 + c_3 2^1 + \cdots + c_t 2^{t-2} \\ &\quad + c_3 2^0 + c_4 2^1 + \cdots + c_t 2^{t-3} \\ &\quad \vdots \\ &\quad + c_t 2^0 \\ &= (2 - 1)c_1 + (2^2 - 1)c_2 + (2^3 - 1)c_3 + \cdots + (2^t - 1)c_t \\ &= n - (c_0 + c_1 + c_2 + \cdots + c_t) \\ &\leq n - 1 \end{aligned}$$

since at least one of the c_i must equal 1. Therefore,

$$\begin{aligned} \text{ord}_2 \left(\binom{2^m}{n} 2^n \right) &\geq m + n - \left(\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \cdots \right) \\ &\geq m + n - (n - 1) \\ &= m + 1. \end{aligned}$$

This is the desired result as noted in (1) above. $\square$

We are now in a position to prove the following theorem:

Theorem 8. *Let $k=(2^m)r$, where m is a nonnegative integer and r is odd. Then, for all positive integers n , we have $\bar{p}_k(n) \equiv 0 \pmod{2^{m+1}}$.*

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} \bar{p}_k(n) q^n &= \prod_{i=1}^{\infty} \left[\frac{1+q^i}{1-q^i} \right]^k \\ &= \prod_{i=1}^{\infty} \left[\frac{1+q^i}{1-q^i} \right]^{(2^m)r} \\ &= \left(\prod_{i=1}^{\infty} \left[\frac{1+q^i}{1-q^i} \right]^{2^m} \right)^r \\ &= \left(\prod_{i=1}^{\infty} \left[1 + \frac{2q^i}{1-q^i} \right]^{2^m} \right)^r \\ &= \left(\prod_{i=1}^{\infty} \left[1 + \sum_{n=1}^{2^m} \binom{2^m}{n} 2^n \left(\frac{q^i}{1-q^i} \right)^n \right] \right)^r \\ &\equiv 1 \pmod{2^{m+1}} \text{ by Lemma 7.} \end{aligned} \quad \square$$

The following theorem is inspired by Theorem 2. As with Theorem 8, it primarily hinges upon the use of the binomial theorem.

Theorem 9. *Let $k=(2^m)r$, $m > 0$ and r is odd. Then, for all $n \geq 1$,*

$$\bar{p}_k(n) \equiv \begin{cases} 2^{m+1} \pmod{2^{m+2}} & \text{if } n \text{ is a square or twice a square,} \\ 0 \pmod{2^{m+2}} & \text{otherwise.} \end{cases}$$

Proof. We prove this result by induction on m .

Basis Step. Let $m = 1$. We must show that

$$\bar{p}_{2r}(n) \equiv \begin{cases} 4 \pmod{8} & \text{if } n \text{ is a square or twice a square,} \\ 0 \pmod{8} & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \bar{p}_{2r}(n)q^n &= \prod_{i=1}^{\infty} \left(\frac{1+q^i}{1-q^i} \right)^{2r} \\ &= \left(\left[\sum_{n=0}^{\infty} \bar{p}(n)q^n \right]^2 \right)^r \\ &= \left(\left[1 + \sum_{\substack{n>0 \\ \text{square}}} \bar{p}(n)q^n + \sum_{\substack{n>0 \\ \text{not square}}} \bar{p}(n)q^n \right]^2 \right)^r \\ &= \left[1 + 2 \left(\sum_{\substack{n>0 \\ \text{square}}} \bar{p}(n)q^n \right) + \left(\sum_{\substack{n>0 \\ \text{square}}} \bar{p}(n)q^n \right)^2 \right. \\ &\quad \left. + 2 \left(\sum_{\substack{n>0 \\ \text{square}}} \bar{p}(n)q^n \right) \left(\sum_{\substack{n>0 \\ \text{not square}}} \bar{p}(n)q^n \right) \right. \\ &\quad \left. + 2 \left(\sum_{\substack{n>0 \\ \text{not square}}} \bar{p}(n)q^n \right) + \left(\sum_{\substack{n>0 \\ \text{not square}}} \bar{p}(n)q^n \right)^2 \right]^r \end{aligned}$$

