



**PLACE-DIFFERENCE-VALUE PATTERNS: A GENERALIZATION
OF GENERALIZED PERMUTATION AND WORD PATTERNS**

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Abstract

Motivated by the study of Mahonian statistics, in 2000, Babson and Steingrímsson introduced the notion of a “generalized permutation pattern” (GP) which generalizes the concept of “classical” permutation pattern introduced by Knuth in 1969. The invention of GPs led to a large number of publications related to properties of these patterns in permutations and words. Since the work of Babson and Steingrímsson, several further generalizations of permutation patterns have appeared in the literature, each bringing a new set of permutation or word pattern problems and often new connections with other combinatorial objects and disciplines. For example, Bousquet-Mélou *et al.* introduced a new type of permutation pattern that allowed them to relate permutation patterns theory to the theory of partially ordered sets.

In this paper we introduce yet another, more general definition of a pattern, called place-difference-value patterns (PDVP) that covers all of the most common definitions of permutation and/or word patterns that have occurred in the literature. PDVPs provide many new ways to develop the theory of patterns in permutations and words. We shall give several examples of PDVPs in both permutations and words that cannot be described in terms of any other pattern conditions that have been introduced previously. Finally, we discuss several bijective questions linking our patterns to other combinatorial objects.

1. Introduction

In the last decade, several hundred papers have been published on the subject of patterns in words and permutations. This is a new, but rapidly growing, branch of combinatorics which has its roots in the works by Rotem, Rogers, and Knuth in the 1970s and early 1980s. However, the first systematic study of permutation patterns was not undertaken until the paper by Simion and Schmidt [22] which appeared in 1985. The field has experienced explosive growth since 1992. The

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notion of patterns in permutations and words has proved to be a useful language in a variety of seemingly unrelated problems including the theory of Kazhdan-Lusztig polynomials, singularities of Schubert varieties, Chebyshev polynomials, rook polynomials for Ferrers board, and various sorting algorithms including sorting stacks and sortable permutations.

A (“classical”) *permutation pattern* is a permutation $\sigma = \sigma_1 \dots \sigma_k$ in the symmetric group S_k viewed as a word without repeated letters. We say that σ *occurs* in a permutation $\pi = \pi_1 \dots \pi_n$ if there is a subsequence $1 \leq i_1 < \dots < i_k \leq n$ such that $\pi_{i_1} \pi_{i_2} \dots \pi_{i_k}$ is order isomorphic to σ . We say that π *avoids* σ if there is no occurrence of σ in π . One of the fundamental questions in the area of permutation patterns is to determine the number of permutations (words) of length n containing k occurrences of a given pattern p . That is, we want to find an explicit formula or a *generating function* for such permutations. It is also of great interest to find bijections between classes of permutations and/or words that satisfy some sort of pattern condition and other combinatorial structures that preserve as many statistics as possible. For example, we say that two permutations $\sigma, \tau \in S_k$ are *Wilf equivalent* if for each n , the number of permutations in S_n that avoid σ equals the number of permutations in S_n that avoid τ . If σ and τ are Wilf equivalent, then it is natural to ask for a bijection between the set of permutations of S_n which avoid σ and the set of permutations of S_n which avoid τ which preserves as many classical permutation statistics as possible. Such statistics-preserving bijections not only reveal structural similarities between different combinatorial objects, but they often also reveal previously unknown properties of the structures being studied.

In [1] Babson and Steingrímsson introduced *generalized permutation patterns* (GPs) that allow for the requirement that two adjacent letters in a pattern must be adjacent in the permutation. If we write, say 2-31, then we mean that if this pattern occurs in a permutation π , then the letters in π that correspond to 3 and 1 are adjacent. For example, the permutation $\pi = 516423$ has only one occurrence of the GP 2-31, namely the subword 564, whereas the GP 2-3-1 occurs, in addition, in the subwords 562 and 563. Note that a pattern containing a dash between each pair of consecutive letters is a classical pattern.

The motivation for introducing these patterns in [1] was the study of *Mahonian statistics*. Many interesting results on GPs appear in the literature (see [24] for a survey). In particular, [4] provides relations of generalized patterns to several well studied combinatorial structures, such as set partitions, *Dyck paths*, *Motzkin paths* and involutions. We refer to [13] for a survey over results on patterns discussed so far.

Further generalizations and refinements of GPs have appeared in the literature. For example, one can study occurrences of a pattern σ in a permutation π where

one pays attention to the parity of the elements in the subsequences of π which are order isomorphic to σ . For instance, Kitaev and Remmel [16] studied descents (the GP 21) where one fixes the parity of exactly one element of a descent pair. Explicit formulas for the distribution of these (four) new patterns were provided. The new patterns are shown in [16] to be connected to the *Genocchi numbers*, the study of which goes back to Euler. In [17], Kitaev and Remmel generalized the results of [16] to classify descents according to equivalence mod k for $k \geq 3$ of one of the descent pairs. As a result of this study, one obtains, in particular, remarkable binomial identities. Liese [20, 21] studied enumerating descents where the difference between descent pairs is a fixed value. More precisely, study of the set $Des_k(\sigma) = \{i | \sigma_i - \sigma_{i+1} = k\}$ is done in [21, Chpt 7]. Hall and Remmel [7] further generalized the studies in [16, 17, 20, 21]. The main focus of [7] is to study the distribution of descent pairs whose top σ_i lies in some fixed set X and whose bottom σ_{i+1} lies in some fixed set Y (such descents are called “ (X, Y) -descents”). Explicit inclusion-exclusion type formulas are given for the number of n -permutations with k (X, Y) -descents.

A new type of permutation pattern condition was introduced by Bousquet-Mélou et al. in [2]. In [2], the authors considered restrictions on both the places and the values where a pattern can occur. For example, they considered pattern diagrams as pictured in Figure 1.



Figure 1: A new 2 3 1 pattern.

The vertical line between the 2 and 3 of the pattern means that the occurrence of the 2 and 3 in a subsequence must be consecutive and the horizontal line between 2 and the 1 in the pattern means that values corresponding to the 1 and 2 in an occurrence must be consecutive. Thus an occurrence of the pattern pictured above in a permutation $\pi = \pi_1 \dots \pi_n \in S_n$, is a sequence $1 \leq i_1 < i_2 < i_3 \leq n$ such that $\pi_{i_1} \pi_{i_2} \pi_{i_3}$ is order isomorphic to 231, $i_1 + 1 = i_2$ and $\pi_{i_3} + 1 = \pi_{i_1}$. For example, if $\pi = 31524$, then there is an occurrence of 231 in π , namely, 352, but it is not an occurrence of the pattern in Figure 1 because 3 and 5 do not occur consecutively in π . However if $\tau = 32541$, then 251 is an occurrence of the pattern as pictured in Figure 2.

An attractive property of these new patterns is that, like classical patterns (but not like GPs!), they are closed under the action of D_8 , the symmetry group of the square. More precisely, the authors in [2] studied permutations that either avoid the



Figure 2: Permutations which avoid and contain the new 2 3 1 pattern.

pattern 2-3-1 or in an occurrence $\pi_i\pi_j\pi_k$ of 2-3-1 in a permutation $\pi_1\pi_2 \dots \pi_n$ where one either has $j \neq i+1$ or $\pi_i \neq \pi_k+1$. It turns out that there is a bijection preserving several statistics between $(\mathbf{2+2})$ -free posets and permutations avoiding the pattern in the previous sentence (see [2]).

The outline of this paper is as follows. In Section 2, we define *place-difference-value patterns (PDVPs)* in both permutations and words. These patterns cover under one roof most of the commonly used pattern restrictions that have occurred in the literature on generalizations of GPs. In Sections 3 and 4, we consider several examples of PDVPs. Some of these examples show connections to other combinatorial objects, which cannot be obtained using the languages of most general notions of GPs studied so far. Our work gives rise to four bijective questions linking our patterns to other combinatorial objects; see Problems 1–4. Two of them were solved by Alexander Valyuzhenich in [26] and another is solved in this paper via a suggestion due to an anonymous referee. Finally, in Section 5, we sketch some directions of further research.

