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NEW PROOFS FOR THE P, Q-ANALOGUE OF CHU-VANDERMONDE'S IDENTITY

George Teşeleanu
Department of Mathematics, Bucharest University, Romania
george.teseleanu@yahoo.com

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Abstract

In this paper we provide five new proofs for the p,q-analogue of Chu-Vandermonde's identity. The presented proofs can be considered as generalizations of the q-version and $q = 1$-version.

1. Introduction

The binomial coefficients are an important tool in combinatorics and they are denoted by $\binom{n}{k}$. They naturally occur as the coefficients in the binomial expansion of $(x+y)^n$; that is,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

If we set $y = 1$ the above relation is called the horizontal generating function for the binomial coefficients. Their explicit value is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

if $0 \leq k \leq n$ and zero otherwise. Some of the most important relations satisfied by the binomial coefficients are the triangular recurrence relation or Pascal's identity:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},$$

and Chu-Vandermonde's identity:

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{k-j} \binom{n}{j}. \tag{1}$$

The relations for the binomial coefficient and other identities can be found for example in [4].

The q-binomial coefficients are a generalization of the binomial coefficients introduced by C.F. Gauss. Choosing particular values for the integer parameters, they evaluate to polynomials in $\mathbb{Z}[q]$. One of their natural occurrences is as a counting tool for the number of k-dimensional subspaces of an n-dimensional vector space over a finite field of order q. Their explicit values are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where we use the notation $[k]_q = 1 + q + q^2 + \ldots + q^{k-1}$ and where $[k]_q!$ denotes the q-factorial which is given by $[k]_q! = [k]_q[k-1]_q \ldots [2]_q[1]_q$. It can easily be verified that, when $q = 1$, the q-binomial coefficients turn into the usual binomial coefficients. Unlike the usual binomial coefficients, the q-binomial coefficients have a limited array of identities. The most basic ones are the triangular recurrence relations:

$$\begin{aligned} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q &= q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q, \\ \begin{bmatrix} n+1 \\ k \end{bmatrix}_q &= \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q, \end{aligned}$$

and the q-analogues of Chu-Vandermonde's identity:

$$\begin{bmatrix} m+n \\ k \end{bmatrix}_q = \sum_{j=0}^{k} q^{j(m-k+j)} \begin{bmatrix} m \\ k-j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q, \tag{2}$$

$$\begin{bmatrix} m+n \\ k \end{bmatrix}_q = \sum_{j=0}^{k} q^{(k-j)(n-j)} \begin{bmatrix} m \\ k-j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q. \tag{3}$$

We also have a generalization of the horizontal generating function that is satisfied by the q-binomial coefficient:

$$\begin{aligned} \prod_{i=0}^{n-1} (q^i + x) &= \sum_{k=0}^{n} q^{(n-k-1)(n-k)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k, \\ \prod_{i=0}^{n-1} (1 + q^i x) &= \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k. \end{aligned}$$

The relations for the q-binomial coefficient and other identities can be found for example in [3] or [5].

The p,q-binomial coefficients are a generalization of the q-binomial coefficient introduced by Robert B. Corcino in [1]. Choosing particular values for the integer parameters they evaluate to polynomials in $\mathbb{Z}[p,q]$. Their explicit value is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!},$$

where $[k]_{p,q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + \ldots + q^{k-1}$, and $[k]_{p,q}!$ denotes the p,q-factorial which is given by $[k]_{p,q}! = [k]_{p,q}[k-1]_{p,q} \ldots [2]_{p,q}[1]_{p,q}$. It can easily be verified that, when $p = 1$, the p,q-binomial coefficients turn into the q-binomial coefficients. It can be noticed that the p,q-binomial coefficients is symmetrical, i.e. $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ k \end{bmatrix}_{q,p}$. Because they are symmetrical, through the rest of the paper, we will write and prove only one variant of a relation. Some of the identities satisfied by the p,q-binomial coefficients are the triangular recurrence relation:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = q^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}, \tag{4}$$

Chu-Vandermonde's identity:

$$\begin{bmatrix} m+n \\ k \end{bmatrix}_{p,q} = \sum_{j=0}^{k} p^{(k-j)(n-j)} q^{j(m-k+j)} \begin{bmatrix} m \\ k-j \end{bmatrix}_{p,q} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q}, \tag{5}$$

and the horizontal generating function:

$$\prod_{i=0}^{n-1} (q^i + p^i x) = \sum_{k=0}^{n} q^{(n-k-1)(n-k)/2} p^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k. \tag{6}$$

We may also see that the following basic identity occurs:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q}. \tag{7}$$

These relations and more identities for the p,q-binomial coefficients can be found in [1] and [7]. Also in [7] there is a combinatorial interpretation of the p,q-binomial coefficients.

