MULTI-POLY-BERNOULLI-STAR NUMBERS AND FINITE MULTIPLE ZETA-STAR VALUES

Kohtaro Imatomi
Graduate School of Mathematics, Kyushu University, Nishi-ku, Fukuoka, Japan
k-imatomi@math.kyushu-u.ac.jp

Received: 11/27/13, Revised: 5/3/14, Accepted: 7/22/14, Published: 10/8/14

Abstract
We define the multi-poly-Bernoulli-star numbers which generalize classical Bernoulli numbers. We study the basic properties for these numbers and establish sum formulas and a duality theorem, and discuss a connection to the finite multiple zeta-star values. As an application, we present alternative proofs of some relations on the finite multiple zeta-star values.

1. Introduction

For any multi-index $(k_1, \ldots, k_r)$ with $k_i \in \mathbb{Z}$, we define two kinds of multi-poly-Bernoulli-star numbers $B_{n,\star}^{(k_1,\ldots,k_r)}$, $C_{n,\star}^{(k_1,\ldots,k_r)}$ by the following generating series:

$$\frac{Li^{\star}_{k_1,\ldots,k_r}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_{n,\star}^{(k_1,\ldots,k_r)} \frac{t^n}{n!},$$
$$\frac{Li^{\star}_{k_1,\ldots,k_r}(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_{n,\star}^{(k_1,\ldots,k_r)} \frac{t^n}{n!},$$

where $Li^{\star}_{k_1,\ldots,k_r}(z)$ is the non-strict multiple polylogarithm given by

$$Li^{\star}_{k_1,\ldots,k_r}(z) = \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{z^{m_1}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

When $r = 1$, these numbers are poly-Bernoulli numbers studied in [1], [9]. Further, when $r = 1$ and $k_1 = 1$, both numbers are classical Bernoulli numbers since $Li^{\star}_1(1-e^{-t}) = t$. We note that $B_{n,\star}^{(1)} = C_{n,\star}^{(1)} \quad (n \neq 1)$ with $B_{1,\star}^{(1)} = 1/2$ and $C_{1,\star}^{(1)} = -1/2$. We call $k = k_1 + \cdots + k_r$ the weight of multi-index $(k_1, \ldots, k_r)$.

We set

$$\mathcal{A} := \frac{\prod_p \mathbb{Z}/p\mathbb{Z}}{\bigoplus_p \mathbb{Z}/p\mathbb{Z}} = \{(a_p)_p; a_p \in \mathbb{Z}/p\mathbb{Z}\}/\sim,$$

where $(a_p)_p \sim (b_p)_p$ is equivalent to the equalities $a_p = b_p$ for all but finitely many primes p. The finite multiple zeta-star values are defined by

$$\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_r) := \big(H^{\star}_p(k_1,\dots,k_r) \bmod p\big)_p \in \mathcal{A},$$

where $H^{\star}_n(k_1,\dots,k_r)$ is the non-strict multiple harmonic sum defined by

$$H^{\star}_n(k_1,\dots,k_r) = \sum_{n-1\ge m_1\ge\cdots\ge m_r\ge 1} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}.$$

For more details on the finite multiple zeta(-star) values, we refer the reader to [7], [10] and [12]. We use "star" to indicate that the inequalities in the sum are non-strict in contrast to the strict ones usually adopted in the references above. Each of these is expressed as a linear combination of the other.

This article is organized as follows. In §2, we give fundamental properties for the multi-poly-Bernoulli-star numbers. In §3, we describe the sum formula and the duality relation for the multi-poly-Bernoulli-star numbers. In §4, we study connections between the finite multiple zeta-star values and the multi-poly-Bernoulli-star numbers. As a result, we obtain alternative proofs of some relations for the finite multiple zeta-star values.

2. Basic Properties for the Multi-Poly-Bernoulli-Star Numbers

In this section, we introduce basic results for the multi-poly-Bernoulli-star numbers. We first give the recurrence relations for the multi-poly-Bernoulli-star numbers $B_{n,\star}^{(k_1,\dots,k_r)}$, $C_{n,\star}^{(k_1,\dots,k_r)}$. Before stating them, we provide the following identity for the non-strict multiple polylogarithm, whose proof is straightforward and is omitted.

