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COLUMN-TO-ROW OPERATIONS ON PARTITIONS: THE ENVELOPES

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Abstract

Conjugation and the Bulgarian solitaire move are considered as extreme cases of several column-to-row operations on integer partitions. Each operation generates a state diagram on the partitions of n, which leads to the questions: How many Garden of Eden states are there? How many cycle states? How many connected components? All of these questions are answered for partitions of n when at least $\frac{n-1}{2}$ columns are switched to rows.

1. Introduction

Conjugation is the fundamental operation on integer partitions. Write a partition λ as $(\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ where $\ell(\lambda)$ denotes the partition's length, its number of parts. The conjugate partition λ' is defined as $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ where λ'_i is the number of parts $\{\lambda_i\}$ greater than or equal to i. This is more easily understood in terms of the Ferrers diagram: the dots are reflected along the diagonal, so that columns and rows are swapped; see Figure 3.

We write $P(n)$ for the set of partitions of n. Since conjugation is an involution, the state diagram of $P(n)$ determined by conjugation consists of singletons and pairs, i.e., self-conjugate partitions and conjugate pairs. See Figure 1 for an example, which also introduces the superscript notation for partitions, e.g., writing 21^3 for $(2, 1, 1, 1)$.

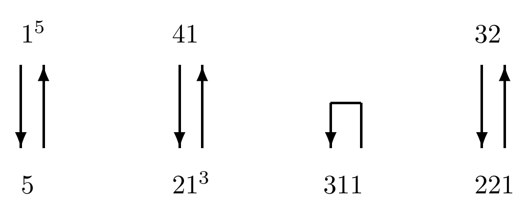

Figure 1: Conjugation on $P(5)$; all 7 partitions are in cycles and there are 4 components.

Consider the effect of conjugation on $P(n)$ as a state diagram. Notice in Figure 1 that all seven partitions of 5 are in cycles and the diagram has four connected components.

Bulgarian solitaire is an operation on partitions introduced by Brandt in 1982 [3]. We define it as $D^1(\lambda) = (\lambda'_1, \lambda_1 - 1, \ldots, \lambda_{\ell(\lambda)} - 1)$ where any zeros are removed and the parts may not be in the standard non-increasing order. In terms of the Ferrers diagram, the operation takes the first (leftmost) column and makes it a row; see Figure 3. Figure 2 shows the effect of D^1 on partitions of 5.

$$\begin{tikzcd}
1^5 \arrow[d] & 21^3 \arrow[d] & & 311 \arrow[dl] \\
5 \arrow[r] & 41 \arrow[r] & 32 \arrow[r] & 221 \arrow[u]
\end{tikzcd}$$

Figure 2: Bulgarian solitaire on $P(5)$; there are 2 Garden of Eden partitions, 3 cycle partitions, and one component.

Like conjugation, the D^1 operation also produces a state diagram on $P(n)$. Notice in Figure 2 that three partitions of 5 are in cycles and the diagram consists of a single component. There are also two partitions that have no pre-image under the operation (1^5 and 21^3); these are called Garden of Eden partitions (subsequently abbreviated GE-partitions).

In this article, we introduce a sequence of column-to-row operations; conjugation and the Bulgarian solitaire operation are the extreme cases. Bulgarian solitaire has been the subject of several articles; Hopkins-Jones [4] includes a fairly complete bibliography. Many of the questions concern state diagram concepts: partitions in cycles, partitions with no preimages, and number of connected components. In this article, we consider these same questions for all generalized column-to-row operations. We determine the number of GE-partitions, the number of cycle partitions, and the number of connected components for approximately half of all possible cases.

2. General Row-to-Column Operations

Conjugation can be thought of as moving all columns to rows; Bulgarian solitaire moves one column to a row. We connect these ideas by introducing the sequence of operations

$$D^k(\lambda) = (\lambda'_1, \ldots, \lambda'_k, \lambda_1 - k, \ldots, \lambda_{\ell(\lambda)} - k)$$

where any nonpositive numbers are removed and the parts may not be in the standard non-increasing order. In terms of the Ferrers diagram, the operation takes the first k columns and makes them rows. Figure 3 shows a partition and its images under various D^k.