The aim of this paper is to provide new proofs for Chu-Vandermonde's identity (5). The identity is already proven in [7] as Identity 3.6 and in [2] as Proposition 4. In the first proof we homogenize the q-variant of the Chu-Vandermonde relation (2). In the second and third proof we generalize some proofs for the classical Chu-Vandermonde relation (1); specifically, in the second proof we make repeated use of Pascal's identity (4), an idea found for the classical relation (1) in [6] as Property 10.7, and for the third proof we use a generalization of the generating horizontal relation (6), an idea for (1) found in [4, Chapter 5.4]. In the fourth proof we use a generalization of quantum calculus ideas found in [5] as Propositions IV.2.2 and IV.2.3. In the last proof we use induction to prove (1) directly.

Through the rest of the paper we will use the following conventions: $\begin{bmatrix} n \\ 0 \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ n \end{bmatrix}_{p,q} = 1$; $\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} = 0$, for $j > n$ or $j < 0$. We will also use the notations LHS and RHS for the left-hand side, respectively right-hand side of an equality.

2. Chu-Vandermonde's Identity

Theorem 2.1. (Chu-Vandermonde identity) *The p,q-binomial coefficients satisfy the identity:*

$$\begin{bmatrix} m+n \\ k \end{bmatrix}_{p,q} = \sum_{j=0}^{k} p^{(k-j)(n-j)} q^{j(m-k+j)} \begin{bmatrix} m \\ k-j \end{bmatrix}_{p,q} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q}.$$

Proof 1. For this proof of (5) we will work in the rational function field $\mathbb{Q}(p,q)$. We will use the Chu-Vandermonde identity (2) for r-binomial coefficients with the parameter $r = \frac{p}{q}$ to prove our identity. Due to our choice of parameter we have the following relation:

$$[k]_r = \frac{r^k - 1}{r - 1} = \frac{\left(\frac{p}{q}\right)^k - 1}{\frac{p}{q} - 1} = q^{-(k-1)} \cdot \frac{p^k - q^k}{p - q} = q^{-(k-1)} \cdot [k]_{p,q}. \tag{8}$$

Expanding the Chu-Vandermonde's identity (2) for r-binomial coefficients,

$$\begin{bmatrix} m+n \\ k \end{bmatrix}_r = \sum_{j=0}^{k} r^{j(m-k+j)} \begin{bmatrix} m \\ k-j \end{bmatrix}_r \begin{bmatrix} n \\ j \end{bmatrix}_r,$$

we obtain:

$$\frac{[m+n]_r!}{[k]_r![m+n-k]_r!} = \sum_{j=0}^{k} r^{j(m-k+j)} \frac{[m]_r!}{[k-j]_r![m-k+j]_r!} \, \frac{[n]_r!}{[j]_r![n-j]_r!}. \tag{9}$$

Now making use of (8), relation (9) becomes:

$$\begin{aligned} LHS &= \frac{q^{1+2+\ldots+(k-1)} \cdot q^{1+2+\ldots+(m+n-k-1)}}{q^{1+2+\ldots+(m+n-1)}} \, \frac{[m+n]_{p,q}!}{[k]_{p,q}![m+n-k]_{p,q}!} \\ &= q^{[k(k-1)+(m+n-k)(m+n-k-1)-(m+n)(m+n-1)]/2} \, \frac{[m+n]_{p,q}!}{[k]_{p,q}![m+n-k]_{p,q}!}, \end{aligned}$$

