THE GENERATING FUNCTION OF THE GENERALIZED FIBONACCI SEQUENCE

Armando Gonçalves
Department of Mathematics, University of Coimbra, Coimbra, Portugal
adsg@mat.uc.pt

M. N. de Jesus[1]
Polytechnic Institute of Viseu, Viseu, Portugal
mnasce@estv.ipv.pt

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Abstract

Using tools of the theory of orthogonal polynomials we obtain the generating function of the generalized Fibonacci sequence established by Petronilho for a sequence of real or complex numbers $\{Q_n\}_{n=0}^{\infty}$ defined by $Q_0 = 0,\ Q_1 = 1,\ Q_m = a_j Q_{m-1} + b_j Q_{m-2},\ m \equiv j \,(\mathrm{mod}\ k)$, where $k \geq 3$ is a fixed integer, and $a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_{k-1}$ are $2k$ given real or complex numbers, with $b_j \neq 0$ for $0 \leq j \leq k-1$. For this sequence some convergence proprieties are obtained.

1. Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications in almost every field of science and art (see e.g. [6]). The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is a well-known sequence of integers. It is defined recursively by the relation

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2 \tag{1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

The Fibonacci number F_{n+1} can be expressed as a determinant of a tridiagonal Toeplitz matrix of order n (see e.g. [10])

$$\begin{vmatrix} 1 & 1 & & & \\ -1 & 1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1 & 1 \\ & & & -1 & 1 \end{vmatrix} = F_{n+1}\,, \quad n \geq 0\,. \tag{2}$$

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Furthermore it is well-known (and easy to check) that the generating function for the Fibonacci sequence (1) is given by

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} \ . \tag{3}$$

There are many generalizations of the Fibonacci sequence [1, 2, 3, 9, 11, 12]. One of them was given in [9] by J.Petronilho as follows:

$$Q_0 = 0,\ Q_1 = 1,\ Q_m = a_j Q_{m-1} + b_j Q_{m-2},\ m \equiv j \,(\mathrm{mod}\ k), \tag{4}$$

where $k \geq 2$ is a fixed integer, and $a_0, a_1, \ldots, a_{k-1}$, $b_0, b_1, \ldots, b_{k-1}$ are $2k$ given real or complex numbers, with $b_j \neq 0$ for $0 \leq j \leq k-1$. In [9] a Binet's-type formula was established for Q_m using an appropriate polynomial mapping in the framework of the theory of orthogonal polynomials. His approach was based on results obtained in [4, 5, 7, 8]. In this paper we present the generating function for $\{Q_n\}_{n=0}^{\infty}$.

Throughout this paper we denote by $\{U_n(x)\}_{n=0}^{\infty}$ the sequence of the Chebyshev polynomials of second kind, which are defined by the three-term recurrence relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \ , \quad n \geq 0 \ ,$$

with initial conditions $U_{-1}(x) \equiv 0$ and $U_0(x) \equiv 1$.

The generating function of the $\{U_n(x)\}_{n=0}^{\infty}$ is given by

$$\sum_{n=0}^{\infty} U_n(x) t^n = \frac{1}{1-2tx+t^2} \tag{5}$$

and for $z \in \mathbb{C} \backslash [-1, 1]$

$$\lim_{n \to +\infty} \frac{U_{n-1}(z)}{U_n(z)} = z - \left(z^2 - 1\right)^{1/2} \ , \tag{6}$$

where

$$\left(z^2-1\right)^{1/2} = \begin{cases} -\sqrt{z^2-1} & \text{if} \quad z \in (-\infty, -1] \\ i\sqrt{1-z^2} & \text{if} \quad z \in [-1, 1] \\ \sqrt{z^2-1} & \text{if} \quad z \in [1, +\infty) \ . \end{cases}$$

The present paper is organized as follows. In Section 2 we present the relation, obtained in [9], between generalized Fibonacci sequences $\{Q_n\}_{n=0}^{\infty}$ and a sequence of orthogonal polynomials $\{R_n(x)\}_{n=0}^{\infty}$. We also generalize the main theorem in [11] and [1, Theorem 11]. In Section 3 we present the generation function for $\{R_n(x)\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$. In Section 4 we discuss the convergence of the ratios of the terms of these sequences. In Section 5, with three application examples, we recover some well-known results.

2. Generalized Fibonacci Sequence Via Orthogonal Polynomials

In what follows, the conventions

$$a_k := a_0 , \quad b_k := b_0 ,$$

will apply.