2. Place-Difference-Value Patterns

In what follows, \mathbb{P} denotes the set of positive integers and $k\mathbb{P}$ denotes the set of all positive multiples of k .

Definition 1. A *place-difference-value pattern (PDVP for short)* is a quadruple $P = (p, X, Y, Z)$ where p is a permutation of length m , X is an $(m + 1)$ -tuple of non-empty, possibly infinite, sets of positive integers, Y is a set of triples $(s, t, Y_{s,t})$ where $0 \leq s < t \leq m + 1$ and $Y_{s,t}$ is a non-empty, possibly infinite, set of positive integers, and Z is an m -tuple of non-empty, possibly infinite, sets of positive integers. A PDVP $P = (p_1p_2 \dots p_m, (X_0, X_1, \dots, X_m), Y, (Z_1, \dots, Z_m))$ occurs in a permutation $\pi = \pi_1\pi_2 \dots \pi_n$, if π has a subsequence $\pi_{i_1}\pi_{i_2} \dots \pi_{i_m}$ with the following properties:

1. $\pi_{i_k} < \pi_{i_\ell}$ if and only if $p_k < p_\ell$ for $1 \leq k < \ell \leq m$;
2. $i_{k+1} - i_k \in X_k$ for $k = 0, 1, \dots, m$, where we assume $i_0 = 0$ and $i_{m+1} = n + 1$;
3. for each $(s, t, Y_{s,t}) \in Y$, $|\pi_{i_s} - \pi_{i_t}| \in Y_{s,t}$ where we assume $\pi_{i_0} = \pi_0 = 0$ and $\pi_{i_{m+1}} = \pi_{n+1} = n + 1$; and
4. $\pi_{i_k} \in Z_k$ for $k = 1, \dots, m$.

For example, let \mathbb{E} and \mathbb{O} denote the set of even and odd numbers, respectively. Then the PDVP $(12, (\{1\}, \{3, 4\}, \{1, 2, 3\}), \{(1, 2, \mathbb{E})\}, (\mathbb{E}, \mathbb{P}))$ occurs in the permutation $\pi = 23154$ once as the subsequence 24. Indeed, each such occurrence must start at position 1 as required by the set X_0 and the second element of the sequence must occur either at position 4 or 5 as required by X_1 . X_2 says the 4 must occur in one of the last three positions of the permutation. The condition that $Z_1 = \mathbb{E}$ says that the value in position 1 must be even. Finally the condition that $(1, 2, \mathbb{E}) \in Y$ rules out 25 as an occurrence of the pattern.

Classical patterns are PDVPs of the form $(p, (\mathbb{P}, \mathbb{P}, \dots), \emptyset, (\mathbb{P}, \mathbb{P}, \dots))$, whereas the GPs introduced in [1] have the property that X_i is either \mathbb{P} or $\{1\}$, $Y = \emptyset$, and $Z_i = \mathbb{P}$ for all i . Also, the patterns introduced in [2] have the property that each of the X_i 's are either \mathbb{P} or $\{1\}$, $Z_i = \mathbb{P}$ for all i , and all the elements of Y are of the form $(i, j, \{1\})$. Similarly, the occurrences of the pattern $(21, (\mathbb{P}, \{1\}, \mathbb{P}), \emptyset, (X, Y))$ in a permutation π correspond to the (X, Y) -descents in π considered by Hall and Remmel [7] and the occurrences of the pattern $(21, (\mathbb{P}, \{1\}, \mathbb{P}), \{(1, 2, \{k\})\}, (\mathbb{P}, \mathbb{P}))$ in π correspond to elements of $\text{Des}_k(\pi)$ as studied by Liese [20, 21].

We should note that there is often more than one way to specify the same pattern. For instance, we can restrict ourselves to occurrences of patterns that involve only even numbers by either setting $Z_i = \mathbb{E}$ for all i or by setting $Y = \{(i, i + 1, \mathbb{E}) : i = 0, \dots, m - 1\}$.

In Table 1, we list how several pattern conditions that have appeared in the literature can be expressed in terms of PDVPs.

The place-difference-value patterns in case of words can be defined in a similar manner.

Definition 2. A *place-difference-value (word) pattern, PDVP*, is a quadruple $P = (p, X, Y, Z)$ where p is a word of length m having an occurrence of each of the letters $1, 2, \dots, k$ for some k , X is an $(m + 1)$ -tuple of non-empty, possibly infinite, sets of positive integers, the elements of Y are of the form $(s, t, Y_{s,t})$ where $0 \leq i < j \leq m + 1$ and $Y_{i,j}$ is a non-empty, possibly infinite, set of non-negative

Object in the literature	PDVP $P = (p, X, Y, Z)$
Classical patterns	$X_i = Z_j = \mathbb{P}$ for all i and j and $Y = \emptyset$.
GPs in [1]	X_i is either \mathbb{P} or $\{1\}$ for all i , $Z_j = \mathbb{P}$ for all j , $Y = \emptyset$.
Descents conditioned on the parity of the elements in the descent pairs as in [16]	$p = 21$ and $X_0 = \mathbb{P}$, $X_1 = \{1\}$, $X_2 = \mathbb{P}$, $Y = \emptyset$, and (Z_1, Z_2) equals (\mathbb{E}, \mathbb{P}) , (\mathbb{O}, \mathbb{P}) , (\mathbb{P}, \mathbb{E}) , or (\mathbb{P}, \mathbb{O}) .
Patterns in [17]	Similar to the last patterns, except we allow Z_i 's of the form $k\mathbb{P}$ where $k \geq 3$.
Patterns in [20, 21]	$(21, (\mathbb{P}, \{1\}, \mathbb{P}), \{(1, 2, \{k\})\}, (\mathbb{P}, \mathbb{P}))$, where $k \geq 1$.
Patterns in [7]	$(21, (\mathbb{P}, \{1\}, \mathbb{P}), \emptyset, (X, Y))$, X and Y are any fixed sets
Patterns in [2]	X_i is either \mathbb{P} or $\{1\}$, the elements of Y are of the form $(i, j, \{1\})$, and $Z_i = \mathbb{P}$ for all i .

Table 1: Objects studied in the literature using the language of place-difference-value patterns.

integers, and Z is an m -tuple of non-empty, possibly infinite, sets of positive integers. A PDVP $P = (p_1 p_2 \dots p_m, (X_0, X_1, \dots, X_m), Y, (Z_1, \dots, Z_m))$ occurs in a word $w = w_1 w_2 \dots w_n$ over the alphabet $\{1, 2, \dots, t\}$, if w has a subsequence $w_{i_1} w_{i_2} \dots w_{i_m}$ with the following properties:

1. $w_{i_k} < w_{i_\ell}$ if and only if $p_k < p_\ell$ for $1 \leq k < \ell \leq m$;
2. $w_{i_k} = w_{i_\ell}$ if and only if $p_k = p_\ell$ for $1 \leq k < \ell \leq m$;
3. $i_{k+1} - i_k \in X_k$ for $k = 0, 1, \dots, m$, where we assume $i_0 = 0$ and $i_{m+1} = n + 1$;
4. for each $(s, t, Y_{s,t}) \in Y$, $|w_{i_s} - w_{i_t}| \in Y_{s,t}$ where we assume $w_{i_0} = w_0 = 0$ and $w_{i_{m+1}} = w_{n+1} = t$; and
5. $w_{i_j} \in Z_j$ for $j = 1, \dots, m$.