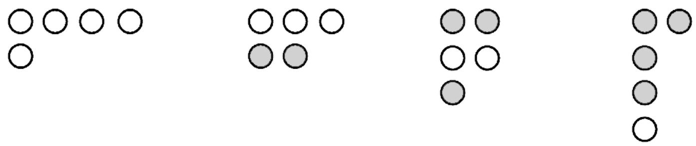

Figure 3: Ferrers diagrams for $\lambda = (4, 1)$, $D^1(\lambda) = (3, 2)$, $D^2(\lambda) = (2, 2, 1)$, and $D^3(\lambda) = D^4(\lambda) = \lambda' = (2, 1, 1, 1)$, with shaded dots showing which rows came from columns of λ.

These operations all generate state diagrams on $P(n)$. Figures 4 and 5 show $P(5)$ under D^2 and D^3, respectively.

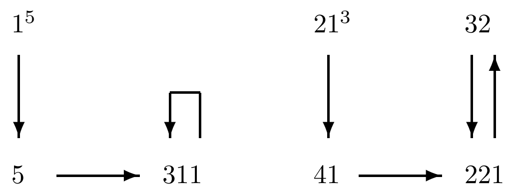

Figure 4: The D^2 operation on $P(5)$; there are 2 GE-partitions, 3 total partitions in cycles, and 2 components.

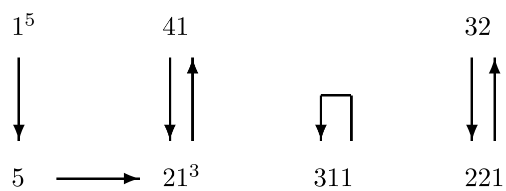

Figure 5: The D^3 operation on $P(5)$; there is 1 GE-partition, 5 cycle partitions, and 3 components.

Notice that for $P(5)$, conjugation (Figure 1) is equivalent to the operation D^4. This is an example of a general fact.

Lemma 1. *For a partition λ with $\lambda_1 \leq k$ or with $\lambda_1 = k + 1$ and $\lambda_2 \leq k$, the operation D^k is equivalent to conjugation. In particular, D^{n-1} is equivalent to conjugation on $P(n)$.*

Proof. For λ with k or fewer columns, the claim is evident. Assume $\lambda_1 = k+1$ and $\lambda_2 \leq k$, i.e., that λ has $k + 1$ columns with $(k + 1)$st having height 1. Moving k columns to rows leaves a single row of length 1, so that the effect of D^k is equivalent to moving all columns to rows. Every $\lambda \in P(n)$ has $\lambda_1 \leq n-1$ except the single-part partition (n), which satisfies the other condition since the second part of (n) is 0. Therefore, for all partitions of n, $D^{n-1}(\lambda) = \lambda'$. $\square$

3. Results on Partitions with Many Parts

This section consists of results about partitions of n with at least $(n-1)/2$ or $n/2$ parts. One result is well-known and others are particular to the purposes of this article. First, we introduce some notation. Capital letters signify sets, corresponding lower-case letters the number of elements in the set, e.g., $p(5) = 7$. Recall the convention that $p(0) = 1$. We will use part-wise addition on partitions, e.g., $(3,1,1) + (2,2) = (5,3,1)$.

Let $P(n,j)$ denote the set of partitions of n with exactly j parts; from the examples, we see $p(5,3) = 2$. Table 1 shows the $p(n,j)$ values for $1 \leq n, j \leq 12$. Notice that, reading right to left, roughly half of each row are initial values of $p(n)$, i.e., $1, 1, 2, 3, 5, 7, \ldots$ We call that portion of the triangle the envelope.

