THE EULER SERIES TRANSFORMATION AND THE BINOMIAL IDENTITIES OF LJUNGGREN, MUNARINI AND SIMONS

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Abstract

We point out that the curious identity of Simons follows immediately from Euler's series transformation formula and also from an identity due to Ljunggren. We also mention its relation to Legendre's polynomials. At the end we use the generalized Euler series transformation to obtain two recent binomial identities of Munarini.

1. Introduction

In 2001 Simons [11] presented the identity

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}(x+1)^k, \qquad (1)$$

which has the interesting property that the binomial coefficients on both sides are the same. This identity has attracted a lot of attention. Independent proofs and extensions have appeared in [1], [6], [7], [8], [10]. Munarini [6], for instance, presented two very interesting extensions. In this article we have included four notes: first, the Simons identity follows naturally from Euler's series transformation formula; second, it also follows from an older identity of Ljunggren; third, it is equivalent to the odd/even property of the Legendre polynomials; finally, in Section 5 we show that the generalized Euler series transformation implies Munarini's identities.

The Simons identity can be viewed as a binomial transform. Given a sequence $\{a_k\}$, its *binomial transform* $\{b_k\}$ is the sequence defined by

$$b_m = \sum_{k=0}^{m}\binom{m}{k}a_k, \quad \text{with inversion} \quad a_m = \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}b_k. \qquad (2)$$

We notice that the right-hand side of (1) is a particular value of the binomial transform of the sequence

$$a_k = \binom{n+k}{k} x^k. \tag{3}$$

For the computation of binomial transforms a very efficient tool is Euler's formula

$$\frac{1}{1-t} f\left(\frac{t}{1-t}\right) = \sum_{m=0}^{\infty} t^m b_m = \sum_{m=0}^{\infty} t^m \left\{ \sum_{k=0}^{m} \binom{m}{k} a_k \right\}, \tag{4}$$

which holds for any function

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \tag{5}$$

analytic about $t = 0$. Instead of functions we can consider formal power series. In the next section we shall derive identity (1) from (4).

2. A Simple Proof of Simons' Identity

Lemma 1 *For $f(t)$ as in (5) and for any α we have*

$$(1 + zt)^\alpha f(t) = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{\alpha}{n-k} a_k z^{n-k} \right\} t^n, \tag{6}$$

where z is an appropriate small parameter assuring convergence.

Proof. The proof is easy. Expand $(1 + zt)^\alpha$ and change the order of summation (or use Cauchy's rule for multiplication of power series); details are left to the reader.   □

Using Euler's transform we shall prove now an extension to (1).

Proposition 2 *For any x, α and any positive integer n,*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{\alpha+k}{k} x^k = \sum_{k=0}^{n} \binom{\alpha}{n-k}\binom{\alpha+k}{k} (-1)^{n-k} (x+1)^k. \tag{7}$$

When $\alpha = n$, this is identity (1).

Proof. We shall compute the binomial transform of the sequence $a_k = \binom{\alpha+k}{k}x^k$. For this purpose we consider the function

$$f(t) = \sum_{k=0}^{\infty} \binom{\alpha+k}{k}x^k t^k = \frac{1}{(1-xt)^{\alpha+1}}. \tag{8}$$

Then

$$
\begin{aligned}
\frac{1}{1-t}f\left(\frac{t}{1-t}\right) &= \frac{(1-t)^{\alpha}}{(1-(x+1)t)^{\alpha+1}} = (1-t)^{\alpha}\sum_{k=0}^{\infty}\binom{\alpha+k}{k}(x+1)^k t^k \\
&= \sum_{m=0}^{\infty} t^m \left\{ \sum_{k=0}^{m}\binom{\alpha}{m-k}\binom{\alpha+k}{k}(-1)^{m-k}(x+1)^k \right\} \tag{9}
\end{aligned}
$$

(by using the lemma). Now (7) follows from (4). □

Identity (7) was proved by Munarini [6] by using Cauchy's integral formula. Munarini, in fact, obtained the more general identity

$$\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\beta+k}{k}x^k y^{n-k} = \sum_{k=0}^{n}\binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k}(-y)^{n-k}(x+y)^k. \tag{10}$$

We shall prove this identity in Section 5.

3. Ljunggren's Identity

In 1947 Wilhelm Ljunggren [5] published the identity

$$\sum_{k=0}^{n}\binom{n}{k}\binom{q}{k}x^{n-k}y^k = \sum_{k=0}^{n}\binom{n}{k}\binom{q+k}{k}(x-y)^{n-k}y^k \tag{11}$$

(true for any two positive integers $q \geq n$), which remained relatively unknown, as the paper was written in Norwegian and its main topic was different. The identity, however, is listed in the table [2] as number (3.18). The proof is short and elegant – Ljunggren evaluated in two different ways the coefficients in the expansion on t of the expression

$$(xt+y)^n(1+t)^q. \tag{12}$$

The method and the identity were recently rediscovered by Pohoaţă [7].

Setting $q = n$, $y = x + 1$ in (11) we find

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{n-k}(x+1)^k = \sum_{k=0}^{n} \binom{n}{k}^2 x^{n-k}(x+1)^k. \tag{13}$$

Also, with $q = n$, $x = y + 1$, $y = x$ (11) turns into

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}x^k = \sum_{k=0}^{n} \binom{n}{k}^2 x^{n-k}(x+1)^k. \tag{14}$$

Now (13) and (14) together imply (1). Therefore, the Simons' identity follows from Ljunggren's identity.