We would like to point out that, of course, the notion of PDVPs can be generalized even further, e.g., increasing the number of dimensions (as it is done in [14, 18]) or, for example, by having the differences or values be dependent on the place. We will discuss some of these extensions in Section 5. In any case, the PDVPs are the closest objects to those most popular pattern restrictions that have appeared in the current literature on patterns.

Another thing to point out is that particular cases of patterns introduced by us already appear in the literature, without any general framework though. For

example, in [25], Tauraso found the number of permutations of size n avoiding simultaneously the PDVPs

$$(12, (\mathbb{P}, \{d\}, \mathbb{P}), \{(1, 2, \{d\})\}, (\mathbb{P}, \mathbb{P}))$$

and

$$(21, (\mathbb{P}, \{d\}, \mathbb{P}), \{(1, 2, \{d\})\}, (\mathbb{P}, \mathbb{P})),$$

where $2 \leq d \leq n - 1$. Also, see [23, A110128] for related objects.

3. Some Results on Place-Difference-Value Patterns in Permutations

Recall that $\mathbb{E} = \{0, 2, 4, \dots\}$ and $\mathbb{O} = \{1, 3, 5, \dots\}$ denote the set of even and odd numbers, respectively. Also, let \mathcal{S}_n denote the set of permutations of length n .

3.1. Distribution of Certain PDVPs on Permutations

Suppose $P = (p, (\mathbb{O}, \mathbb{E}, \dots, \mathbb{E}), \emptyset, (\mathbb{E}, \dots, \mathbb{E}))$, where p is any permutation on t elements. We will show an easy connection between distributions of p , viewed as a classical pattern, and the pattern P .

The restriction on the X 's says that we want our pattern to occur at the odd positions. The restriction on the Z 's says that we are worried about the even numbers that appear at the odd positions.

Let $A_{n,m}$ (resp. $B_{n,m}$) be the number of permutations in \mathcal{S}_n that contain m occurrences of the pattern p (resp. P). There is an easy way to express $B_{n,m}$ in terms of $A_{n,m}$, as it is shown below.

Consider \mathcal{S}_{2n} and suppose a permutation contains m occurrences of P and k even numbers in odd positions, where $0 \leq k \leq n$. We can choose these even numbers that appear in odd positions in $\binom{n}{k}$ ways, and then we can choose the positions of the even numbers in $\binom{n}{k}$ ways. Once we have chosen those numbers and those positions, we have to arrange the even numbers so that the permutation built by them contains m occurrences of p . This can be done in $A_{k,m}$ ways. Then we have to choose the odd numbers that appear in the odd positions in $\binom{n}{n-k}$ ways and those odd numbers can be arranged in $(n - k)!$ ways. Finally the numbers which occupy the even positions can be arranged in $n!$ ways. Thus,

$$B_{2n,m} = \sum_{k=0}^n n!(n - k)! \binom{n}{k}^3 A_{k,m}. \tag{1}$$

The case for \mathcal{S}_{2n+1} is similar.

Thus, whenever we know the distribution of a classical pattern p (it is known only in a few cases), we can find the distribution of P . On the other hand, we can use the same formula for avoidance matters, in which case we can get more applications of it. For example, for $p = 12$, $A_{k,0}$ becomes 1, as the only way to avoid 1-2 is to arrange the corresponding even elements in decreasing order. Thus, in this case, we have

$$B_{2n,0} = \sum_{k=0}^n n!(n-k)! \binom{n}{k}^3.$$

Another example is when $p = 123$. It is well-known that the number of n -permutations avoiding the pattern 1-2-3 is given by the n -th Catalan number C_n , and thus, in this case, we have

$$B_{2n,0} = \sum_{k=0}^n n!(n-k)! \binom{n}{k}^3 C_k.$$

For a slightly more complicated example, suppose that $p = 12$, $X = (\mathbb{O}, 4\mathbb{P}, \mathbb{P})$, $Y = \{(1, 2, 4\mathbb{P})\}$, and $Z = (\mathbb{O}, \mathbb{P})$. Let $A = \{1, 5, 9, 13, \dots\}$ and $B = \{3, 7, 11, 15, \dots\}$. The restriction imposed by our choice of X says that we are only interested in subsequences that occur at positions in A or subsequences that occur at positions in B . The restrictions imposed by our choice of Y and Z says that we are only interested in subsequences that involve values in A or subsequences that involve values in B . Suppose we want to find the number K_n of permutations in \mathcal{S}_n that avoid our pattern. Then choose k_1 to be the number of elements of A that occur in positions in A and k_2 to be the number of elements of A that occur in positions in B . Similarly choose l_1 to be the number of elements of B that occur in positions in A and l_2 to be the number of elements of B that occur in positions in B . To avoid our pattern, the k_1 elements of A that occur in the positions of A must be in decreasing order and k_2 elements of A that occur in the positions of B must occur in decreasing order. Next, the l_1 elements of B that occur in the positions of A must be in decreasing order and l_2 elements of B that occur in the positions of B must occur in decreasing order. Then we can arrange the remaining elements that occur in the positions of A in any order we want and we can arrange in the remaining elements that occur in the positions of B any order we want. Finally, we can arrange the elements that lie in positions outside of A and B in any order that we want. Thus, our final answer is determined by the number of ways to choose the elements that correspond to k_1 , k_2 , l_1 and l_2 and the number of ways to choose their corresponding positions in A and B . Thus, for example, one can easily check that

$$\begin{aligned}
 K_{4n} &= (2n)! \sum_{\substack{0 \leq k_1, k_2, l_1, l_2 \\ k_1 + k_2 \leq n, l_1 + l_2 \leq n}} \binom{n}{k_1, k_2, n - k_1 - k_2} \binom{n}{l_1, l_2, n - l_1 - l_2} \\
 &\quad \times \binom{n}{k_1, l_1, n - k_1 - l_1} \binom{n}{k_2, l_2, n - k_2 - l_2} (n - k_1 - l_1)! (n - k_2 - l_2)! \\
 &= (2n)! \sum_{\substack{0 \leq k_1, k_2, l_1, l_2 \\ k_1 + k_2 \leq n, l_1 + l_2 \leq n}} \frac{(n!)^4}{(k_1!)^2 (k_2!)^2 (l_1!)^2 (l_2!)^2 (n - k_1 - k_2)! (n - l_1 - l_2)!}.
 \end{aligned}$$

3.2. One More Result on PDVPs on Permutations

In this subsection, we consider the permutations which simultaneously avoid the GPs 231 and 132 and the PDVP $P = (12, (\mathbb{P}, \{k\}, \mathbb{P}), \{(1, 2, \{1\})\}, (\mathbb{P}, \mathbb{P}))$, where $k \geq 1$. Let $a_{n,k}$ be the number of such permutations of length n . We will show that

$$a_{n,k} = \begin{cases} F(n) & \text{if } k = 1, \\ 2^{n-1} & \text{if } k \geq 2 \text{ and } n \leq k, \\ 3 \cdot 2^{n-3} & \text{if } k \geq 2 \text{ and } n \geq k + 1 \end{cases}$$

where $F(n)$ is the n -th Fibonacci number. The sequence of $a_{n,2}$'s — 3, 6, 12, 24, 48, 96, 192, . . . — appears in [23, A042950].

Notice, that avoiding just the GPs 231 and 132 gives 2^{n-1} permutations of length n ([8]), and the structure of such permutations is a decreasing word followed by an increasing word (1 is staying between the words and it is assumed to belong to both of them). Suppose first that $k \geq 2$. If $n \leq k$, then there is no chance for P to occur thus giving 2^{n-1} possibilities. On the other hand, assuming $n = k + 1$, the number of permutations avoiding the three patterns is given by $2^k - 2^{k-2}$ as whenever the first letter is $n - 1$ and the last letter is n , we get an occurrence of P (there are 2^{k-2} such cases). Finally, if we increase the number of letters in a “good” permutation of length $k + 1$, one by one, we always have two places to insert a current largest element: at the very beginning or at the very end, which gives in total $(2^k - 2^{k-2})2^{n-k-1} = 3 \cdot 2^{n-3}$ possibilities, as claimed.