Lemma 2.1. *For any multi-index $(k_1,\dots,k_r)$ with $k_i \in \mathbb{Z}$, we have*

$$\frac{d}{dt}Li^{\star}_{k_1,\dots,k_r}(t) = \begin{cases} \dfrac{1}{t}Li^{\star}_{k_1-1,k_2,\dots,k_r}(t) & (k_1 \neq 1), \\ \dfrac{1}{t(1-t)}Li^{\star}_{k_2,\dots,k_r}(t) & (k_1 = 1, r \neq 1), \\ \dfrac{1}{1-t} & (k_1 = r = 1). \end{cases}$$

Proposition 2.1. *For any multi-index $(k_1,\dots,k_r)$, we have the following recursions:*

(i) When $k_1 \neq 1$,

$$\begin{aligned} B_{n,\star}^{(k_1,\dots,k_r)} &= \frac{1}{n+1}\left(B_{n,\star}^{(k_1-1,k_2,\dots,k_r)} - \sum_{j=1}^{n-1}\binom{n}{j-1}B_{j,\star}^{(k_1,\dots,k_r)}\right), \\ C_{n,\star}^{(k_1,\dots,k_r)} &= \frac{1}{n+1}\left(C_{n,\star}^{(k_1-1,k_2,\dots,k_r)} - \sum_{j=0}^{n-1}\binom{n+1}{j}C_{j,\star}^{(k_1,\dots,k_r)}\right). \end{aligned}$$

(ii) When $k_1 = 1$,

$$\begin{aligned} B_{n,\star}^{(1,k_2,\dots,k_r)} &= \frac{1}{n+1}\left(\sum_{j=0}^{n}\binom{n}{j}B_{j,\star}^{(k_2,\dots,k_r)} - \sum_{j=1}^{n-1}\binom{n}{j-1}B_{j,\star}^{(1,k_2,\dots,k_r)}\right), \\ C_{n,\star}^{(1,k_2,\dots,k_r)} &= \frac{1}{n+1}\left(C_{n,\star}^{(k_2,\dots,k_r)} - \sum_{j=1}^{n-1}(-1)^{n-j}\binom{n}{j-1}C_{j,\star}^{(1,k_2,\dots,k_r)}\right), \end{aligned}$$

where an empty sum is understood to be 0.

Proof. We prove the relations for $B_{n,\star}^{(k_1,\dots,k_r)}$: those for $C_{n,\star}^{(k_1,\dots,k_r)}$ are similar.

$$Li^{\star}_{k_1,\dots,k_r}(1-e^{-t}) = (1-e^{-t})\sum_{n=0}^{\infty}B_{n,\star}^{(k_1,\dots,k_r)}\frac{t^n}{n!}. \tag{1}$$

We differentiate both sides of (1): When $k_1 \neq 1$, we obtain

$$\begin{aligned} \text{(LHS)} &= \frac{e^{-t}}{1-e^{-t}}Li^{\star}_{k_1-1,k_2,\dots,k_r}(1-e^{-t}), \\ \text{(RHS)} &= e^{-t}\sum_{n=0}^{\infty}B_{n,\star}^{(k_1,\dots,k_r)}\frac{t^n}{n!} + (1-e^{-t})\sum_{n=0}^{\infty}B_{n+1,\star}^{(k_1,\dots,k_r)}\frac{t^n}{n!}. \end{aligned}$$

So we have

$$\begin{aligned} \sum_{n=0}^{\infty}\left(B_{n,\star}^{(k_1-1,k_2,\dots,k_r)} - B_{n,\star}^{(k_1,\dots,k_r)}\right)\frac{t^n}{n!} &= (e^t-1)\sum_{n=0}^{\infty}B_{n+1,\star}^{(k_1,\dots,k_r)}\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\binom{n}{j}B_{j+1,\star}^{(k_1,\dots,k_r)}\frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $t^n/n!$ on both sides, we obtain (i). When $k_1 = 1$,

$$\begin{aligned} \text{(LHS)} &= \frac{1}{1-e^{-t}}Li^{\star}_{k_2,\dots,k_r}(1-e^{-t}), \\ \text{(RHS)} &= \sum_{n=0}^{\infty}B_{n+1,\star}^{(1,k_2,\dots,k_r)}\frac{t^n}{n!} + e^{-t}\sum_{n=0}^{\infty}\left(B_{n,\star}^{(1,k_2,\dots,k_r)} - B_{n+1,\star}^{(1,k_2,\dots,k_r)}\right)\frac{t^n}{n!}. \end{aligned}$$