Then we set

$$\Delta_{\mu,\nu}(x) := \begin{vmatrix} x+a_\mu & 1 & & & \\ -b_{\mu+1} & x+a_{\mu+1} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{\nu-1} & x+a_{\nu-1} & 1 \\ & & & -b_\nu & x+a_\nu \end{vmatrix} \quad \text{if} \quad 0 \le \mu < \nu \le k .$$

If $\mu \ge \nu$, this tridiagonal matrix determinant has the following value

$$\Delta_{\mu,\nu}(x) := \begin{cases} 0 , & \text{if} \quad \mu > \nu+1 \\ 1 , & \text{if} \quad \mu = \nu+1 \\ x+a_\mu , & \text{if} \quad \mu = \nu . \end{cases}$$

We also define

$$\varphi_k(x) := \begin{vmatrix} x+a_2 & 1 & & & & & 1 \\ -b_3 & x+a_3 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -b_{k-1} & x+a_{k-1} & 1 & & \\ & & & -b_0 & x+a_0 & 1 & \\ -b_2 & & & & -b_1 & x+a_1 \end{vmatrix}$$

(in this determinant, the involved matrix is of order k, all the entries that do not appear are zero and the matrix associated with the principal minor of order $k-1$ is tridiagonal).

Let $\{R_n(x)\}_{n=0}^{\infty}$ be the sequence of polynomials defined by the three-term recurrence relation

$$R_{n+1}(x) = (x-\beta_n)R_n(x) - \gamma_n R_{n-1}(x) , \quad n \ge 0 ,$$

with initial conditions $R_{-1}(x) = 0$ and $R_0(x) = 1$, where

$$\beta_{nk+j} := -a_{j+2} , \quad \gamma_{nk+j} := -b_{j+2} , \quad 0 \le j \le k-1 , \quad n \ge 0 .$$

Obviously,

$$Q_n = R_{n-1}(0) , \quad n \ge 0 . \tag{7}$$

According to [5, Theorem 5.1], if we set

$$\widetilde{U}_n(x) := d^n \, U_n\left(\tfrac{x-c}{2d}\right) , \quad n \ge 0 , \tag{8}$$

where d satisfies $d^2 = b := (-1)^k \prod_{i=0}^{k-1} b_i$ and $c := (-1)^k\,(b_2 + b/b_2)$, we can deduce

$$R_{nk+j}(x) = \Delta_{2,j+1}(x)\,\widetilde{U}_n(\varphi_k(x)) + (-1)^{j+1}\left(\prod_{i=2}^{j+2} b_i\right)\,\Delta_{j+3,k}(x)\,\widetilde{U}_{n-1}(\varphi_k(x)) \tag{9}$$

for $0 \le j \le k-1$ and $n \ge 0$.

Throughout this paper we use the following notation

$$\Delta_{\mu,\nu} := \Delta_{\mu,\nu}(0) \quad \text{and} \quad \Delta_k := \varphi_k(0)\,.$$

The following results are immediate consequences of (9) and (7).

Lemma 1. *For $k \ge 3$ and $0 \le j \le k-1$ we have*

(a) $R_j(x) = \Delta_{2,j+1}(x)$;

(b) $R_{k+j}(x) - (\varphi_k(x) - c)R_j(x) = (-1)^{j+1}\left(\prod_{i=2}^{j+2} b_i\right)\,\Delta_{j+3,k}(x)$;

(c) $Q_{j+1} = \Delta_{2,j+1}$;

(d) $Q_{k+j+1} - (\Delta_k - c)Q_{j+1} = (-1)^{j+1}\left(\prod_{i=2}^{j+2} b_i\right)\,\Delta_{j+3,k}$.

Proof. To prove (a) and (b) we consider in (9) $n = 0$ and $n = 1$ respectively. By (a) and (b) and (7) we obtain (c) and (d). □

Theorem 1. *Let $n \ge 2$ and k be positive integers. Then for $0 \le j \le k-1$ we have*

$$R_{nk+j}(x) = (\varphi_k(x) - c)\,R_{(n-1)k+j}(x) - d^2 R_{(n-2)k+j}(x)\,. \tag{10}$$

Proof. It is easy to prove that for $n \ge 0$ and $k \ge 3$,

$$\widetilde{U}_{n+1}\left(\varphi_k(x)\right) = (\varphi_k(x) - c)\,\widetilde{U}_n\left(\varphi_k(x)\right) - d^2\widetilde{U}_{n-1}\left(\varphi_k(x)\right)\,.$$