From Theorem 2, we know that $\bar{p}(n) \equiv 2$ or $6 \pmod{8}$ when n is a square and $\bar{p}(n) \equiv 0$ or $4 \pmod{8}$ otherwise. Since 2×0 , 2×4 , 6×0 , 6×4 , 0×0 , 0×4 , and 4×4 are all congruent to $0 \pmod{8}$,

$$\begin{aligned} 2 \left(\sum_{\substack{n>0 \\ \text{square}}} \bar{p}(n)q^n \right) \left(\sum_{\substack{n>0 \\ \text{not square}}} \bar{p}(n)q^n \right) &\equiv 0 \pmod{8}, \\ 2 \left(\sum_{\substack{n>0 \\ \text{not square}}} \bar{p}(n)q^n \right) &\equiv 0 \pmod{8}, \\ \text{and } \left(\sum_{\substack{n>0 \\ \text{not square}}} \bar{p}(n)q^n \right)^2 &\equiv 0 \pmod{8}. \end{aligned}$$

This gives

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{p}_{2r}(n)q^n &\equiv \left[1 + 2 \left(\sum_{n=1}^{\infty} \bar{p}(n^2)q^{n^2} \right) + \left(\sum_{n=1}^{\infty} \bar{p}(n^2)q^{n^2} \right)^2 \right]^r \pmod{8} \\ &\equiv \left[1 + 4 \left(\sum_{n=1}^{\infty} q^{n^2} \right) + 4 \left(\sum_{n=1}^{\infty} q^{n^2} \right)^2 \right]^r \pmod{8}\end{aligned}$$

again thanks to Theorem 2.

Given that $(q^{n_1} + q^{n_2} + \dots)^2 = (q^{2n_1} + q^{2n_2} + \dots) + 2(q^{n_1+n_2} + \dots)$, we then have

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{p}_{2r}(n)q^n &\equiv \left[1 + 4 \left(\sum_{n=1}^{\infty} q^{n^2} \right) + 4 \left(\sum_{n=1}^{\infty} q^{2n^2} + 2 \sum_{\substack{n_1, n_2 > 0 \\ n_1 \neq n_2}} q^{n_1^2 + n_2^2} \right) \right]^r \pmod{8} \\ &\equiv \left[1 + 4 \left(\sum_{m=1}^{\infty} q^{m^2} + \sum_{n=1}^{\infty} q^{2n^2} \right) \right]^r \pmod{8} \\ &= \sum_{j=0}^{\infty} \binom{r}{j} 4^j \left(\sum_{m=1}^{\infty} q^{m^2} + \sum_{n=1}^{\infty} q^{2n^2} \right)^j \\ &\equiv 1 + 4 \left(\sum_{m=1}^{\infty} q^{m^2} + \sum_{n=1}^{\infty} q^{2n^2} \right) \pmod{8} \text{ since } r \text{ is odd.}\end{aligned}$$

This proves the result needed for the basis step.

Induction Step. Assume that

$$\bar{p}_{(2^m)r}(n) \equiv \begin{cases} 2^{m+1} \pmod{2^{m+2}} & \text{if } n \text{ is a square or twice a square,} \\ 0 \pmod{2^{m+2}} & \text{otherwise.} \end{cases}$$

We must show that

$$\bar{p}_{(2^{m+1})r}(n) \equiv \begin{cases} 2^{m+2} \pmod{2^{m+3}} & \text{if } n \text{ is a square or twice a square,} \\ 0 \pmod{2^{m+3}} & \text{otherwise.} \end{cases}$$