$$\begin{aligned} RHS &= \sum_{j=0}^{k} \frac{q^{1+2+\ldots+(k-j-1)} \cdot q^{1+2+\ldots+(m-k+j-1)}}{q^{1+2+\ldots+(m-1)}} \frac{[m]_{p,q}!}{[k-j]_{p,q}![m-k+j]_{p,q}!} \\ &\quad \frac{q^{1+2+\ldots+(j-1)} \cdot q^{1+2+\ldots+(n-j-1)}}{q^{1+2+\ldots+(n-1)}} \frac{[n]_{p,q}!}{[j]_{p,q}![n-j]_{p,q}!} \left(\frac{p}{q}\right)^{j(m-k+j)} \\ &= \sum_{j=0}^{k} p^{j(m-k+j)} \cdot q^{A} \frac{[m]_{p,q}!}{[k-j]_{p,q}![m-k+j]_{p,q}!} \frac{[n]_{p,q}!}{[j]_{p,q}![n-j]_{p,q}!}, \end{aligned}$$

where $A = [(k-j)(k-j-1)+(m-k+j)(m-k+j-1)-m(m-1)+j(j-1)+(n-j)(n-j-1)-n(n-1)]/2 - j(m-k+j)$. By multiplying the LHS and RHS with $q^{-[k(k-1)+(m+n-k)(m+n-k-1)-(m+n)(m+n-1)]/2}$, we obtain identity (5). □

Proof 2. In this proof we make repeated use of Pascal's identity (4). So, we have:

$$\begin{aligned} \begin{bmatrix} m+n \\ k \end{bmatrix}_{p,q} &= q^{k} \begin{bmatrix} m+n-1 \\ k \end{bmatrix}_{p,q} + p^{n+m-k} \begin{bmatrix} m+n-1 \\ k-1 \end{bmatrix}_{p,q} \\ &= q^{k} \left\{ q^{k} \begin{bmatrix} m+n-2 \\ k \end{bmatrix}_{p,q} + p^{n+m-1-k} \begin{bmatrix} m+n-2 \\ k-1 \end{bmatrix}_{p,q} \right\} \\ &\quad + p^{n+m-k} \left\{ q^{k-1} \begin{bmatrix} m+n-2 \\ k-1 \end{bmatrix}_{p,q} + p^{n+m-k} \begin{bmatrix} m+n-2 \\ k-2 \end{bmatrix}_{p,q} \right\} \\ &= q^{2k} \begin{bmatrix} m+n-2 \\ k \end{bmatrix}_{p,q} + p^{n+m-k-1} q^{k-1} [2]_{p,q} \begin{bmatrix} m+n-2 \\ k-1 \end{bmatrix}_{p,q} \\ &\quad + p^{2(n+m-k)} \begin{bmatrix} m+n-2 \\ k-2 \end{bmatrix}_{p,q} \\ &= q^{3k} \begin{bmatrix} m+n-3 \\ k \end{bmatrix}_{p,q} + p^{n+m-k-2} q^{2(k-1)} [3]_{p,q} \begin{bmatrix} m+n-3 \\ k-1 \end{bmatrix}_{p,q} \\ &\quad + p^{2(n+m-k-1)} q^{k-2} [3]_{p,q} \begin{bmatrix} m+n-3 \\ k-2 \end{bmatrix}_{p,q} + p^{3(n+m-k)} \begin{bmatrix} m+n-3 \\ k-3 \end{bmatrix}_{p,q} \\ &= q^{4k} \begin{bmatrix} m+n-4 \\ k \end{bmatrix}_{p,q} + p^{n+m-k-3} q^{3(k-1)} [4]_{p,q} \begin{bmatrix} m+n-4 \\ k-1 \end{bmatrix}_{p,q} \\ &\quad + p^{2(n+m-k-2)} q^{2(k-2)} \frac{[3]_{p,q} \cdot [4]_{p,q}}{[2]_{p,q}} \begin{bmatrix} m+n-4 \\ k-2 \end{bmatrix}_{p,q} \\ &\quad + p^{3(n+m-k-1)} q^{k-3} [4]_{p,q} \begin{bmatrix} m+n-4 \\ k-3 \end{bmatrix}_{p,q} + p^{4(n+m-k)} \begin{bmatrix} m+n-4 \\ k-4 \end{bmatrix}_{p,q} \\ &= \ldots \\ &= q^{mk} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \begin{bmatrix} m \\ 0 \end{bmatrix}_{p,q} + p^{n-k-1} q^{(m-1)(k-1)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \begin{bmatrix} m \\ 1 \end{bmatrix}_{p,q} \end{aligned}$$