Then, using (9), we obtain for $0 \le j \le k-1$ and $n \ge 2$,

$$\begin{aligned} R_{nk+j}(x) &= \Delta_{2,j+1}(x)\left\{(\varphi_k(x) - c)\,\widetilde{U}_{n-1}\left(\varphi_k(x)\right) - d^2\widetilde{U}_{n-2}\left(\varphi_k(x)\right)\right\} + (-1)^{j+1}\times \\ &\quad \left(\prod_{i=2}^{j+2} b_i\right)\,\Delta_{j+3,k}(x)\left\{(\varphi_k(x) - c)\,\widetilde{U}_{n-2}\left(\varphi_k(x)\right) - d^2\widetilde{U}_{n-3}\left(\varphi_k(x)\right)\right\} \\ &= (\varphi_k(x) - c)\,R_{(n-1)k+j}(x) - d^2 R_{(n-2)k+j}(x)\,. \end{aligned}$$

The case $k = 2$ can be proved using the same reasoning and the results in [9, Section 2]. If $k = 1$ the result is trivial. □

Corollary 1. *Let $n \ge 2$ and k be positive integers. Then for $0 \le j \le k-1$*

$$Q_{nk+j} = (\Delta_k - c)\,Q_{(n-1)k+j} - d^2 Q_{(n-2)k+j}\,. \tag{11}$$

Remark 1. We note that by [5, equality (5.2)] we have

$$\varphi_k(x) - c = \Delta_{2,k+1}(x) + b_2\Delta_{3,k}(x) = \Delta_{1,k}(x) + b_1\Delta_{2,k-1}(x)\,.$$

Thus we can conclude that the Corollary 1 generalizes the main theorem in [11] and [1, Theorem 11].

3. Generating Function of the Generalized Fibonacci Sequences

The generating function of the generalized Fibonacci sequence $\{Q_n\}_{n=0}^{\infty}$ defined by (4), has been found in [12] for the case $k = 2$ and in [1] and [11] for the case $b_0 = b_1 = \ldots = b_{k-1} = 1$. In this section we give the generating function of the generalized Fibonacci sequences $\{Q_n\}_{n=0}^{\infty}$ for the case $k \geq 3$.

We consider the formal power series representation of the generating function for $\{R_n(x)\}_{n=0}^{\infty}$,

$$F(x,t) = R_0(x) + R_1(x)t + R_2(x)t^2 + \ldots + R_n(x)t^n + \ldots = \sum_{m=0}^{\infty} R_m(x)t^m \ .$$

We rewrite $F(x,t)$ as

$$F(x,t) = \sum_{j=0}^{k-1} \left(\sum_{n=0}^{\infty} R_{nk+j}(x) t^{nk+j} \right) \ . \tag{12}$$

Using (9) we obtain

$$\begin{aligned}
F(x,t) &= \sum_{j=0}^{k-1} \left(\sum_{n=0}^{\infty} \Big\{ \Delta_{2,j+1}(x)\, \widetilde{U}_n(\varphi_k(x)) + \right. \\
&\qquad \left. + \ (-1)^{j+1} \left(\textstyle\prod_{i=2}^{j+2} b_i \right) \ \Delta_{j+3,k}(x)\, \widetilde{U}_{n-1}(\varphi_k(x)) \ \Big\}\, t^{nk+j} \right) \\
&= \sum_{j=0}^{k-1} \left(\sum_{n=0}^{\infty} \Big\{ \Delta_{2,j+1}(x)\, d^n U_n \left(\frac{\varphi_k(x) - c}{2d} \right) + \right. \\
&\qquad \left. + \ (-1)^{j+1} \left(\textstyle\prod_{i=2}^{j+2} b_i \right) \ \Delta_{j+3,k}(x)\, d^{n-1} U_{n-1} \left(\frac{\varphi_k(x) - c}{2d} \right) \ \Big\}\, t^{nk+j} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{k-1} \Delta_{2,j+1}(x)\, t^j \sum_{n=0}^{\infty} U_n \left(\frac{\varphi_k(x) - c}{2d} \right) \left(t^k d \right)^n + \\
&\quad + \sum_{j=0}^{k-1} (-1)^{j+1} \left(\textstyle\prod_{i=2}^{j+2} b_i \right) \ \Delta_{j+3,k}(x)\, t^{k+j} \ \sum_{n=1}^{\infty} U_{n-1} \left(\frac{\varphi_k(x) - c}{2d} \right) \ \left(dt^k \right)^{n-1} \\
&= \sum_{j=0}^{k-1} \frac{\Delta_{2,j+1}(x)\, t^j}{1 - (\varphi_k(x) - c)t^k + d^2 t^{2k}} + \sum_{j=0}^{k-1} \frac{(-1)^{j+1} \left(\prod_{i=2}^{j+2} b_i \right) \ \Delta_{j+3,k}(x)\, t^{k+j}}{1 - (\varphi_k(x) - c)t^k + d^2 t^{2k}} \\
&= \sum_{j=0}^{k-1} t^j \frac{\Delta_{2,j+1}(x) + (-1)^{j+1} \left(\prod_{i=2}^{j+2} b_i \right) \ \Delta_{j+3,k}(x)\, t^k}{1 - (\varphi_k(x) - c)t^k + d^2 t^{2k}} \ .
\end{aligned}$$