Consider the generating function for $\bar{p}_{2^{m+1}}(n)$:

$$\begin{aligned}
 \sum_{n=0}^{\infty} \bar{p}_{(2^{m+1})r}(n)q^n &= \prod_{i=1}^{\infty} \left(\frac{1+q^i}{1-q^i} \right)^{(2^{m+1})r} \\
 &= \left(\prod_{i=1}^{\infty} \left(\frac{1+q^i}{1-q^i} \right)^{2^m r} \right)^2 \\
 &= \left(\sum_{n=0}^{\infty} \bar{p}_{(2^m)r}(n)q^n \right)^2 \\
 &= \left(1 + \sum_{\substack{n>0 \\ \text{not sq.} \\ \text{and not} \\ \text{twice sq.}}} \bar{p}_{(2^m)r}(n)q^n + \sum_{\substack{n>0 \\ \text{square or} \\ \text{twice sq.}}} \bar{p}_{(2^m)r}(n)q^n \right)^2 \\
 &= 1 + 2 \left(\sum_{\substack{n>0 \\ \text{square or} \\ \text{twice sq.}}} \bar{p}_{(2^m)r}(n)q^n \right) + \left(\sum_{\substack{n>0 \\ \text{square or} \\ \text{twice sq.}}} \bar{p}_{(2^m)r}(n)q^n \right)^2 \\
 &\quad + 2 \left(\sum_{\substack{n>0 \\ \text{square or} \\ \text{twice sq.}}} \bar{p}_{(2^m)r}(n)q^n \right) \left(\sum_{\substack{n>0 \\ \text{not sq.} \\ \text{and not} \\ \text{twice sq.}}} \bar{p}_{(2^m)r}(n)q^n \right) \\
 &\quad + 2 \left(\sum_{\substack{n>0 \\ \text{not sq.} \\ \text{and not} \\ \text{twice sq.}}} \bar{p}_{(2^m)r}(n)q^n \right) + \left(\sum_{\substack{n>0 \\ \text{not sq.} \\ \text{and not} \\ \text{twice sq.}}} \bar{p}_{(2^m)r}(n)q^n \right)^2.
 \end{aligned}$$

Using a very similar argument about the coefficients to that of the basis step, we use the induction hypothesis to conclude that

$$\begin{aligned}
 \sum_{n=0}^{\infty} \bar{p}_{(2^{m+1})r}(n)q^n &\equiv 1 + 2 \sum_{n=1}^{\infty} \bar{p}_{(2^m)r}(n^2)q^{n^2} + \sum_{s=1}^{\infty} \bar{p}_{(2^m)r}(2s^2)q^{2s^2} \\
 &\quad + \left(\sum_{n=1}^{\infty} \bar{p}_{(2^m)r}(n^2)q^{n^2} + \sum_{s=1}^{\infty} \bar{p}_{(2^m)r}(2s^2)q^{2s^2} \right)^2 \pmod{2^{m+3}} \\
 &\equiv 1 + 2 \left(\sum_{n=1}^{\infty} \bar{p}_{(2^m)r}(n^2)q^{n^2} + \sum_{s=1}^{\infty} \bar{p}_{(2^m)r}(2s^2)q^{2s^2} \right) \pmod{2^{m+3}}.
 \end{aligned}$$

We know that all coefficients of the last term are congruent to 2^{m+1} or $2^{m+1} + 2^{m+2} \pmod{2^{m+3}}$ from the induction hypothesis. But the last term is multiplied by 2. So then all coefficients are congruent to $2^{m+2} \pmod{2^{m+3}}$ or $2^{m+2} + 2^{m+3} \equiv 2^{m+2} \pmod{2^{m+3}}$, which implies

$$\sum_{n=0}^{\infty} \bar{p}_{(2^{m+1})_r}(n)q^n \equiv 1 + 2^{m+2} \left(\sum_{n=1}^{\infty} q^{n^2} + \sum_{s=1}^{\infty} q^{2s^2} \right) \pmod{2^{m+3}}.$$

This completes the induction and proves the theorem. $\square$

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