$$+ \ldots + p^{(k-1)(n-1)} q^{m-k+1} \begin{bmatrix} n \\ 1 \end{bmatrix}_{p,q} \begin{bmatrix} m \\ k-1 \end{bmatrix}_{p,q} + p^{kn} \begin{bmatrix} n \\ 0 \end{bmatrix}_{p,q} \begin{bmatrix} m \\ k \end{bmatrix}_{p,q}.$$

Taking the first and last term of the equalities we obtain identity (5). □

For the next proof of Theorem 2.1 we utilize the following proposition.

Proposition 2.1. *The p,q-binomial coefficients satisfy the following identity:*

$$\prod_{i=0}^{n-1} (q^{a+i} + p^i x) = \sum_{k=0}^{n} q^{(n-k-1+2a)(n-k)/2} p^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k, \quad (10)$$

for any nonnegative integer a.

Remark. When $a = 0$, relation (10) becomes relation (6) that can be found in [1] as Theorem 3 and in [7] as Identity 3.3.

Proof. We will make use of induction and Pascal's identity (4) to prove this identity. When $n = 1$ we have $q^a + x = q^a \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{p,q} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{p,q} x$, which is true. When $n = 2$ we have $(q^a + x)(q^{a+1} + px) = q^{2a+1} + q^a(p+q)x + px^2 = q^{2a+1} \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{p,q} + q^a \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q} x + p \begin{bmatrix} 2 \\ 2 \end{bmatrix}_{p,q} x^2$, which is true. Now we assume that the identity holds for n and we prove the identity for $n + 1$. So we have:

$$\begin{aligned}
\prod_{i=0}^{n} (q^{a+i} + p^i x) &= \prod_{i=0}^{n-1} (q^{a+i} + p^i x) \cdot (q^{a+n} + p^n x) \\
&= \Big[\sum_{k=0}^{n} q^{(n-k-1+2a)(n-k)/2} p^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k \Big] (q^{a+n} + p^n x) \\
&= \sum_{k=0}^{n} q^{((n-k-1+2a)(n-k)/2)+a+n} p^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k \\
&+ \sum_{k=0}^{n} q^{(n-k-1+2a)(n-k)/2} p^{(k(k-1)/2)+n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^{k+1}.
\end{aligned}$$

We observe that the coefficients $q^{(n+2a)(n+1)/2}$ of x^0 and $p^{n(n+1)/2}$ of x^{n+1}, correspond with the ones in the formula (10). Now let us check the other coefficients. Let us take a j, $1 \leq j \leq n$, and calculate the coefficient of x^j:

$$\begin{aligned} C &= q^{((n-j-1+2a)(n-j)/2)+a+n} p^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \\ &\quad + q^{(n-j+2a)(n-j+1)/2} p^{((j-1)(j-2)/2)+n} \begin{bmatrix} n \\ j-1 \end{bmatrix}_{p,q} \\ &= q^{(n-j+2a)(n-j+1)/2} p^{(j(j-1)/2)} \Big(q^j \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} + p^{n-(j-1)} \begin{bmatrix} n \\ j-1 \end{bmatrix}_{p,q} \Big) \\ &= q^{(n-j+2a)(n-j+1)/2} p^{(j(j-1)/2)} \begin{bmatrix} n+1 \\ j \end{bmatrix}_{p,q}, \end{aligned}$$

where for the last equality we used Pascal's identity (4). So we have proved identity (10). □

Proof 3. In this proof we make use of Proposition 2.1 for $a = 0$ and $a = n$. So, we have:

$$\prod_{k=0}^{n+m-1} (q^k + p^k x) = \prod_{i=0}^{n-1} (q^i + p^i x) \cdot \prod_{j=0}^{m-1} .(q^{n+j} + p^j (p^n x)).$$