After the renaming $x - y \to y$, $y \to x$, identity (11) can also be put in the form

$$\sum_{k=0}^{n} \binom{n}{k}\binom{q+k}{k}x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{q}{k}(x+y)^{n-k}x^k, \tag{15}$$

or, setting now $z = x/y$ we can write (11) with one variable

$$\sum_{k=0}^{n} \binom{n}{k}\binom{q+k}{k}z^k = \sum_{k=0}^{n} \binom{n}{k}\binom{q}{k}(z+1)^{n-k}z^k. \tag{16}$$

Comparing this to (7) we conclude that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{q}{k}(x+1)^{n-k}x^k = \sum_{k=0}^{n} \binom{q}{n-k}\binom{q+k}{k}(-1)^{n-k}(x+1)^k. \tag{17}$$

In fact, identity (17) is true for arbitrary q. This can be proved by using Euler's formula for the sequence

$$a_k = \binom{q}{k}\left(\frac{x}{x+1}\right)^k, \tag{18}$$

with arbitrary q.

Identity (15) was proved independently by Munarini [6]. Christian Krattenthaler indicated to the author that Ljunggren's identity results also from Pfaff's transformation property [12, p. 110] of the hypergeometric function $_2F_1$.

4. The Legendre Polynomials

Defined by the Rodrigues formula, the Legendre polynomials are

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx}\right)^n (x^2 - 1)^n. \tag{19}$$

Gould's table [2] includes the following representation, entry (3.135),

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k, \tag{20}$$

and therefore, identity (1) says that

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\left(\frac{x+1}{2}\right)^k, \tag{21}$$

which expresses the known even/odd property $P_n(-x) = (-1)^n P_n(x)$ of the Legendre polynomials.

On its part, identity (14) yields

$$P_n(x) = \left(\frac{x-1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x+1}{x-1}\right)^k \tag{22}$$

which is also in Gould's table, entry (3.134).

5. The Generalized Euler Series Transformation

It is worth noting that formula (4) extends to the following (cf. [4, p. 178], [9]).

Proposition 3 *For every* α, x *and* $f(t)$ *as in* (5) *we have*

$$\frac{1}{(1-xt)^{\alpha+1}}f\left(\frac{t}{1-xt}\right) = \sum_{n=0}^{\infty} t^n \left\{\sum_{k=0}^{n} \binom{\alpha+n}{n-k}x^{n-k}a_k\right\}. \tag{23}$$

Proof. Starting from the left side we have

$$\frac{1}{(1-xt)^{\alpha+1}}\sum_{k=0}^{\infty} a_k \left(\frac{t}{1-xt}\right)^k = \sum_{k=0}^{\infty} a_k \frac{t^k}{(1-xt)^{\alpha+k+1}}$$

$$= \sum_{k=0}^{\infty} t^k a_k (1-xt)^{-\alpha-k-1}$$

$$= \sum_{k=0}^{\infty} t^k a_k \sum_{m=0}^{\infty} \binom{\alpha+k+m}{m} x^m t^m = \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} \binom{\alpha+n}{n-k} x^{n-k} a_k \right\}, \qquad (24)$$

by changing the order of summation and setting $n = m + k$.

This is a powerful formula. For example, it can be used to prove Munarini's identity (10).

Replacing α by $\beta - \alpha$ and x by $-y$ in (23) we have

$$\frac{1}{(1+yt)^{\beta-\alpha+1}} f\left(\frac{t}{1+yt}\right) = \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} \binom{\beta-\alpha+n}{n-k} (-y)^{n-k} a_k \right\}. \qquad (25)$$

Now take the function

$$f(t) = \sum_{k=0}^{\infty} \binom{\beta+k}{k} (x+y)^k t^k = \frac{1}{(1-(x+y)t)^{\beta+1}}. \qquad (26)$$

Simple algebra gives

$$\frac{1}{(1+yt)^{\beta-\alpha+1}} f\left(\frac{t}{1+yt}\right) \quad = \quad \frac{(1+yt)^\alpha}{(1-xt)^{\beta+1}} \quad = \quad (1+yt)^\alpha \sum_{k=0}^{\infty} \binom{\beta+k}{k} x^k t^k$$

$$= \quad \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} \binom{\alpha}{n-k} \binom{\beta+k}{k} x^k y^{n-k} \right\} \qquad (27)$$

by Lemma 1. Identity (10) follows now from (25) and (27).

If we apply the generalized Euler formula (23) in the form

$$\frac{1}{(1-xt)^{\beta-\alpha+1}} f\left(\frac{t}{1-xt}\right) = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{\beta-\alpha+n}{n-k} x^{n-k} a_k \right\} t^n \qquad (28)$$

to the function

$$f(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (x+y)^k t^k = (1 + (x+y)t)^\alpha, \qquad (29)$$

in the same way we obtain another interesting identity of Munarini [6]

$$\sum_{k=0}^{n} \binom{\alpha}{n-k} \binom{\beta+k}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{\beta-\alpha+n}{n-k} \binom{\alpha}{k} x^{n-k} (x+y)^k, \qquad (30)$$

as the left side in (28) becomes $\dfrac{(1+yt)^\alpha}{(1-xt)^{\beta+1}}$, the same as in (27). $\qquad \square$

Acknowledgement

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