Theorem 2. *For $k \geq 3$, the generating function of the generalized Fibonacci sequence $\{Q_n\}_{n=0}^{\infty}$ defined by (4) is given by*

$$G(t) = \sum_{j=0}^{k-1} \frac{t^j \left(\Delta_{2,j} + (-1)^j \left(\prod_{i=2}^{j+1} b_i \right) \Delta_{j+2,k}\, t^k \right)}{1 - (\Delta_k - c)t^k + d^2 t^{2k}} \quad , \tag{13}$$

with $d^2 = b := (-1)^k \prod_{i=0}^{k-1} b_i$ and $c := (-1)^k \, (b_2 + b/b_2)$.

Proof. We note that

$$\begin{aligned} Q_0 + Q_1 t + Q_2 t^2 + \ldots &= \sum_{m=0}^{+\infty} R_m(0) t^{m+1} = tF(0,t) \\ &= \sum_{j=0}^{k-1} t^{j+1} \frac{\Delta_{2,j+1} + (-1)^{j+1} \left(\prod_{i=2}^{j+2} b_i \right) \Delta_{j+3,k}\, t^k}{1 - (\Delta_k - c)t^k + d^2 t^{2k}} \\ &= \sum_{j=1}^{k} t^j \frac{\Delta_{2,j} + (-1)^j \left(\prod_{i=2}^{j+1} b_i \right) \Delta_{j+2,k}\, t^k}{1 - (\Delta_k - c)t^k + d^2 t^{2k}} \\ &= \sum_{j=0}^{k-1} t^j \frac{\Delta_{2,j} + (-1)^j \left(\prod_{i=2}^{j+1} b_i \right) \Delta_{j+2,k}\, t^k}{1 - (\Delta_k - c)t^k + d^2 t^{2k}} \ . \end{aligned}$$

□

Remark 2. From the previous theorem and from Lemma 1 we have that

$$G(t) = \sum_{j=0}^{k-1} \frac{t^j \left(Q_j + \{ Q_{k+j} - (\Delta_k - c) Q_j \} \, t^k \right)}{1 - (\Delta_k - c)t^k + d^2 t^{2k}} \ ,$$

with $d^2 = b := (-1)^k \prod_{i=0}^{k-1} b_i$ and $c := (-1)^k \, (b_2 + b/b_2)$, is an equivalent expression for the generating function of $\{Q_n\}_{n=0}^{\infty}$.

Remark 3. If $\alpha(x) = \frac{\varphi_k(x) - c - \sqrt{(\varphi_k(x)-c)^2 - 4b}}{2}$ and $\beta(x) = \frac{\varphi_k(x) - c + \sqrt{(\varphi_k(x)-c)^2 - 4b}}{2}$ are the roots of the quadratic equation

$$z^2 + (\, c - \varphi_k(x)\,)\, z + b = 0 \ , \tag{14}$$

with

$$b := (-1)^k \prod_{i=0}^{k-1} b_i \ , \quad c := (-1)^k \, (b_2 + b/b_2) \ , \tag{15}$$

then

$$\begin{cases} \alpha(x)\beta(x) = b \\ \alpha(x) + \beta(x) = \varphi_k(x) - c \ . \end{cases} \tag{16}$$