So we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(B_{n,\star}^{(1,k_2,\ldots,k_r)}-B_{n+1,\star}^{(1,k_2,\ldots,k_r)}\right)\frac{t^n}{n!} &= e^t\sum_{n=0}^{\infty}\left(B_{n,\star}^{(k_2,\ldots,k_r)}-B_{n+1,\star}^{(1,k_2,\ldots,k_r)}\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\left(B_{j,\star}^{(k_2,\ldots,k_r)}-B_{j+1,\star}^{(1,k_2,\ldots,k_r)}\right)\frac{t^n}{n!}\end{aligned}$$

and by this we obtain (ii). □

We proceed to describe explicit formulas for the multi-poly-Bernoulli-star numbers in terms of the Stirling numbers of the second kind. We recall that the Stirling numbers of the second kind are the integers $\left\{ {n \atop m} \right\}$ for all integers m, n satisfying the following recursions and the initial values:

$$\left\{ {n+1 \atop m} \right\} = \left\{ {n \atop m-1} \right\} + m\left\{ {n \atop m} \right\} \quad (\forall n, m \in \mathbb{Z}),$$

$$\left\{ {0 \atop 0} \right\} = 1, \left\{ {n \atop 0} \right\} = \left\{ {0 \atop m} \right\} = 0 \quad (n, m \neq 0).$$

Proposition 2.2. *For any multi-index $(k_1,\ldots,k_r)$, $k_i \in \mathbb{Z}$, we have*

$$B_{n,\star}^{(k_1,\ldots,k_r)} = \sum_{n+1\geq m_1\geq\cdots\geq m_r\geq 1}\frac{(-1)^{m_1+n-1}(m_1-1)!}{m_1^{k_1}\cdots m_r^{k_r}}\left\{ {n \atop m_1-1} \right\}$$

and

$$C_{n,\star}^{(k_1,\ldots,k_r)} = \sum_{n+1\geq m_1\geq\cdots\geq m_r\geq 1}\frac{(-1)^{m_1+n-1}(m_1-1)!}{m_1^{k_1}\cdots m_r^{k_r}}\left\{ {n+1 \atop m_1} \right\}.$$

Proof. Using the following identity (cf. [3, eqn. 7.49])

$$(e^t-1)^m = m!\sum_{j=m}^{\infty}\left\{ {j \atop m} \right\}\frac{t^j}{j!} \quad (m \geq 0), \tag{2}$$

we have

$$\begin{aligned}&\sum_{n=0}^{\infty}B_{n,\star}^{(k_1,\ldots,k_r)}\frac{t^n}{n!}\\ =\ & \frac{Li^{\star}_{k_1,\ldots,k_r}(1-e^{-t})}{1-e^{-t}}\\ =\ & \sum_{m_1\geq\cdots\geq m_r\geq 1}\frac{(1-e^{-t})^{m_1-1}}{m_1^{k_1}\cdots m_r^{k_r}}\\ =\ & \sum_{m_1\geq\cdots\geq m_r\geq 1}\sum_{j=m_1-1}^{\infty}\frac{(-1)^{m_1+j-1}(m_1-1)!}{m_1^{k_1}\cdots m_r^{k_r}}\left\{ {j \atop m_1-1} \right\}\frac{t^j}{j!}\end{aligned}$$

$$= \sum_{j=0}^{\infty} \sum_{j+1 \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{(-1)^{m_1+j-1}(m_1-1)!}{m_1^{k_1} \cdots m_r^{k_r}} \left\{ {j \atop m_1 - 1} \right\} \frac{t^j}{j!}.$$

Thus comparing the coefficients of $t^n/n!$ on both sides, we obtain the explicit formula for $B_{n,\star}^{(k_1,\ldots,k_r)}$.