Now using Proposition 2.1 we have:

$$\begin{aligned} LHS &= \sum_{k=0}^{n+m} q^{(n+m-k-1)(n+m-k)/2} p^{k(k-1)/2} \begin{bmatrix} n+m \\ k \end{bmatrix}_{p,q} x^k, \\ RHS &= \Big(\sum_{i=0}^{n} q^{(n-i-1)(n-i)/2} p^{i(i-1)/2} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} x^i \Big) \\ &\quad \cdot \Big(\sum_{j=0}^{m} q^{(m-j-1+2n)(m-j)/2} p^{[j(j-1)/2]+nj} \begin{bmatrix} m \\ j \end{bmatrix}_{p,q} x^j \Big) \\ &= \sum_{i,j=0}^{m} q^B p^{[i(i-1)+j(j-1)+2nj]/2} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} \begin{bmatrix} m \\ j \end{bmatrix}_{p,q} x^{i+j}, \end{aligned}$$

where $B = [(n-i-1)(n-i) + (m-j-1+2n)(m-j)]/2$. Now we identify the coefficient of x^k, $0 \le k \le n+m$, on both sides and we obtain:

$$q^{(n+m-k-1)(n+m-k)/2} p^{k(k-1)/2} \begin{bmatrix} n+m \\ k \end{bmatrix}_{p,q} = \sum_{i=0, j=k-i}^{n} q^V p^W \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} \begin{bmatrix} m \\ k-i \end{bmatrix}_{p,q}, \quad (11)$$

where $V = [(n-i-1)(n-i) + (m-k+i-1+2n)(m-k+i)]/2$ and $W = [i(i-1) + (k-i)(k-i-1) + 2n(k-i)]/2$. Now by multiplying relation (11) by $q^{-(n+m-k-1)(n+m-k)/2} p^{-k(k-1)/2}$, we obtain identity (5). □

For the fourth proof of Theorem 2.1. we need Proposition 2.2. To prepare the stage, the following remark is crucial.

Remark. For Proposition 2.2 and Proof 4 we will work in the quotient algebra $R = k\{X,Y\}/I_{p,q}$, where k is a ring, $k\{X,Y\}$ is the free algebra of two indeterminates and $I_{p,q}$ is the two-sided ideal of the free algebra generated by $qYX - pXY,\ p,q \in k$. Clearly $qyx = pxy$, where x and y are the equivalence classes of X and Y in R.

Proposition 2.2. *The p,q-binomial coefficients satisfy the following identity:*

$$q^{n(n-1)/2}(x+y)^n = \sum_{j=0}^{n} q^{[j(j-1)+(n-j)(n-j-1)]/2} \begin{bmatrix} n \\ n-j \end{bmatrix}_{p,q} x^{n-j}y^j. \tag{12}$$

Proof. We will prove the above relation using induction on n and Pascal's identity (4). For $n = 2$ we check that

$$\begin{aligned} q(x+y)^2 &= qx^2 + qxy + qyx + qy^2 = q(x^2+y^2) + (q+p)xy \\ &= q\begin{bmatrix} 2 \\ 0 \end{bmatrix}_{p,q} x^2 + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q} xy + q\begin{bmatrix} 2 \\ 2 \end{bmatrix}_{p,q} y^2. \end{aligned}$$

Now we assume that the identity holds for n and we prove the identity for $n+1$. So we have:

$$\begin{aligned} q^{n(n+1)/2}(x+y)^{n+1} &= q^n(x+y)q^{n(n-1)/2}(x+y)^n \\ &= q^n(x+y)\Bigg(q^{n(n-1)/2}(x^n+y^n) \\ &\quad + \sum_{j=1}^{n-1} q^{[j(j-1)+(n-j)(n-j-1)]/2} \begin{bmatrix} n \\ n-j \end{bmatrix}_{p,q} x^{n-j}y^j\Bigg) \\ &= q^{n(n+1)/2}(x^n+y^n) + q^{n(n+1)/2}xy^n + q^{n(n+1)/2}yx^n \\ &\quad + \sum_{j=1}^{n-1} q^{[j(j-1)+(n-j)(n-j-1)+2n]/2} \begin{bmatrix} n \\ n-j \end{bmatrix}_{p,q} x^{n-j+1}y^j \\ &\quad + \sum_{j=1}^{n-1} q^{[j(j-1)+(n-j)(n-j-1)+2n]/2} \begin{bmatrix} n \\ n-j \end{bmatrix}_{p,q} yx^{n-j}y^j. \end{aligned}$$