In the case $k = 1$, we think of counting good permutations by starting with the letter 1, and inserting, one by one, the letters 2,3, . . . If P would not be prohibited, we would always have two choices to insert a current largest element. However, inserting n , the configuration $(n - 1)n$ is prohibited, which leads immediately to a recursion for the Fibonacci numbers.

Remark 3. Because of the structure of permutations avoiding GPs 231 and 132, one can see that the maximum number of occurrences of P in such permutations is 1. Thus, we actually found distribution of P on 231- and 132-avoiding permutations, as the number of such permutations having exactly one occurrence of P is $2^{n-1} - a_{n,k}$.

An interpretation of the sequence [23, A042950], based on a result in [15, Section 6.2], suggests the first of our bijective problems.

Problem 4. For $k \geq 2$ and $n \geq k + 1$, find a bijection between permutations of length n which simultaneously avoid the GPs 231 and 132 and the PDVP $P = (12, (\mathbb{P}, \{k\}, \mathbb{P}), \{(1, 2, \{1\})\}, (\mathbb{P}, \mathbb{P}))$ and the set of rises (occurrence of the GP 12) after n iterations of the morphism $1 \rightarrow 123, 2 \rightarrow 13, 3 \rightarrow 2$, starting with element 1. For example, for $k = 2$ and $n = 3$, there are 3 permutations avoiding the prohibitions, 123, 312, and 321, and there are 3 rises in 1231323.

In this case, we can construct the desired bijection essentially following the bijection description provided by the anonymous referee. Fix $k \geq 2$. The first step is to give a coding for the set \mathcal{A}_n of permutations in S_n which avoid both the GPs 231 and 132. Clearly, $\mathcal{A}_1 = \{1\}$ and in general, we can construct \mathcal{A}_{n+1} from \mathcal{A}_n by adding $n + 1$ to both the right and the left of each element of \mathcal{A}_n . Thus each element $\sigma \in \mathcal{A}_n$, where $n \geq 2$, can be coded by a word $w(\sigma) \in \{L, R\}^{n-1}$. That is, if $\sigma \in \mathcal{A}_n$, $w(\sigma) = w_1 \dots w_{n-1}$ where then $w_i = R$ if i is to the right of 1 in σ and $w_i = L$ if i is to the left of 1 in σ . Next observe that the PDVP $P = (12, (\mathbb{P}, \{k\}, \mathbb{P}), \{(1, 2, \{1\})\}, (\mathbb{P}, \mathbb{P}))$ occurs in $\sigma \in \mathcal{A}_n$ if and only if $w(\sigma) = w_1 \dots w_{n-1}$ where $w_k = L$ and $w_{k+1} = R$. Thus for $n \geq k + 1$, the set of permutations $\mathcal{B}_n^{(k)}$ which simultaneously avoid the GPs 231 and 132 and the PDVP $P = (12, (\mathbb{P}, \{k\}, \mathbb{P}), \{(1, 2, \{1\})\}, (\mathbb{P}, \mathbb{P}))$ in bijection with the set of words $\sigma \in \mathcal{A}_n$ such that $w(\sigma) = w_1 \dots w_{n-1}$ where $w_k w_{k+1}$ equals LL, RL or RR . For each $\sigma \in \mathcal{B}_n^{(k)}$, we let $\bar{w}(\sigma)$ be the word $w_k w_{k+1} w_1 \dots w_{k-1} w_{k+2} \dots w_{n-1}$ if $w(\sigma) = w_1 \dots w_n$. In this way, we have a bijection between $\mathcal{B}_n^{(k)}$ and the sequences of the form $LL\{L, R\}^{n-3} \cup RL\{L, R\}^{n-3} \cup RR\{L, R\}^{n-3}$.

Now let ϕ be the morphism such that on any word $w = w_1 \dots w_n \in \{1, 2, 3\}^n$, $\phi(w) = \phi(w_1) \dots \phi(w_n)$ where $\phi(1) = 123, \phi(2) = 13$, and $\phi(3) = 2$. Then define a sequence of words U_n for $n \geq 1$ by induction by letting $U_1 = \phi(1) = 123$ and $U_{n+1} = \phi(U_n)$ for all $n \geq 1$. Thus for example,

$$U_1 = 123,$$

$$U_2 = 123132,$$

$$U_3 = 123132123213, \text{ and}$$

$$U_4 = 123132123213123132131232.$$

First we claim that U_n is a rearrangement of 2^{n-1} 1's, 2^{n-1} 2's, and 2^{n-1} 3's. Clearly, this is true for $n = 1$. If by induction, we assume our claim is true for n , then each 1 in U_n gives rise to 2^{n-1} 1's, 2's and 3's in U_{n+1} , each 2 in U_n gives rise to 2^{n-1} 1's and 3's in U_{n+1} , and each 3 in U_n gives rise to 2^{n-1} 2's in U_{n+1} . Thus there will be $2^{n-1} + 2^{n-1} = 2^n$ 1's, 2's and 3's in U_{n+1} . Moreover, each letter in U_{n+1} arises from some $\phi(u_i)$ where $U_n = u_1 \dots u_{3 \cdot 2^{n-1}}$. By iterating this idea, we can define what we call the *trace* of each letter in U_{n+1} . That is, for U_1 , $trace(i) = i$ for $i = 1, 2, 3$. Now suppose that $U_{n+1} = w_1 \dots w_{3 \cdot 2^n}$ and $U_n = u_1 \dots u_{3 \cdot 2^{n-1}}$ and we have defined $trace(u_i)$ for $i = 1, \dots, 3 \cdot 2^{n-1}$. Then for $j = 1, \dots, 3 \cdot 2^n$, we define $trace(w_j) = w_j trace(u_i)$ if w_j arises from $\phi(u_i)$. For example, suppose that $n = 4$, if we consider the third 2 from the right in U_4 , we have underlined the sequence of letters that correspond to the trace of that 2 below:

$$U_1 = 1\underline{2}3$$

$$U_2 = 123\underline{1}32$$

$$U_3 = 1231321\underline{2}3213$$

$$U_4 = 12313212321312313\underline{2}131232.$$

Thus the trace of the third 2 from the right in U_n is 2312. Now it is easy to see that the set of all possible traces of letters in U_{n+1} corresponds to three binary trees of height n . That is, if $v_1 v_2 \dots v_{n+1}$ is the trace of a letter in U_{n+1} , then $v_1 \in \{1, 2, 3\}$ and for each i , $v_{i-1} \in \{1, 2\}$ if $v_i = 1$, $v_{i-1} \in \{1, 3\}$ if $v_i = 2$, and $v_{i-1} \in \{1, 2\}$ if $v_i = 3$. Next, consider the rises in U_{n+1} where we say that w_i is a rise in U_{n+1} if $w_i < w_{i+1}$. It is easy to check that the only rises that occur in U_{n+1} are the rises which are internal to the image of some $\phi(u_i)$. That is, our morphism is such that there is never a rise between the last letter of $\phi(u_i)$ and the first letter of $\phi(u_{i+1})$ for any i . This means that for $i = 1, \dots, 3 \cdot 2^n - 1$, w_i is a rise in U_{n+1} if and only if w_i arose from some u_j where either $u_j = 1$ and $w_i = 1$ or $w_i = 2$, or $u_j = 2$ and $w_i = 1$. Thus the traces of such w_i are of the form $11 \dots$, $21 \dots$, or $12 \dots$. In each case, the \dots part corresponds to a path from root to leaf in a binary tree. For example, in the case of $n = 4$, the traces of the twelve rises in U_4 would correspond to the twelve paths from root to leaf in the three trees pictured in Figure 3.