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	1	1										
3	1	1	1									
4	1	2	1	1								
5	1	2	2	1	1							
6	1	3	3	2	1	1						
7	1	3	4	3	2	1	1					
8	1	4	5	5	3	2	1	1				
9	1	4	7	6	5	3	2	1	1			
10	1	5	8	9	7	5	3	2	1	1		
11	1	5	10	11	10	7	5	3	2	1	1	
12	1	6	12	15	13	11	7	5	3	2	1	1

Table 1: $p(n,k)$, the number of partitions of n with k parts.

Lemma 2. *Let a positive integer n be given. For each integer $j \geq \frac{n}{2}$, $p(n,j) = p(n-j)$.*

Proof. We demonstrate a bijection between $P(n,j)$ and $P(n-j)$. Any $\lambda \in P(n,j)$ can be written as $\lambda = 1^j + \mu$ where $\mu \in P(n-j)$; let $\lambda \mapsto \mu$. Any $\mu \in P(n-j)$ has at most j parts, since the restriction on j implies $n-j \leq j$, so that $\lambda = 1^j + \mu \in P(n,j)$; let $\mu \mapsto \lambda$. Clearly these are inverse maps. $\square$

This result shows that, in some sense, the difficulty of studying partitions lies in the partitions with fewer than $n/2$ parts, which correspond to roughly the left-hand half of each row in Table 1. There are direct formulas for $p(n,j)$ with $j \leq 5$ (see, e.g., [2] and [6]), but they quickly become complicated. Notice that the $\lambda \mapsto \mu$ relation determined by $\lambda = 1^j + \mu$ is equivalent to removing the first column of the Ferrers diagram of λ, i.e., the Bulgarian solitaire D^1 operation without including the part λ'_1.

Lemma 3. *Let a positive integer n be given. For each integer $j \geq \frac{n-1}{2}$, the following hold.*

(a) All $\lambda \in P(n,k)$ with $k \geq j+2$ have $\lambda_j = 1$.

(b) All $\mu \in P(n)$ with $\ell(\mu) \leq \mu'_1$ have $\mu_1 \leq n - \ell(\mu) + 1$.

(c) All $\nu \in P(n)$ with $\nu_1 = j+1$ have $\nu_2 \leq j$. That is, all $\nu \in P(n)$ with $\nu_1 \leq j+1$ satisfy one of the conditions of Lemma 1.

Proof. (a) Assume that $\lambda \in P(n, j+2)$. Since λ has $j+2$ parts, $\lambda_j \neq 0$. Suppose that $\lambda_j \geq 2$. Then the sum of the parts of λ would be at least $2j+1+1 \geq n+1$, a contradiction. For $k > j+2$, the sum of the parts has a higher lower bound, so the result follows.

(b) The first row and the first column share a dot in the Ferrers diagram, so their sum is at most $n+1$.

(c) If $\nu_1 = \nu_2 = j+1$, then $\nu_1 + \nu_2 > n$, contradicting $\mu \in P(n)$. □

Note that (a) is "sharp" in the sense that, for example, every $\lambda \in P(9,6)$ has $\lambda_4 = 1$ but $P(9,5)$ includes 32211 and $2^4 1$ whose third parts are 2, not 1.

4. Garden of Eden Partitions

Let $GE(n,k)$ denote the GE-partitions of n under the operation D^k. From the examples of the previous section, we know $ge(5,1) = ge(5,2) = 2$, $ge(5,3) = 1$, and $ge(5,4) = 0$. Those data correspond to the $n = 5$ row of Table 2.

The diagonal of zeros corresponds to the fact that every partition has a pre-image under conjugation. Notice that other diagonals seem to eventually stabilize at some value; these limiting values comprise the envelope.

What is the sequence of values $0, 1, 2, 4, 7, 12, 19, 30, 34, 67, 97, \ldots$ in the envelope? One possibility is the partial sum of partition numbers (A000070 in [8]). Let

$$s(n) = \sum_{i=0}^{n} p(i)$$

and $s(-1) = 0$. Before we can verify that this sequence describes the envelope, we need to characterize GE-partitions.