Furthermore, for $0 \le j \le k-1$ the generating function for the subsequence $\{R_{nk+j}(x)\}_{n=0}^{\infty}$ is given by

$$\begin{aligned} F_j(x,t) &= \frac{t^j \left(\Delta_{2,j+1}(x) + (-1)^{j+1} \left(\prod_{i=2}^{j+2} b_i\right) \Delta_{j+3,k}(x)\, t^k \right)}{1-(\varphi_k(x)-c)t^k + d^2 t^{2k}} \\ &= \frac{t^j}{\alpha(x)-\beta(x)} \sum_{n=0}^{\infty} \left(\Delta_{2,j+1}(x) d^{2n+2} t^{kn} \left(\frac{1}{\beta^{n+1}(x)} - \frac{1}{\alpha^{n+1}(x)} \right) + \right. \\ &\qquad \left. + (-1)^{j+1} \left(\prod_{i=2}^{j+2} b_i\right) d^{2n} t^{kn} \left(\frac{1}{\beta^n(x)} - \frac{1}{\alpha^n(x)} \right) \Delta_{j+3,k}(x) \right) \\ &= \sum_{n=0}^{\infty} \frac{\Delta_{2,j+1}(x)\left(\alpha^{n+1}(x)-\beta^{n+1}(x)\right) + (-1)^{j+1}\left(\prod_{i=2}^{j+2} b_i\right) \Delta_{j+3,k}(x)\left(\alpha^n(x)-\beta^n(x)\right)}{\alpha(x)-\beta(x)} t^{nk+j}. \end{aligned}$$

Thus

$$R_{nk+j}(x) = \frac{\Delta_{2,j+1}(x)\left(\alpha^{n+1}(x)-\beta^{n+1}(x)\right)+(-1)^{j+1}\left(\prod_{i=2}^{j+2} b_i\right) \Delta_{j+3,k}(x)(\alpha^n(x)-\beta^n(x))}{\alpha(x)-\beta(x)}$$

for $0 \le j \le k-1$ and $n \ge 0$. Therefore, we obtain

$$R_{nk+j}(x) = \frac{(A_j(x)\alpha(x)+B_j(x))\,\alpha^n(x)-(A_j(x)\beta(x)+B_j(x))\,\beta^n(x)}{\alpha(x)-\beta(x)}\ , \quad 0 \le j \le k-1\ , n \ge 0$$

where

$$A_j(x) := \Delta_{2,j+1}(x)\ , \quad B_j(x) := (-1)^{j+1} \left(\prod_{i=2}^{j+2} b_i\right) \Delta_{j+3,k}(x)\ , \quad 0 \le j \le k-1\ . \tag{17}$$

If $m = nk+j$ with $0 \le j \le k-1$ and $n \ge 0$, then $n = \lfloor \frac{m}{k} \rfloor$ and $j = m - k\lfloor \frac{m}{k} \rfloor$, that is, if $m \equiv j \pmod k$

$$\begin{aligned} A_j(x) &= A_{m-k\lfloor \frac{m}{k} \rfloor}(x) = \Delta_{2,1+m-k\lfloor \frac{m}{k} \rfloor}(x) := A_{k,m}(x) \\ B_j(x) &= B_{m-k\lfloor \frac{m}{k} \rfloor}(x) = (-1)^{1+m-k\lfloor \frac{m}{k} \rfloor} \left(\prod_{i=2}^{2+m-k\lfloor \frac{m}{k} \rfloor} b_i\right) \Delta_{3+m-k\lfloor \frac{m}{k} \rfloor,k}(x) := B_{k,m}(x) \end{aligned}$$

and we obtain

$$R_n(x) = \frac{(A_{k,n}(x)\alpha(x)+B_{k,n}(x))\ \alpha^{\lfloor \frac{n}{k} \rfloor}(x) - (A_{k,n}(x)\beta(x)+B_{k,n}(x))\ \beta^{\lfloor \frac{n}{k} \rfloor}(x)}{\alpha(x)-\beta(x)}\ . \tag{18}$$

Denoting $\alpha := \alpha(0)$, $\beta := \beta(0)$, setting $x=0$ in (18) and taking into account (7) we recover Theorem 3.1 in [9].

4. Convergence Properties

For the classical Fibonacci sequence, it is well-known (and easy to check) that

$$\lim_{n \to +\infty} \frac{F_{n+1}}{F_n} = \Phi = \frac{1+\sqrt{5}}{2}\ .$$

Now, we study the convergence of ratios on the terms of the subsequence $\{R_{nk+j}(x)\}_{n=0}^{\infty}$ and consequently on the subsequence $\{Q_{nk+j+1}\}_{n=0}^{\infty}$.