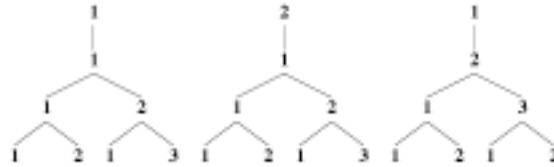


Figure 3: The trees corresponding to the traces of the rises in U_4 .

Our problem is to find a bijection between the elements of $\mathcal{B}_{n+1}^{(k)}$ and the rises in U_n . We have shown that both sets correspond to the paths in the union of three trees of height $n-2$. So we map the trees $RR\{L, R\}^{n-2}$ onto the trees corresponding to the traces of the rises in U_n whose traces are of the form $11\cdots$ where we interpret L as taking the left branch at each node and R as taking the right branch at each node. Similarly, we map the trees $RL\{L, R\}^{n-2}$ onto the trees corresponding to the traces of the rises in U_n whose traces are of the form $21\cdots$ and we map the trees $LL\{L, R\}^{n-2}$ onto the trees corresponding to the traces of the rises in U_n whose traces are of the form $12\cdots$.

4. Some Results on Place-Difference-Value Patterns on Words

In this section, we consider examples of PDVPs on words involving both distance and value, that cannot be expressed in terms of pattern conditions that have appeared in the literature so far.

4.1. The PDVP $(12, (\mathbb{P}, \{2\}, \mathbb{P}), \{(1, 2, \{2\})\}, (\mathbb{P}, \mathbb{P}))$ on Words.

Consider words $w \in \{1, \dots, k\}^*$. If $w = w_1 \dots w_n$, then let $S(w) = \{i : w_{i+2} - w_i = 2\}$ and $s(w) = |S(w)|$.

Our goal is to compute the generating function

$$A_k(q, z) = \sum_{w \in \{1, \dots, k\}^*} q^{|w|} z^{s(w)}. \tag{2}$$

Let

$$A_k(i_1 \dots i_j; q, z) = \sum_{w \in \{1, \dots, k\}^*} q^{|i_1 \dots i_j w|} z^{s(i_1 \dots i_j w)}. \tag{3}$$

Then, for example, when $k = 3$, we easily obtain the following recursions for $A_3(ij; q, z) = A(ij; q, z)$.

$$\begin{aligned}
 A(11; q, z) &= q^2 + qA(11; q, z) + qA(12; q, z) + qzA(13; q, z) \\
 A(12; q, z) &= q^2 + qA(21; q, z) + qA(22; q, z) + qzA(23; q, z) \\
 A(13; q, z) &= q^2 + qA(31; q, z) + qA(32; q, z) + qzA(33; q, z) \\
 A(21; q, z) &= q^2 + qA(11; q, z) + qA(12; q, z) + qA(13; q, z) \\
 A(22; q, z) &= q^2 + qA(21; q, z) + qA(22; q, z) + qA(23; q, z) \\
 A(23; q, z) &= q^2 + qA(31; q, z) + qA(32; q, z) + qA(33; q, z) \\
 A(31; q, z) &= q^2 + qA(11; q, z) + qA(12; q, z) + qA(13; q, z) \\
 A(32; q, z) &= q^2 + qA(21; q, z) + qA(22; q, z) + qA(23; q, z) \\
 A(33; q, z) &= q^2 + qA(31; q, z) + qA(32; q, z) + qA(33; q, z).
 \end{aligned}$$

Thus if we let

$$\begin{aligned}
 Q &= (-q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2) \text{ and} \\
 A &= (A(11; q, z), A(12; q, z), A(13; q, z), A(21; q, z), A(22; q, z), A(23; q, z), \\
 &\quad A(31; q, z), A(32; q, z), A(33; q, z)),
 \end{aligned}$$

then we see that

$$Q^T = MA^T$$

where

$$M = \begin{pmatrix}
 q-1 & q & zq & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 0 & q & q & qz & 0 & 0 & 0 \\
 0 & 0 & -1 & 0 & 0 & 0 & q & q & qz \\
 q & q & q & -1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & q & q-1 & q & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -1 & q & q & q \\
 q & q & q & 0 & 0 & 0 & -1 & 0 & 0 \\
 0 & 0 & 0 & q & q & q & 0 & -1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & q & q & q-1
 \end{pmatrix}.$$

Thus $A^T = M^{-1}Q^T$ and our desired generating function is given by

$$\begin{aligned} A_3(q, z) &= 1 + 3q + (1, 1, 1, 1, 1, 1, 1, 1, 1)A^T \\ &= \frac{1}{(1 - q^2(1 - z))(1 - (3q + q^2(z - 1)))}. \end{aligned}$$

where we used Mathematica for the last equation.

Note that

$$\begin{aligned} \frac{1}{1 - (3q + q^2(z - 1))} &= \sum_{m \geq 0} \sum_{k=0}^m \binom{m}{k} 3^k q^k q^{2m-2k} (z - 1)^{m-k} \\ &= \sum_{m \geq 0} \sum_{k=0}^m \binom{m}{k} 3^k (z - 1)^{m-k} q^{2m-k}. \end{aligned}$$

Now it is easy to see that $\sum_{k=0}^m \binom{m}{k} 3^k (z - 1)^{m-k} q^{2m-k}$ involves powers of q that range from m to $2m$. Thus, since $2m - k = 2n$ if and only if $k = 2m - 2n$, we have

$$\begin{aligned} \left. \frac{1}{1 - (3q + q^2(z - 1))} \right|_{q^{2n}} &= \sum_{m=n}^{2n} \binom{m}{2m - 2n} 3^{2m-2n} (z - 1)^{m-(2m-2n)} \\ &= \sum_{m=0}^n \binom{m+n}{2m} 3^{2m} (z - 1)^{n-m}. \end{aligned}$$

It follows that

$$\begin{aligned} \left. \frac{1}{(1 - q^2(1 - z))(1 - (3q + q^2(z - 1)))} \right|_{q^{2n}} &= \sum_{r=0}^n (1 - z)^{n-r} \sum_{m=0}^r \binom{m+r}{2m} 3^{2r} (z - 1)^{r-m} \\ &= \sum_{r=0}^n \sum_{m=0}^r (-1)^{n-r} \binom{m+r}{2m} 3^{2m} (z - 1)^{n-m} \end{aligned}$$

and

$$\begin{aligned} \left. \frac{1}{(1 - q^2(1 - z))(1 - (3q + q^2(z - 1)))} \right|_{q^{2n}z^s} &= \sum_{r=0}^n \sum_{m=0}^r (-1)^{m+r+s} \binom{m+r}{2m} \binom{n-m}{s} 9^m. \end{aligned}$$

Similarly, one can show that

$$\begin{aligned} & \frac{1}{1 - (3q + q^2(z - 1))} \Big|_{q^{2n+1}} \\ &= \sum_{m=n+1}^{2n+1} \binom{m}{2m - 2n - 1} 3^{2m-2n-1} (z - 1)^{m-(2m-2n-1)} \\ &= \sum_{m=0}^n \binom{m+n+1}{2m+1} 3^{2m+1} (z - 1)^{n-m}. \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{1}{(1 - q^2(1 - z))(1 - (3q + q^2(z - 1)))} \Big|_{q^{2n+1}} \\ &= \sum_{r=0}^n (1 - z)^{n-r} \sum_{m=0}^r \binom{m+r+1}{2m+1} 3^{2r+1} (z - 1)^{r-m} \\ &= \sum_{r=0}^n \sum_{m=0}^r (-1)^{n-r} \binom{m+r+1}{2m+1} 3^{2m+1} (z - 1)^{n-m} \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{(1 - q^2(1 - z))(1 - (3q + q^2(z - 1)))} \Big|_{q^{2n+1}z^s} \\ &= \sum_{r=0}^n \sum_{m=0}^r (-1)^{m+r+s} \binom{m+r+1}{2m+1} \binom{n-m}{s} 3^{2m+1}. \end{aligned} \tag{4}$$