Lemma 4. *A partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \in P(n)$ is in $GE(n,k)$ precisely when $\lambda_k - \ell(\lambda) \leq -1 - k$.*

Proof. In terms of the Ferrers diagram, λ has a pre-image under D^k for every set of k rows each greater than or equal to $\ell(\lambda) - k$, i.e., long enough to be moved to become columns to the left side of the remaining dots. This fails when $\lambda_k < \ell(\lambda) - k$. □

	1	2	3	4	5	6	7	8	9	10
2	0									
3	1	0								
4	1	1	0							
5	2	2	1	0						
6	3	3	2	1	0					
7	5	5	4	2	1	0				
8	7	8	6	4	2	1	0			
9	10	12	10	7	4	2	1	0		
10	14	18	15	11	7	4	2	1	0	
11	20	25	23	17	12	7	4	2	1	0
12	27	35	33	26	18	12	7	4	2	1
13	37	48	47	38	28	19	12	7	4	2
14	49	66	65	55	41	29	19	12	7	4
15	66	88	89	77	60	43	30	19	12	7
16	86	118	120	107	85	63	44	30	19	12
17	113	155	161	145	119	90	65	45	30	19
18	147	203	213	196	163	127	93	66	45	30
19	190	263	280	260	222	175	132	95	67	45
20	243	340	364	344	297	240	183	135	96	67
21	311	435	471	449	394	323	252	188	137	97

Table 2: $ge(n, k)$, the number of GE-partitions in $P(n)$ under D^k.

This generalizes the initial lemma and corollary of Hopkins-Jones [4] for Bulgarian solitaire (D^1).

We now show that, in the envelope, the GE-partitions are precisely the partitions with many parts.

Theorem 1. *Let a positive integer n be given. For each integer $\frac{n-1}{2} \le j \le n-1$, we have $ge(n, j) = s(n-j-2)$.*

Proof. If $j = n-1$, then the operation is equivalent to conjugation and there are no GE-partitions, matching $ge(n, n-1) = s(-1) = 0$. So assume that $\frac{n-1}{2} \le j \le n-2$. By the preceding lemma, $GE(n, j)$ consists of all $\lambda \in P(n)$ with $\lambda_j - \ell(\lambda) \le -1-j$. This means that any GE-partition λ must have $\ell(\lambda) \ge \lambda_j + j + 1$, so $\lambda_j \ne 0$ and in fact $\ell(\lambda) \ge j+2$.

But by Lemma 3a, all $\mu \in P(n)$ with $j+2$ or more parts have $\mu_j = 1$, so that $\mu_j - \ell(\mu) = 1 - \ell(\mu) \le 1-(j+2) = -1-j$. That is, $GE(n, j)$ is exactly $P(n, j+2) \cup \cdots \cup P(n, n)$. By Lemma 2, $p(n, j+2) = p(n-j-2), \ldots, p(n, n) = p(0)$. We conclude that $ge(n, j) = \sum p(i) = s(n-j-2)$. $\square$

As with the $p(n, j)$ values of Table 1, one would like to have formulas for the columns of Table 2. Hopkins-Sellers [5] provides two proofs of the following result.

Theorem 2.

$$ge(n,1) = p(n-3) - p(n-9) + p(n-18) - \cdots = \sum_{j\geq 1} (-1)^{j+1} p\left(n - \frac{3j^2+3j}{2}\right)$$

We make the following conjectures about the next few columns.