The explicit formula for $C_{n,\star}^{(k_1,\ldots,k_r)}$ is obtained similarly by using the identity

$$\frac{e^{-t}(1-e^{-t})^{m-1}}{(m-1)!} = \sum_{j=m-1}^{\infty} (-1)^{j+m+1} \left\{ {j+1 \atop m} \right\} \frac{t^j}{j!},$$

which follows from (2) by differentiation. □

We finish this section by giving some simple relations among the multi-poly-Bernoulli-star numbers.

Proposition 2.3. *For any multi-index $(k_1, \ldots, k_r)$ with $k_i \in \mathbb{Z}$, we have*

$$B_{n,\star}^{(k_1,\ldots,k_r)} = \sum_{j=0}^{n} \binom{n}{j} C_{j,\star}^{(k_1,\ldots,k_r)}$$

and

$$C_{n,\star}^{(k_1,\ldots,k_r)} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_{j,\star}^{(k_1,\ldots,k_r)}.$$

Proof. The generating functions of $B_{n,\star}^{(k_1,\ldots,k_r)}$, $C_{n,\star}^{(k_1,\ldots,k_r)}$ differ by the factor e^t, and the above identities follow immediately. □

Proposition 2.4. *For any multi-index $(k_1, \ldots, k_r)$, we have*

$$C_{n,\star}^{(k_1,\ldots,k_r)} = B_{n,\star}^{(k_1,\ldots,k_r)} - C_{n-1,\star}^{(k_1-1,k_2,\ldots,k_r)}.$$

Proof. By the explicit formula for $C_{n,\star}^{(k_1,\ldots,k_r)}$, we obtain

$$\begin{aligned} C_{n,\star}^{(k_1,\ldots,k_r)} &= \sum_{n+1 \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{(-1)^{m_1+n-1}(m_1-1)!}{m_1^{k_1} \cdots m_r^{k_r}} \left\{ {n+1 \atop m_1} \right\} \\ &= \sum_{n+1 \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{(-1)^{m_1+n-1}(m_1-1)!}{m_1^{k_1} \cdots m_r^{k_r}} \left\{ {n \atop m_1 - 1} \right\} \\ &+ \sum_{n+1 \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{(-1)^{m_1+n-1}(m_1-1)!}{m_1^{k_1-1} m_2^{k_2} \cdots m_r^{k_r}} \left\{ {n \atop m_1} \right\} \\ &= B_{n,\star}^{(k_1,\ldots,k_r)} - C_{n-1,\star}^{(k_1-1,k_2,\ldots,k_r)}. \end{aligned}$$

The second equality above is by the recursion for the Stirling numbers of the second kind. □

3. Main Results

We give the following sum formulas for multi-poly-Bernoulli-star numbers.

Theorem 3.1. *We have*

$$\sum_{\substack{k_1+\cdots+k_r=k \\ 1\le r\le k, k_i\ge 1}} (-1)^r B_{n,\star}^{(k_1,\ldots,k_r)} = \frac{(-1)^k}{k}\binom{n}{k-1} B_{n-k+1,\star}^{(1)} \tag{3}$$

and

$$\sum_{\substack{k_1+\cdots+k_r=k \\ 1\le r\le k, k_i\ge 1}} (-1)^r C_{n,\star}^{(k_1,\ldots,k_r)} = \frac{(-1)^k}{k}\binom{n}{k-1} C_{n-k+1,\star}^{(1)}. \tag{4}$$

Proof. We multiply both sides of (3) by $t^n/n!$ and sums on n. Hence we have

$$\begin{aligned}
\text{(LHS)} &= \sum_{n=0}^{\infty} \sum_{\substack{k_1+\cdots+k_r=k \\ 1\le r\le k, k_i\ge 1}} (-1)^r B_{n,\star}^{(k_1,\ldots,k_r)} \frac{t^n}{n!} \\
&= \sum_{\substack{k_1+\cdots+k_r=k \\ 1\le r\le k, k_i\ge 1}} (-1)^r \frac{Li^{\star}_{k_1,\ldots,k_r}(1-e^{-t})}{1-e^{-t}},
\end{aligned}$$