Thus we have shown that, in particular, in case of avoidance,

$$A_3(q, 0)|_{q^{2n}} = \sum_{r=0}^n \sum_{m=0}^r (-1)^{m+r} \binom{m+r}{2m} 3^{2m} \text{ and} \tag{5}$$

$$A_3(q, 0)|_{q^{2n+1}} = \sum_{r=0}^n \sum_{m=0}^r (-1)^{m+r} \binom{m+r+1}{2m+1} 3^{2m+1}. \tag{6}$$

A check in [23] shows that $A_3(q, 0)|_{q^{2n}} = (F(2n))^2$ and $A_3(q, 0)|_{q^{2n+1}} = F(2n)F(2n + 2)$ where $F(n)$ is the n -th Fibonacci number (see [23, A006190]). Here is a proof of that fact. Clearly if we have a word $w = w_1 \dots w_{2n}$ such that $s(w) = 0$, then $u = w_1 w_3 \dots w_{2n-1}$ and $v = w_2 w_4 \dots w_{2n}$ must be words in $\{1, 2, 3\}^*$ that never have a 3 following a 1. The map $\{1 \rightarrow 01, 2 \rightarrow 00, 3 \rightarrow 10\}$

gives a bijection from the set of words of length n avoiding 13 and the set of binary words of length $2n$ avoiding 11 and known to be counted by $F(2n)$.

For another way to understand the same result, we first find the distribution of consecutive occurrences of 13 over words in $\{1, 2, 3\}^*$. For any word $u = u_1 \dots u_n \in \{1, 2, 3\}^*$, let $T(w) = \{i : w_{i+1} = 2 + w_i\}$ and $t(w) = |T(w)|$. Then we wish to compute

$$B_3(q, z) = \sum_{w \in \{1,2,3\}^*} q^{|w|} z^{t(w)}. \tag{7}$$

Let

$$B_3(i_1 \dots i_j; q, z) = \sum_{w \in \{1,2,3\}^*} q^{|i_1 \dots i_j w|} z^{t(i_1 \dots i_j w)}. \tag{8}$$

Then it is easy to see that

$$\begin{aligned} B_3(1; q, z) &= q + qB_3(1; q, z) + qB_3(2; q, z) + qzB_3(3; q, z) \\ B_3(2; q, z) &= q + qB_3(1; q, z) + qB_3(2; q, z) + qB_3(3; q, z) \\ B_3(3; q, z) &= q + qB_3(1; q, z) + qB_3(2; q, z) + qB_3(1; q, z). \end{aligned}$$

Thus if $\bar{Q} = (-q, -q, -q)$ and $B = (B_3(1; q, z), B_3(2; q, z), B_3(3; q, z))$, then $\bar{Q}^T = RB^T$ where

$$R = \begin{pmatrix} q-1 & q & qz \\ q & q-1 & q \\ q & q & q-1 \end{pmatrix}.$$

Thus $B^T = R^{-1}\bar{Q}^T$ and hence

$$B_3(q, z) = 1 + B_3(1; q, z) + B_3(2; q, z) + B_3(3; q, z) \tag{9}$$

$$= \frac{1}{1 - 3q - q^2(z - 1)}. \tag{10}$$

To derive the avoidance case algebraically, notice that the generating function for the Fibonacci numbers (with a proper shift of indices) is

$$F(q) = \sum_{n \geq 0} F(n)q^n = \frac{1 + q}{1 - q - q^2}. \tag{11}$$

Thus,

$$\begin{aligned}
 f(q) &= \sum_{n \geq 0} F(2n)q^n \\
 &= \frac{F(q^{1/2}) + F(-q^{1/2})}{2} \\
 &= \frac{1}{1 - 3q + q^2} \\
 &= B_3(q, 0).
 \end{aligned}$$

This shows again that the number of words $w \in \{1, 2, 3\}^*$ of length n such that $t(w) = 0$ is equal to $F(2n)$. It easily follows that $w \in \{1, 2, 3\}^*$ of length $2n$ such that $s(w) = 0$ is equal to $(F(2n))^2$ and the number of word $w \in \{1, 2, 3\}^*$ of length $2n + 1$ such that $s(w) = 0$ is equal to $F(2n)F(2n + 2)$.

One can do a similar calculations when $k = 4$. In that case, Mathematica shows that

$$A_4(q, z) = \frac{1}{1 - 4q - 8q^3(z - 1) - 4q^4(z - 1)^2} \tag{12}$$

and thus,

$$A_4(q, 0) = 1 + 4q + 16q^2 + 56q^3 + 196q^4 + 672q^5 + 2304q^6 + \dots$$

As before, if we have a word of $w = w_1 \dots w_{2n}$ such that $s(w) = 0$, then $u = w_1 w_3 \dots w_{2n-1}$ and $v = w_2 w_4 \dots w_{2n}$ must be words in $\{1, 2, 3, 4\}^*$ that never have a 3 following a 1 or a 4 following a 2. For any word $u = u_1 \dots u_n \in \{1, 2, 3, 4\}^*$, let, as before, $T(w) = \{i : w_{i+1} = 2 + w_i\}$ and $t(w) = |T(w)|$. Then we can compute

$$B_4(q, z) = \sum_{w \in \{1, 2, 3, 4\}^*} q^{|w|} z^{t(w)} \tag{13}$$

in the same way that we computed $B_3(q, z)$. In this case,

$$B_4(q, z) = \frac{1}{1 - 4q - 2q^2(z - 1)}. \tag{14}$$

Then

$$B_4(q, 0) = 1 + 4q + 14q^2 + 48q^3 + 164q^4 + 560q^5 + 1912q^6 + \dots$$

It should be noted that $B_4(q, 0) = \frac{1}{1 - 4q + 2q^2}$ is a generating function that beyond the objects listed in [23, A007070] counts the number of independent sets

in certain “almost regular” graphs G_3^n (see [3]). We leave establishing a bijection between the objects in question as an open problem, instead considering the following bijective question.

Problem 5. Find a bijection between the set A_n of words $w = w_1w_2 \dots w_n \in \{1, 2, 3, 4\}^*$ that avoid the pattern $(12, (\mathbb{P}, \{1\}, \mathbb{P}), \{(1, 2, \{2\})\}, (\mathbb{P}, \mathbb{P}))$ and the set B_n of words $w_0w_1 \dots w_{2n+3}$ over $\{1, 2, \dots, 7\}^*$ with $w_0 = 1$ and $w_{2n+3} = 4$ and $|w_i - w_{i-1}| = 1$.

Problem 5 was solved by Alexander Valyuzhenich in [26] who found a recursive bijection between the objects involved.

4.2. The PDVP $(12, (\mathbb{P}, \{1, 2\}, \mathbb{P}), \{(1, 2, \{2\})\}, (\mathbb{P}, \mathbb{P}))$ on Words.

Let $U(w) = \{i : w_{i+1} - w_i = 2\}$ and $V(w) = \{i : w_{i+2} - w_i = 2\}$ and let $p(w) = |U(w)| + |V(w)|$. In that case, we can use essentially the same methods to calculate $D_k(q, z) = \sum_{w \in \{1, \dots, k\}^*} q^{|w|} z^{p(w)}$.