Conjectures

$$\begin{aligned} ge(n,2) &= p(n-4) + p(n-5) - p(n-11) - p(n-12) - p(n-13) + p(n-21) + \cdots \\ &= \sum_{j\geq 1} \sum_{k=0}^{j} (-1)^{j+1} p\left(n - \frac{3j^2+3j}{2} - j - k\right) \end{aligned}$$

$$\begin{aligned} ge(n,3) &= p(n-5) + p(n-6) + p(n-7) - p(n-13) - p(n-14) - 2p(n-15) \\ &\quad - p(n-16) - p(n-17) + p(n-24) + \cdots \end{aligned}$$

$$\begin{aligned} ge(n,4) &= p(n-6) + p(n-7) + p(n-8) + p(n-9) - p(n-15) - p(n-16) \\ &\quad - 2p(n-17) - 2p(n-18) - 2p(n-19) - p(n-20) - p(n-21) \\ &\quad + p(n-27) + p(n-28) + 2p(n-29) + 3p(n-30) + 3p(n-31) + \cdots \end{aligned}$$

where complete expressions for $ge(n,3)$ and $ge(n,4)$ involve q-binomial coefficients. These are consistent with Theorem 1, since only the initial positive terms arise in the envelope. These conjectures will be considered in future work with Louis Kolitsch.

5. Cycle Partitions

Let $CP(n,k)$ denote the partitions of n in cycles under the operation D^k. From the examples of the previous section, we know $cp(5,1) = cp(5,2) = 3$, $cp(5,3) = 5$, and $cp(5,4) = 7$. Those data correspond to the $n = 5$ row on the left-hand side of Table 3.

Under conjugation, we know every partition is in a cycle, either self-conjugate or half of a conjugate pair. By Lemma 1, then, $cp(n, n-1) = p(n)$. The right-hand side of Table 3 shows $p(n) - cp(n,k)$. The envelope of this triangle of differences appears to be $2s(n)$ for $s(n)$ defined in Section 4.

Theorem 3. *Let a positive integer n be given. For each integer $\frac{n-1}{2} \leq j \leq n-1$, $cp(n,j) = p(n) - 2s(n-j-2)$.*

Proof. Given $\lambda \in GE(n,j)$, we claim that its iterated images under D^j have the form

$$\lambda \xrightarrow{\prime} \lambda' \longrightarrow \mu \xleftrightarrow{\prime} \mu'$$

where $\xrightarrow{\prime}$ indicates that the operation D^j coincides with conjugation and we allow the possibility $\mu = \mu'$.

	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
2	2								0							
3	1	3							2	0						
4	3	3	5						2	2	0					
5	3	3	5	7					4	4	2	0				
6	1	5	7	9	11				10	6	4	2	0			
7	4	6	7	11	13	15			11	9	8	4	2	0		
8	6	8	10	14	18	20	22		16	14	12	8	4	2	0	
9	4	6	11	16	22	26	28	30	26	24	19	14	8	4	2	0
10	1	5	15	20	28	34	38	40	41	37	27	22	14	8	4	2
11	5	5	15	23	32	42	48	52	51	51	41	33	24	14	8	4
12	10	10	20	28	41	53	63	69	67	67	57	49	36	24	14	8
13	10	16	22	32	46	63	77	87	91	85	79	69	55	38	24	14
14	5	23	29	37	56	77	97	111	130	112	106	98	79	58	38	24
15	1	28	35	42	63	91	116	138	175	148	141	134	113	85	60	38
16	6	33	41	49	75	108	143	171	225	198	190	182	156	123	88	60
17	15	35	45	57	83	124	168	207	282	262	252	240	214	173	129	90
18	20	42	48	68	98	145	202	253	365	343	337	317	287	240	183	132
19	15	39	45	79	107	166	233	301	475	451	445	411	383	324	257	189
20	6	41	43	93	126	190	275	360	621	586	584	534	501	437	352	267
21	1	46	42	108	142	215	314	423	791	746	750	684	650	577	478	369

Table 3: On the left, $cp(n,k)$, the number of cycle partitions. On the right, $p(n) - cp(n,k)$, the number of partitions not in cycles.