$$\begin{aligned}
\text{(RHS)} &= \frac{(-1)^k}{k} \sum_{n=0}^{\infty} \binom{n}{k-1} B_{n-k+1,\star}^{(1)} \frac{t^n}{n!} \\
&= \frac{(-1)^k}{k!} \sum_{n=0}^{\infty} B_{n-k+1,\star}^{(1)} \frac{t^n}{(n-k+1)!} \\
&= \frac{(-1)^k}{k!} \sum_{n=0}^{\infty} B_{n,\star}^{(1)} \frac{t^{n+k-1}}{n!} \\
&= \frac{(-1)^k}{k!} \frac{t^k}{1-e^{-t}}.
\end{aligned}$$

Since both sides have the same denominator, it suffices to prove the following identity:

$$\sum_{\substack{k_1+\cdots+k_r=k \\ 1\le r\le k, k_i\ge 1}} (-1)^r Li^{\star}_{k_1,\ldots,k_r}(1-e^{-t}) = \frac{(-1)^k}{k!} t^k. \tag{5}$$

This equality is proved by induction on the weight. When $k=1$, the left-hand side is $-Li^{\star}_1(1-e^{-t}) = -t$ and is equal to the right-hand side. Next we assume the identity holds when the weight is k. Then by differentiating the left-hand side of the identity of weight $k+1$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{\substack{k_1+\cdots+k_r=k+1\\1\le r\le k+1,k_i\ge 1}}(-1)^r Li^{\star}_{k_1,\dots,k_r}(1-e^{-t})\right)\\
=&\sum_{\substack{k_1+\cdots+k_r=k\\1\le r\le k,k_i\ge 1}}(-1)^r\left(\frac{e^{-t}}{1-e^{-t}}-\frac{1}{1-e^{-t}}\right)Li^{\star}_{k_1,\dots,k_r}(1-e^{-t})\\
=&\sum_{\substack{k_1+\cdots+k_r=k\\1\le r\le k,k_i\ge 1}}(-1)^{r+1} Li^{\star}_{k_1,\dots,k_r}(1-e^{-t})\\
=&\frac{(-1)^{k+1}}{k!}t^k.
\end{aligned}$$

We used the induction hypothesis in the last equality. Therefore we have

$$\sum_{\substack{k_1+\cdots+k_r=k+1\\1\le r\le k+1,k_i\ge 1}}(-1)^r Li^{\star}_{k_1,\dots,k_r}(1-e^{-t})=\frac{(-1)^{k+1}}{(k+1)!}t^{k+1}+C$$

with some constant C, which we find is 0 by putting $t=0$. The equation (4) follows from (5) since the generating function of $C_{n,\star}$ differs from that of $B_{n,\star}$ only by a factor e^{-t}. □

Next we describe the duality relation for the multi-poly-Bernoulli-star numbers. We recall the duality operation of Hoffman [6, p. 65]. We define a function S from the set of multi-indices $(k_1,\dots,k_r)$ with $k_i\ge 1$ and weight k to the power set of $\{1,2,\dots,k-1\}$ by

$$S((k_1,\dots,k_r))=\{k_1,k_1+k_2,\dots,k_1+\cdots+k_{r-1}\}.$$

Obviously, the map S is a one-to-one correspondence. Then $(k'_1,\dots,k'_l)$ is said to be the dual index for $(k_1,\dots,k_r)$ in Hoffman's sense when

$$(k'_1,\dots,k'_l)=S^{-1}(\{1,2,\dots,k-1\}-S((k_1,\dots,k_r))).$$

It is easy to see that Hoffman's duality operation is an involution. Note that $k_1>1$ if and only if $k'_1=1$.

Theorem 3.2. *For any multi-index $(k_1,\dots,k_r)$ with $k_i\ge 1 (1\le i\le r)$, we have*

$$C_{n,\star}^{(k_1,\dots,k_r)}=(-1)^n B_{n,\star}^{(k'_1,\dots,k'_l)},$$

where $(k'_1,\dots,k'_l)$ is the dual index of $(k_1,\dots,k_r)$ in Hoffman's sense.