Let

$$D_k(i_1 \dots i_j; q, z) = \sum_{w \in \{1, \dots, k\}^*} q^{|i_1 \dots i_j w|} z^{p(i_1 \dots i_j w)}. \tag{15}$$

Then for example, when $k = 3$, we easily obtain the following recursions for $D_3(ij; q, z) = D(ij; q, z)$.

$$\begin{aligned} D(11; q, z) &= q^2 + qD(11; q, z) + qD(12; q, z) + qzD(13; q, z) \\ D(12; q, z) &= q^2 + qD(21; q, z) + qD(22; q, z) + qzD(23; q, z) \\ D(13; q, z) &= zq^2 + zqD(31; q, z) + zqD(32; q, z) + qz^2D(33; q, z) \\ D(21; q, z) &= q^2 + qD(11; q, z) + qD(12; q, z) + qD(13; q, z) \\ D(22; q, z) &= q^2 + qD(21; q, z) + qD(22; q, z) + qD(23; q, z) \\ D(23; q, z) &= q^2 + qD(31; q, z) + qD(32; q, z) + qD(33; q, z) \\ D(31; q, z) &= q^2 + qD(11; q, z) + qD(12; q, z) + qD(13; q, z) \\ D(32; q, z) &= q^2 + qD(21; q, z) + qD(22; q, z) + qD(23; q, z) \\ D(33; q, z) &= q^2 + qD(31; q, z) + qD(32; q, z) + qD(33; q, z). \end{aligned}$$

Thus if we let

$$\begin{aligned}
 Q &= (-q^2, -q^2, -zq^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2) \text{ and} \\
 D &= (D(11; q, z), D(12; q, z), D(13; q, z), D(21; q, z), D(22; q, z), D(23; q, z), \\
 &\quad D(31; q, z), D(32; q, z), D(33; q, z)),
 \end{aligned}$$

then we see that

$$Q^T = MD^T$$

where

$$M = \begin{pmatrix} q-1 & q & zq & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & q & q & qz & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & zq & zq & qz^2 \\ q & q & q & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & q-1 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & q & q & q \\ q & q & q & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & q & q & q & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & q & q-1 \end{pmatrix}.$$

Thus $D^T = M^{-1}Q^T$ and our desired generating function is given by

$$\begin{aligned}
 D_3(q, z) &= 1 + 3q + (1, 1, 1, 1, 1, 1, 1, 1, 1)D^T \\
 &= \frac{1}{1 - 3q - q^2(z - 1) - q^3(2z + 1)(z - 1) - q^4(z - 1)^2} \tag{16}
 \end{aligned}$$

where we again used Mathematica for the last equation. In this case,

$$D_3(q, 0) = 1 + 3q + 8q^2 + 20q^3 + 49q^4 + 119q^5 + 288q^6 + \dots$$

A similar computation as that above will show that

$$D_4(q, z) = \frac{1 + 2q^2(1 - z) - 2q^3(z - 1)^2}{1 - 4q - 8q^2(z - 1) - 4q^4(z - 1)^2}. \tag{17}$$

It follows that

$$\begin{aligned}
 D_4(q, 0) &= \frac{1 + 2q^2 - 2q^3}{1 - 4q + 8q^3 - 4q^4} \\
 &= 1 + 4q + 14q^2 + 46q^3 + 156q^4 + 528q^5 + 1800q^6 + \dots
 \end{aligned}$$

The sequence 1, 3, 8, 20, 49, 119, 288, . . . appears in [23, A048739] where it is given an interpretation as the number of words $w = w_0 \dots w_{n+2} \in \{1, 2, 3\}^*$ such that $|w_{i+1} - w_i| \leq 1$, and $w_0 = 1$ and $w_{n+2} = 3$. This leads to the following bijective problem.

Problem 6. Find a bijection between the set A_n of words $w = w_1 w_2 \dots w_n \in \{1, 2, 3\}^*$ that avoid the pattern $(12, (\mathbb{P}, \{1, 2\}, \mathbb{P}), \{(1, 2, \{2\})\}, (\mathbb{P}, \mathbb{P}))$ and the set B_n of words $w = w_0 \dots w_{n+2} \in \{1, 2, 3\}^*$ such that $|w_{i+1} - w_i| \leq 1$, and $w_0 = 1$ and $w_{n+2} = 3$.

Problem 6 was solved by Alexander Valyuzhenich in [26] who found a recursive bijection in question.

4.3. The PDVP $(12, (\mathbb{P}, \{1, 2\}, \mathbb{P}), \{(1, 2, \{2\})\}, (\mathbb{O}, \mathbb{P}))$ on Words.

In this case, an occurrence of our PDVP is either 2 consecutive odd numbers that differ by 2 or two odd numbers at distance 2 from each other that differ by 2. Let $P(w) = \{i : w_{i+1} - w_i = 2 \ \& \ w_{i+1} \in \mathbb{O}\}$ and $Q(w) = \{i : w_{i+2} - w_i = 2 \ \& \ w_{i+2} \in \mathbb{O}\}$ and let $r(w) = |P(w)| + |Q(w)|$. In that case, we can use essentially the same methods to calculate $E_k(q, z) = \sum_{w \in \{1, \dots, k\}^*} q^{|w|} z^{r(w)}$.

Let

$$E_k(i_1 \dots i_j; q, z) = \sum_{w \in \{1, \dots, k\}^*} q^{|i_1 \dots i_j w|} z^{r(i_1 \dots i_j w)}. \tag{18}$$

For example, in the case that $n = 4$, it is easy to see that

$$\begin{aligned} E(11; q, z) &= q^2 + qE(11; q, z) + qE(12; q, z) + qzE(13; q, z) + qE(14 : q, z) \\ E(12; q, z) &= q^2 + qE(21; q, z) + qE(22; q, z) + qzE(23; q, z) + qE(24 : q, z) \\ E(13; q, z) &= q^2 z + qzE(31; q, z) + qzE(32; q, z) + qz^2 E(33; q, z) \\ &\quad + qzE(34 : q, z) \\ E(14; q, z) &= q^2 + qE(41; q, z) + qE(42; q, z) + qzE(13; q, z) + qE(14 : q, z) \end{aligned}$$

and for any $i \in 2, 3, 4$,

$$\begin{aligned} E(i1; q, z) &= q^2 + qE(11; q, z) + qE(12; q, z) + qE(13; q, z) + qE(14 : q, z) \\ E(i2; q, z) &= q^2 + qE(21; q, z) + qE(22; q, z) + qE(23; q, z) + qE(24 : q, z) \\ E(i3; q, z) &= q^2 + qE(31; q, z) + qE(32; q, z) + qE(33; q, z) + qE(34 : q, z) \\ E(i4; q, z) &= q^2 + qE(41; q, z) + qE(42; q, z) + qE(13; q, z) + qE(14 : q, z). \end{aligned}$$

Thus if we let

$$Q = (-q^2, -q^2, -zq^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2, -q^2) \text{ and}$$

$$E = (E(ij; q, z))_{1 \leq i, j \leq 4}$$

where we order the elements of E according to the lexicographic order on the pairs (i, j) , then we see that

$$Q^T = ME^T$$

where

$$M = \begin{pmatrix} q-1 & q & zq & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & q & q & qz & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & qz & qz & qz^2 & qz & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & q & qz & q & 0 & 0 \\ q & q & q & q & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & q-1 & q & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & q & q & q & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & q & q & q & q & 0 & 0 \\ q & q & q & q & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & q & q & q & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & q & q-1 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & q & q & q & q & 0 & 0 \\ q & q & q & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & q & q & q & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & q & q & q & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & q & q & q-1 & 0 \end{pmatrix}.$$

Thus $E^T = M^{-1}Q^T$ and our desired generating function is given by

$$\begin{aligned} E_4(q, z) &= 1 + 4q + (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)E^T \\ &= \frac{1}{1 - 4q - (z - 1)q^2 - 2(z^2 - 1)q^3 - z(z - 1)^2q^4} \end{aligned} \tag{19}$$

where we again used Mathematica for the last equation. Then

$$E_4(q, 0) = 1 + 4q + 15q^2 + 54q^3 + 193q^4 + 688q^5 + \dots$$

4.4. One More Result on PDVPs on Words

In this subsection, we consider the following two PDVP's,

$$P_1 = (12, (\mathbb{P}, \{1\}, \mathbb{P}), \{(1, 2, \{1\})\}, (\mathbb{P}, \mathbb{P})) \text{ and}$$

$$P_2 = (12, (\mathbb{P}, \{2\}, \mathbb{P}), \{(1, 2, \{2\})\}, (\mathbb{P}, \mathbb{P})).$$

Thus an occurrence of P_1 in $w = w_1 \dots w_n$ is a pair $w_i w_{i+1}$ of the form $x x + 1$ and an occurrence of P_2 in w is a pair $w_i w_{i+2}$ of the form $x x + 2$.