First, we know from the proof of Theorem 1 that $\lambda \in GE(n,j)$ has at least $j+2$ parts, and then $\lambda_1 \leq j$ by Lemma 3b. Therefore, by Lemma 1, $D^j(\lambda) = \lambda'$. Let $D^j(\lambda') = \mu$. Since λ is a Garden of Eden partition, $\mu \neq \lambda$ and applying D^j to λ' is not equivalent to conjugation. By the definition of D^j,

$$\mu = D^j(D^j(\lambda)) = D^j(\lambda') = (\lambda_1, \ldots, \lambda_j, \lambda'_1 - j)$$

with $\lambda'_2 - j$ and subsequent terms removed since $\lambda'_2 \leq j$ by Lemma 3c. Therefore $\mu'_1 = j+1$ and, by Lemma 3b, $\mu_1 \leq j+1$. By Lemmas 3c and 1 we conclude that $D^j(\mu) = \mu'$. Likewise, $D^j(\mu') = \mu$. Since conjugation is an involution, the sets $\{\lambda, \lambda'\}$ and $\{\mu, \mu'\}$ are disjoint, completing the claim.

We can now complete the proof of the theorem. If some partition $\lambda \in P(n)$ is not in a cycle, it is either a GE-partition or between a GE-partition and a cycle partition. By the claim above, we know that a GE-partition maps to its conjugate, which maps to a cycle partition. From Theorem 1, we know that there are $s(n-j-2)$ GE-partitions, which are not in cycles. Their conjugates are the other $s(n-j-2)$ partitions not in cycles. $\square$

It is important to realize that the structural results of the proof do not imply that every component of $P(n)$ under D^j in the envelope contains at most four partitions. It is true that $\lambda \in GE(n,j)$ has iterates $\lambda \to \lambda' \to \mu$ for $\mu \in CP(n,j)$, but multiple GE-partitions can lead to the same μ, e.g.,

$$D^3(D^3((21^5)) = D^3(61) = 3211, \qquad D^3(D^3((31^4)) = D^3(511) = 3211.$$

Also, the structural results of the proof do not hold in general outside the envelope. For instance, $D^2(D^2(51^3)) = 3221$ which does not have $k+1 = 3$ parts. Also, Figure 1 shows that $D^1(D^1(1^5)) = 41 \notin CP(5,1)$, so two steps from a GE-partition is not always a cycle partition. Lengths from GE-partitions to cycle partitions for D^1 are among the data tabulated in [4].

A formula for the column $cp(n,1)$ is proven in [3].

Theorem 4. *Write $n = \binom{m+1}{2} - a$ where $0 \le a \le m-1$. Then $cp(n,1) = \binom{m}{a}$.*

Formulas for other columns would seem to require generalizing the characterization of cycle partition for D^1 found in [3]. The proof in the next section describes cycle partitions in the envelope, but does not apply for smaller j.

6. Connected Components

Let $cc(n,k)$ denote the number of connected components in the state diagram of $P(n)$ under the operation D^k. From the examples of the previous section, we know $cc(5,1) = 1$, $cc(5,2) = 2$, $cc(5,3) = 3$, and $cc(5,4) = 4$. Those data correspond to the $n = 5$ row on the left-hand side of Table 4.

The numbers of self-conjugate partitions and conjugate pairs were studied by Osima [7]. It follows that the total number of components of $P(n)$ under conjugation (equivalently D^{n-1}) is given by

$$cc(n) = p(n) - p(n-2) + p(n-8) - p(n-18) + p(n-32) - \cdots = \sum_{k \ge 0} (-1)^k p(n - 2k^2)$$

with initial terms $1, 1, 2, 3, 4, 6, 8, 12, 16, 22, 29...$ (A046682 in [8]). The right-hand side of Table 4 shows $cc(n) - cc(n,k)$. The envelope of this triangle of differences appears once again to be $s(n)$ defined in Section 4.