Proof. As in the previous proof, consider the generating functions of both sides:

$$\begin{aligned} (\text{LHS}) &= \sum_{n=0}^{\infty} C_{n,\star}^{(k_1,\dots,k_r)} \frac{t^n}{n!} = \frac{Li^{\star}_{k_1,\dots,k_r}(1-e^{-t})}{e^t - 1}, \\ (\text{RHS}) &= \sum_{n=0}^{\infty} (-1)^n B_{n,\star}^{(k'_1,\dots,k'_l)} \frac{t^n}{n!} = \frac{Li^{\star}_{k'_1,\dots,k'_l}(1-e^{t})}{1 - e^t}. \end{aligned}$$

Hence we have to show the following identity:

$$Li^{\star}_{k_1,\dots,k_r}(1-e^{-t}) + Li^{\star}_{k'_1,\dots,k'_l}(1-e^{t}) = 0.$$

This identity also follows from induction on the weight. First, it is trivial in the case $k = 1$. Thus, we assume the above identity holds when the weight is k. Since $k_1 = 1$ is equivalent to $k_1^{'} \neq 1$, we may assume $k_1 = 1$ by the symmetry of the identity. Then when the weight is $k+1$, the derivative of the left-hand side yields

$$\begin{aligned} & \frac{d}{dt}\left(Li^{\star}_{1,k_2,\dots,k_r}(1-e^{-t}) + Li^{\star}_{k'_1,\dots,k'_l}(1-e^{t})\right) \\ = \; & \frac{1}{1-e^{-t}} Li^{\star}_{k_2,\dots,k_r}(1-e^{-t}) + \frac{-e^t}{1-e^t} Li^{\star}_{k'_1-1,k'_2,\dots,k'_l}(1-e^{t}) \\ = \; & \frac{1}{1-e^{-t}}\left(Li^{\star}_{k_2,\dots,k_r}(1-e^{-t}) + Li^{\star}_{k'_1-1,k'_2,\dots,k'_l}(1-e^{t})\right) \\ = \; & 0. \end{aligned}$$

Therefore we obtain

$$Li^{\star}_{1,k_2,\dots,k_r}(1-e^{-t}) + Li^{\star}_{k'_1,\dots,k'_l}(1-e^{t}) = C$$

with some constant C, and by putting $t = 0$, we conclude $C = 0$. □

4. Connection to the Finite Multiple Zeta-Star Values

In this section, we give alternative proofs for some relations of the finite multiple zeta-star values using the multi-poly-Bernoulli-star numbers. The following congruence is the "star-version" of the congruence given in [8, Theorem.8], and is proved in exactly the same manner:

$$H_p^{\star}(k_1,\dots,k_r) \equiv -C_{p-2,\star}^{(k_1-1,k_2,\dots,k_r)} \bmod p.$$

Thus we find

$$\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_r) = \left(-C_{p-2,\star}^{(k_1-1,k_2,\cdots,k_r)} \bmod p\right)_p. \tag{6}$$

Corollary 4.1 (M. Hoffman [7]). *For any multi-index $(k_1,\dots,k_r)$ with $k_i \geq 1 (1 \leq i \leq r)$, let $(k_1',\dots,k_l')$ be the dual index for $(k_1,\dots,k_r)$ in Hoffman's sense. Then we have*

$$\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_r) = -\zeta^{\star}_{\mathcal{A}}(k_1',\dots,k_l').$$

Proof. It is sufficient to prove the case $k_1 = 1$. By (6) and the duality relation for the multi-poly-Bernoulli-star numbers, we obtain

$$\begin{aligned}
\text{(LHS)} &= \left(-C^{(0,k_2,\cdots,k_r)}_{p-2,\star} \bmod p\right)_p, \\
\text{(RHS)} &= \left(C^{(k_1'-1,k_2',\cdots,k_l')}_{p-2,\star} \bmod p\right)_p \\
&= \left((-1)^p B^{(k_2,\cdots,k_r)}_{p-2,\star} \bmod p\right)_p.
\end{aligned}$$