Now we check the coefficient of xy^n:

$$\begin{aligned} &q^{n(n+1)/2}xy^n + q^{[(n-1)(n-2)+2n]/2}yxy^{n-1}\begin{bmatrix} n \\ 1 \end{bmatrix}_{p,q} \\ &\quad = \left(q^{n(n+1)/2} + pq^{[(n-1)(n-2)+2(n-1)]/2}\begin{bmatrix} n \\ 1 \end{bmatrix}_{p,q}\right)xy^n \end{aligned}$$

$$= q^{n(n-1)/2}\{q^n + p(q^{n-1} + q^{n-2}p + \ldots + p^{n-1})\}xy^n$$
$$= q^{n(n-1)/2}xy^n \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{p,q},$$

and we see that it is the right coefficient. Similarly for $x^n y$.

We group the terms with similar powers and we see that the coefficient for $x^{n-j}y^{j+1}$, where $1 \le j \le n-2$, is:

$$q^{[j(j+1)+(n-j-1)(n-j-2)+2n]/2}x^{n-j}y^{j+1}\begin{bmatrix} n \\ n-j-1 \end{bmatrix}_{p,q}$$
$$+q^{[j(j-1)+(n-j)(n-j-1)+2n]/2}yx^{n-j}y^j\begin{bmatrix} n \\ n-j \end{bmatrix}_{p,q}$$
$$= q^{[j(j+1)+(n-j)(n-j-1)]/2}\left(q^{j+1}\begin{bmatrix} n \\ n-j-1 \end{bmatrix}_{p,q} + p^{n-j}\begin{bmatrix} n \\ n-j \end{bmatrix}_{p,q}\right)x^{n-j}y^{j+1}$$
$$= q^{[j(j+1)+(n-j)(n-j-1)]/2}x^{n-j}y^{j+1}\begin{bmatrix} n+1 \\ n-j \end{bmatrix}_{p,q},$$

where for the last equality we used Pascal's identity (4). We observe that for this j we end up with the right coefficient and this ends our proof. □

Proof 4. For this proof we use Proposition 2.2. In the ring R we have:

$$q^{(n+m)(n+m-1)/2}(x+y)^{n+m} = q^{nm}\cdot q^{n(n-1)/2}(x+y)^n \cdot q^{m(m-1)/2}(x+y)^m.$$

We expand both sides using Proposition 2.2.,

$$\begin{aligned}
LHS &= \sum_{i=0}^{n+m} q^{[i(i-1)+(n+m-i)(n+m-i-1)]/2}x^{n+m-i}y^i\begin{bmatrix} n+m \\ n+m-i \end{bmatrix}_{p,q},\\
RHS &= q^{nm}\left(\sum_{j=0}^{n} q^{[j(j-1)+(n-j)(n-j-1)]/2}x^{n-j}y^j\begin{bmatrix} n \\ n-j \end{bmatrix}_{p,q}\right)\\
&\quad\cdot\left(\sum_{k=0}^{m} q^{[k(k-1)+(m-k)(m-k-1)]/2}x^{m-k}y^k\begin{bmatrix} m \\ m-k \end{bmatrix}_{p,q}\right)\\
&= \sum_{j=0}^{n+m}\sum_{k=0}^{n+m} q^Z\begin{bmatrix} n \\ n-j \end{bmatrix}_{p,q}\begin{bmatrix} m \\ m-k \end{bmatrix}_{p,q} x^{n-j}y^jx^{m-k}y^k\\
&= \sum_{j=0}^{n+m}\sum_{k=0}^{n+m} q^T p^{j(m-k)}\begin{bmatrix} n \\ n-j \end{bmatrix}_{p,q}\begin{bmatrix} m \\ m-k \end{bmatrix}_{p,q} x^{(n-j)+(m-k)}y^{j+k},
\end{aligned}$$