Our goal is to show that the number a_n of words of length n over $\{1, 2, 3\}$ avoiding simultaneously the PDVP P_1 and P_2 is given by $F(n + 5) - n - 4$, where $F(n)$, as above, is the n -th Fibonacci number. The corresponding sequence — 3, 7, 14, 26, 46, 79, 133, 221, ... — appears in [23, A079921].

It is easy to check that $a_1 = 3$ and $a_2 = 7$. We will show that for $n \geq 3$, $a_n = a_{n-1} + a_{n-2} + n + 1$, thus proving the claim by [23, A079921]. For a given word $w = w_1 w_2 \dots w_n$ over $\{1, 2, 3\}$ avoiding the prohibited patterns, we distinguish 5 non-overlapping cases covering all possibilities:

1. $w_1 = 3$. This 3 has no effect on the rest of w (it cannot be involved in an occurrence of a prohibited pattern) thus giving a_{n-1} possibilities.
2. $w_1 w_2 = 11$. There is only one valid extension of 11 to the right, namely, $1 \dots 1$, as placing 2 (resp. 3) will introduce an occurrence of P_1 (resp. P_2) in w . Thus, the number of possibilities in this case is 1.
3. $w_1 w_2 = 13$. Extending 13 to the right by any legal word $w_3 w_4 \dots w_n$ of length $n - 2$, we will be getting valid words of length n , except for the case when $w_3 = 3$ ($w_1 w_3$ is an occurrence of P_2). The number of “bad” words, according to case (1) above is a_{n-3} . Thus, the number of possibilities in this case is $a_{n-2} - a_{n-3}$.
4. $w = \underbrace{2 \dots 2}_{>0} \underbrace{1 \dots 1}_{\geq 0}$. The number of possibilities in this case is, clearly, n .
5. $w_1 = 2$ and w contains at least one 3. Notice, that the leftmost 3 in w must be preceded by 1 (to avoid P_1), which, in turn, must be preceded by 2 (using the fact that w avoids P_2 and $w_1 = 2$). Thus, in this case, w begins with a word of the form $\underbrace{2 \dots 2}_{>0} 13x$, where x , if it exists, is not equal to 3. To count all such words, we proceed according to the following, obviously reversible, procedure. Consider a good word, say v , of length $n - 3$. If v does not begin with 3, map it to $213v$ to get a proper word of length n in the class in question. On the other hand, if $v = \underbrace{3 \dots 3}_{i>0} xV$, were $x \neq 3$ (assuming

such x exists), map v to $\underbrace{2 \dots 2}_{i>1} 13xV$ getting a proper word of length n in the class. Clearly, we get all words in the class. Thus, the number of possibilities in this case is a_{n-3} .

Summarizing cases 1–5 above we get the desired recurrence.

The problem below involves so called *2-stack sortable permutations*, that is, permutations that can be sorted by passing them twice through a *stack* (where the letters on the stack must be in increasing order). Such permutations were first considered in [27], but have attracted much attention in the literature since then.

Problem 7. Find a bijection between the set A_n of words $w = w_1w_2 \dots w_n \in \{1, 2, 3\}^*$ that avoid simultaneously the PDVPs P_1 and P_2 and the set B_n of 2-stack sortable permutations which avoid the pattern 1-3-2 and contain exactly one occurrence of the pattern 1-2-3. The last object is studied in [5].

5. Beyond PDVPs: Directions of Further Research

Another generalization of the GPs is *partially ordered patterns (POPs)* when the letters of a pattern form a partially ordered set (poset), and an occurrence of such a pattern in a permutation is a linear extension of the corresponding poset in the order suggested by the pattern (we also pay attention to eventual dashes and brackets). For instance, if we have a poset on three elements labeled by $1'$, 1 , and 2 , in which the only relation is $1 < 2$ (see Figure ??), then in an occurrence of $p = 1'-12$ in a permutation π the letter corresponding to the $1'$ in p can be either larger or smaller than the letters corresponding to 12 . Thus, the permutation 31254 has three occurrences of p , namely $3-12$, $3-25$, and $1-25$.

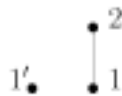


Figure 4: A poset on three elements with the only relation $1 < 2$.

The notion of a POP allows us to collect under one roof (to provide a uniform notation for) several combinatorial structures such as *peaks*, *valleys*, *modified maxima* and *modified minima* in permutations, *Horse permutations* and *p-descents* in permutations. See [9, 10, 11] for results, including a survey paper, on POPs in permutations and [12] on POPs on words.

In the literature on permutation patterns, there are several publications involving so called *barred patterns*. For example, in [2] a conjecture is settled on the number of permutations avoiding the barred pattern $3\bar{1}\bar{5}\bar{2}\bar{4}$. A permutation π avoids $3\bar{1}\bar{5}\bar{2}\bar{4}$ if every occurrence of the pattern 2-3-1 plays the role of 352 in an occurrence of the pattern 3-1-5-2-4. In some cases, barred patterns can be expressed in terms of generalized patterns. E.g., to avoid $4\bar{1}\bar{3}\bar{5}\bar{2}$ is the same as to avoid 3-14-2. However, in many cases, one cannot express the barred patterns in terms of other patterns. The pattern $3\bar{1}\bar{5}\bar{2}\bar{4}$ is an example of such pattern. Another example is the barred pattern $3\bar{5}\bar{2}\bar{4}\bar{1}$ (it is shown in [27] that the set of 2-stack sortable permutations mentioned above is described by avoidance of $3\bar{5}\bar{2}\bar{4}\bar{1}$ and 2-3-4-1). In general, one can consider distributions, rather than just avoidance, of barred patterns. For example, the pattern $2\bar{3}\bar{1}$ occurs in a permutation π k times, if there are exactly k occurrences ba in π of the pattern 2-1 such that there is no element $c > b$ in π between b and a .

It is straightforward to define *place-difference-value partially order patterns*, *PDVPOPs*, or *place-difference-value barred patterns*, *PDVBP*, since our place, difference, and value restrictions just limit where and what values are required for a pattern match. In particular, formula (1) holds for PDVPOPs. We shall not pursue the study of PDVPOPs or PDVBPs in this paper. Instead, we shall leave it as a topic for further research.

Finally, we should observe that our definition of PDVP's does not cover the most general types of restrictions on patterns that one can consider. For example, one can easily imagine cases where there are restrictions on the values in occurrences of patterns that are a function of the places occupied by the occurrence or there are restrictions on the places which an occurrence occupied that are functions of the values in the occurrence. Thus the most general type of restriction for a pattern $p \in S_m$ would be to just give a set \mathcal{S} of $2m$ -tuples $(x_1, \dots, x_m : y_1, \dots, y_m)$ where $1 \leq x_1 < \dots < x_m$ and where y_1, \dots, y_m is order isomorphic to p . In such a situation, we can say (p, \mathcal{S}) occurs in a permutation $\pi = \pi_1 \dots \pi_n$ if and only if there is a $2m$ -tuple $(x_1, \dots, x_m : y_1, \dots, y_m) \in \mathcal{S}$ such that $\pi_{x_i} = y_i$ for $i = 1, \dots, m$. While this is the most general type of pattern condition that we can think of, in most cases this would be a very cumbersome notation. Our definition of PDVP's was motivated by our attempts to cover all the different types of pattern matching conditions that have appeared in the literature that still allows for a relatively compact notation.

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