Theorem 5. *Let a positive integer n be given. For each integer $\frac{n-1}{2} \le j \le n-1$, we have $cc(n,j) = cc(n) - s(n-j-2)$.*

Proof. Recall from the proof of Theorem 3 that a partition of $P(n)$ not in a cycle under D^j is either a GE-partition or the conjugate of a GE-partition. For each $\lambda \in GE(n,j)$, the conjugate pair $\{\lambda, \lambda'\}$ counted in $cc(n)$ is part of another component. The discussion of $\mu = D^j(D^j(\lambda))$ in the proof of Theorem 3 established that the conjugate pair $\{\mu, \mu'\}$ is still a 2-cycle under D^j or the self-conjugate μ is still self-conjugate under D^j.

	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
2	1								0							
3	1	2							1	0						
4	1	2	3						2	1	0					
5	1	2	3	4					3	2	1	0				
6	1	2	4	5	6				5	4	2	1	0			
7	1	2	4	6	7	8			7	6	4	2	1	0		
8	2	3	5	8	10	11	12		10	9	7	4	2	1	0	
9	1	2	5	9	12	14	15	16	15	14	11	7	4	2	1	0
10	1	2	6	10	15	18	20	21	21	20	16	12	7	4	2	1
11	1	2	6	11	17	22	25	27	28	27	23	18	12	7	4	2
12	2	3	7	13	21	28	33	36	38	37	33	27	19	12	7	4
13	2	4	7	14	23	33	40	45	50	48	45	38	29	19	12	7
14	1	5	9	15	27	39	50	57	68	64	60	54	42	30	19	12
15	1	6	11	17	30	46	60	71	89	84	79	73	60	44	30	19
16	1	7	12	18	34	54	73	88	117	111	106	100	84	64	45	30
17	3	8	13	20	37	61	85	106	148	143	138	131	114	90	66	45
18	4	9	14	23	41	69	101	128	191	186	181	172	154	126	94	67
19	3	9	13	25	44	78	116	152	245	239	235	223	204	170	132	96
20	1	8	12	29	49	87	135	181	316	309	305	288	268	230	182	136
21	1	7	12	33	53	97	153	212	399	393	388	367	347	303	247	188

Table 4: On left, $cc(n,k)$, the number of connected components. On right, $cc(n) - cc(n,k)$.

We show that no partitions in $CP(n,j)$ are part of larger cycles. Recall from the proof of Theorem 1 that $GE(n,j) = P(n,j+2) \cup \cdots \cup P(n,n)$. It follows that {conjugates of GE-partitions} is the set of $\lambda \in P(n)$ with $\lambda_1 \geq j+2$. Therefore $CP(n,j)$ is the remainder of $P(n)$, namely, the $\pi \in P(n)$ with $\pi_1 \leq j+1$ and $\pi'_1 \leq j+1$, i.e., the partitions of n that fit inside a $(j+1) \times (j+1)$ square. By Lemmas 3c and 1, the operation D^j is equivalent to conjugation for $CP(n,j)$, which means that it consists of conjugate pairs and self-conjugate partitions.

Of the singletons and pairs counted by $cc(n)$, exactly $ge(n,j)$ pairs are no longer components in $P(n)$ under D^j. By Theorem 1, we conclude $cc(n,j) = cc(n) - s(n-j-2)$. $\square$

Viewing $P(n)$ dynamically under D^{n-1}, then D^{n-2}, ..., some conjugate pairs are "opened" into "$\lambda \to \lambda' \to$" fragments that attach to preserved conjugate pairs or self-conjugate partitions. To the left of the envelope, larger cycles develop and there are longer paths from GE-partitions to cycle partitions, as in Figure 1.

A formula for the column $cc(n,1)$ is proven in [3].

Theorem 6. *Write* $n = \binom{m+1}{2} - a$ *where* $0 \le a \le m-1$. *Then*

$$cc(n,1) = \frac{1}{m} \sum_{d|(m,a)} \varphi(d) \binom{m/d}{a/d}$$

where the summation is over all divisors of the greatest common divisor of m *and* a, *and* φ *is the Euler phi function.*

It would be very interesting to determine formulas for other columns, as they would transition between the number-theoretic formula for $cc(n,1)$ and the formulas involving $p(n)$ for $cc(n,j)$ in the envelope.

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