where $Z = [j(j-1) + (n-j)(n-j-1) + k(k-1) + (m-k)(m-k-1)]/2$ and $T = Z - j(m-k)$. For a given i, the coefficient of $x^{n+m-i}y^i$ in LHS is:

$$C1 = q^{[i(i-1)+(n+m-i)(n+m-i-1)]/2} \begin{bmatrix} n+m \\ n+m-i \end{bmatrix}_{p,q},$$

and in RHS is:

$$C2 = \sum_{j=0,k=i-j}^{n} q^{\alpha} p^{j(m-i+j)} \begin{bmatrix} n \\ n-j \end{bmatrix}_{p,q} \begin{bmatrix} m \\ m-i+j \end{bmatrix}_{p,q},$$

where $2\alpha = j(j-1) + (n-j)(n-j-1) + (i-j)(i-j-1) + (m-i-j)(m-i-j-1) - j(m-i-j)$. Now reducing $C1$ and $C2$ with $q^{[i(i-1)+(n+m-i)(n+m-i-1)]/2}$ we obtain:

$$\begin{bmatrix} n+m \\ n+m-i \end{bmatrix}_{p,q} = \sum_{j=0}^{n} q^{(i-j)(n-j)} p^{j(m-i+j)} \begin{bmatrix} n \\ n-j \end{bmatrix}_{p,q} \begin{bmatrix} m \\ m-i+j \end{bmatrix}_{p,q}.$$

Using identity (7), the conclusion follows. □

Proof 5. In our last proof we make use of induction on n and Pascal's identity (4). When $n = 1$, we have:

$$\begin{aligned} \begin{bmatrix} m+1 \\ k \end{bmatrix}_{p,q} &= p^k \begin{bmatrix} m \\ k \end{bmatrix}_{p,q} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{p,q} + q^{m-k+1} \begin{bmatrix} m \\ k-1 \end{bmatrix}_{p,q} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{p,q} \\ &= p^k \begin{bmatrix} m \\ k \end{bmatrix}_{p,q} + q^{m-k+1} \begin{bmatrix} m \\ k-1 \end{bmatrix}_{p,q}, \end{aligned}$$

which is exactly Pascal's identity (4). We assume that the identity holds for n and we prove it for $n+1$. Alternatively, using (4) and the induction statement, we have:

$$\begin{aligned} \begin{bmatrix} m+n+1 \\ k \end{bmatrix}_{p,q} &= q^k \begin{bmatrix} m+n \\ k \end{bmatrix}_{p,q} + p^{m+n+1-k} \begin{bmatrix} m+n \\ k-1 \end{bmatrix}_{p,q} \\ &= q^k \sum_{j=0}^{k} p^{(k-j)(n-j)} q^{j(m-k+j)} \begin{bmatrix} m \\ k-j \end{bmatrix}_{p,q} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \\ &\quad + p^{m+n+1-k} \sum_{j=0}^{k-1} p^{(k-1-j)(n-j)} q^{j(m-k+1+j)} \begin{bmatrix} m \\ k-1-j \end{bmatrix}_{p,q} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \\ &= \sum_{j=0}^{k-1} p^{(k-j)(n-j)} q^{j(m-k+1+j)} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \Bigg\{ q^{k-j} \begin{bmatrix} m \\ k-j \end{bmatrix}_{p,q} \end{aligned}$$

$$
\begin{aligned}
& +p^{m-k+j+1}\begin{bmatrix} m \\ k-j-1 \end{bmatrix}_{p,q} \Bigg\} + q^{k(m+1)} \begin{bmatrix} m \\ 0 \end{bmatrix}_{p,q} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \\
= \; & \sum_{j=0}^{k-1} p^{(k-j)(n-j)} q^{j(m-k+1+j)} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} m+1 \\ k-j \end{bmatrix}_{p,q} \\
& +q^{k(m+1)} \begin{bmatrix} m+1 \\ 0 \end{bmatrix}_{p,q} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \\
= \; & \sum_{j=0}^{k} p^{(k-j)(n-j)} q^{j(m+1-k+j)} \begin{bmatrix} m+1 \\ k-j \end{bmatrix}_{p,q} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q},
\end{aligned}
$$

so we end up with identity (5). □

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