Hence we complete the proof if we prove $C^{(0,k_2,\cdots,k_r)}_{n,\star} = B^{(k_2,\cdots,k_r)}_{n,\star}$ for all n. We consider the generating functions of these numbers:

$$\begin{aligned}
\sum_{n=0}^{\infty} C^{(0,k_2,\cdots,k_r)}_{n,\star} \frac{t^n}{n!} &= \frac{Li^{\star}_{0,k_2,\dots,k_r}(1-e^{-t})}{e^t-1} \\
&= \frac{1}{e^t-1} \sum_{m_2 \geq \cdots \geq m_r \geq 1} \frac{1}{m_2^{k_2}\cdots m_r^{k_r}} \sum_{m_1=m_2}^{\infty} (1-e^{-t})^{m_1} \\
&= \frac{1}{e^{-t}(e^t-1)} Li^{\star}_{k_2,\dots,k_r}(1-e^{-t}) \\
&= \sum_{n=0}^{\infty} B^{(k_2,\cdots,k_r)}_{n,\star} \frac{t^n}{n!}.
\end{aligned}$$

From this we have

$$-C^{(0,k_2,\cdots,k_r)}_{p-2,\star} = (-1)^p B^{(k_2,\cdots,k_r)}_{p-2,\star}$$

for any odd prime p. □

The following corollary is a weaker version of the sum formula for the finite multiple zeta-star values proved in [11].

Corollary 4.2. *We have*

$$\sum_{\substack{k_1+\cdots+k_r=k \\ r\geq 1, k_1\geq 2, k_i \geq 1}} (-1)^r \zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_r) = \left(B_{p-k} \bmod p\right)_p,$$

where B_n is the classical Bernoulli numbers.

Proof. Equations (4) and (6) yield

$$\sum_{\substack{k_1+\cdots+k_r=k+1\\ r\geq 1, k_1\geq 2, k_i\geq 1}} (-1)^{r+1}\zeta^{\star}_{\mathcal{A}}(k_1,\ldots,k_r) = \left(\frac{(-1)^k}{k}\binom{p-2}{k-1}C^{(1)}_{p-k-1,\star} \bmod p\right)_p$$
$$= \left(-C^{(1)}_{p-k-1,\star} \bmod p\right)_p.$$

So replacing k by $k-1$, we obtain the desired identity. □

Acknowledgments. The author expresses his sincere gratitude to Professor M. Kaneko for useful discussions and invaluable advice.

References

[1] T. Arakawa and M. Kaneko, *Multiple zeta values, poly-Bernoulli numbers, and related zeta functions*, Nagoya Math. J. **153** (1999), 189–209.

[2] A. Bayad and Y. Hamahata, *Multiple Polylogarithms and Multi-Poly-Bernoulli Polynomials*, Funct. Approx. Comment. Math. **46** (2012), 45–61.

[3] R. Graham, D. Knuth, and O. Patashnik, Concrete Mathematics, Addison-Wesley, 1989.

[4] Y. Hamahata and H. Masubuchi, *Special Multi-Poly-Bernoulli Numbers*, J. Integer Seq. **10**, (2007).

[5] Y. Hamahata and H. Masubuchi, *Recurrence Formulae for Multi-Poly-Bernoulli Numbers*, Integers **7** (2007), #A46, 15pp.

[6] M. Hoffman, *Algebraic aspects of multiple zeta values*, Zeta functions, Topology and Quantum Physics, Developments in Math. 14, Springer, New York, 2005, pp. 51–73.

[7] M. Hoffman, *Quasi-symmetric functions and mod p multiple harmonic sums*, preprint, `arXiv:math/0401319v2` [math.NT] 17 Aug. 2007.

[8] K. Imatomi, M. Kaneko and E. Takeda, *Multi-Poly-Bernoulli Numbers and Finite Multiple Zeta Values*, J. Integer Seq. **17** (2014).

[9] M. Kaneko, *Poly-Bernoulli numbers*, J. Théor. Nombres Bordeaux **9** (1997), 199–206.

[10] M. Kaneko and D. Zagier, *Finite Multiple Zeta Values*, preprint.

[11] S. Saito and N. Wakabayashi, *Sum formula for finite multiple zeta values*, J. Math. Soc. Japan, to appear.

[12] J. Zhao, *Wolstenholme type theorem for multiple harmonic sums*, Int. J. Number Theory **4** (2008), 73–106.