From now on $a_0, a_1, \ldots, a_{k-1}$, $b_0, b_1, \ldots, b_{k-1}$ are $2k$ given real numbers, with $b_i \neq 0$ $(i = 0, \ldots, k-1)$ and, for any real number y, $sgn(y)$ denotes the sign of y.

Theorem 3. *If* $|\varphi_k(x) - c| > 2|d|$ *then for* $0 \le j \le k-1$

$$\lim_{n\to+\infty} \frac{R_{(n+1)k+j}(x)}{R_{nk+j}(x)} = \frac{(\varphi_k(x)-c) + sgn\,(\varphi_k(x)-c)\sqrt{(\varphi_k(x)-c)^2 - 4d^2}}{2} \tag{19}$$

and

$$\lim_{n\to+\infty} \frac{R_{nk+j}(x)}{R_{nk+j-1}(x)} = \begin{cases} \frac{R_{k+j}(x)-\beta(x)R_j(x)}{R_{k+j-1}(x)-\beta(x)R_{j-1}(x)} & \text{if} \quad \varphi_k(x)-c < -2|d| \\ \frac{R_{k+j}(x)-\alpha(x)R_j(x)}{R_{k+j-1}(x)-\alpha(x)R_{j-1}(x)} & \text{if} \quad \varphi_k(x)-c > 2|d|\,. \end{cases} \tag{20}$$

Proof. Indeed using (9) we obtain

$$\lim_{n\to+\infty} \frac{R_{(n+1)k+j}(x)}{R_{nk+j}(x)}$$
$$= \lim_{n\to+\infty} \frac{\widetilde{U}_{n+1}(\varphi_k(x))}{\widetilde{U}_n(\varphi_k(x))} \frac{\Delta_{2,j+1}(x) + (-1)^{j+1}\left(\prod_{i=2}^{j+2} b_i\right)\Delta_{j+3,k}(x)\frac{\widetilde{U}_n(\varphi_k(x))}{\widetilde{U}_{n+1}(\varphi_k(x))}}{\Delta_{2,j+1}(x) + (-1)^{j+1}\left(\prod_{i=2}^{j+2} b_i\right)\Delta_{j+3,k}(x)\frac{\widetilde{U}_{n-1}(\varphi_k(x))}{\widetilde{U}_n(\varphi_k(x))}}.$$

Using (6) and (8) we can conclude

$$\lim_{n\to+\infty} \frac{\widetilde{U}_{n+1}(\varphi_k(x))}{\widetilde{U}_n(\varphi_k(x))} = \begin{cases} \frac{d^2}{\beta(x)} & \text{if} \quad \varphi_k(x)-c < -2|d| \\ \frac{d^2}{\alpha(x)} & \text{if} \quad \varphi_k(x)-c > 2|d| \end{cases} = \begin{cases} \alpha(x) & \text{if} \quad \varphi_k(x)-c < -2|d| \\ \beta(x) & \text{if} \quad \varphi_k(x)-c > 2|d|\,. \end{cases} \tag{21}$$

This completes the proof of (19).
By (9) we obtain

$$\lim_{n\to+\infty} \frac{R_{nk+j}(x)}{R_{nk+j-1}(x)} = \lim_{n\to+\infty} \frac{\Delta_{2,j+1}(x) + (-1)^{j+1}\left(\prod_{i=2}^{j+2} b_i\right)\Delta_{j+3,k}(x)\frac{\widetilde{U}_{n-1}(\varphi_k(x))}{\widetilde{U}_n(\varphi_k(x))}}{\Delta_{2,j}(x) + (-1)^{j}\left(\prod_{i=2}^{j+1} b_i\right)\Delta_{j+2,k}(x)\frac{\widetilde{U}_{n-1}(\varphi_k(x))}{\widetilde{U}_n(\varphi_k(x))}}$$

$$= \begin{cases} \frac{\alpha(x)\Delta_{2,j+1}(x) + (-1)^{j+1}\left(\prod_{i=2}^{j+2} b_i\right)\Delta_{j+3,k}(x)}{\alpha(x)\Delta_{2,j}(x) + (-1)^{j}\left(\prod_{i=2}^{j+1} b_i\right)\Delta_{j+2,k}(x)} & \text{if} \quad \varphi_k(x)-c < -2|d| \\ \frac{\beta(x)\Delta_{2,j+1}(x) + (-1)^{j+1}\left(\prod_{i=2}^{j+2} b_i\right)\Delta_{j+3,k}(x)}{\beta(x)\Delta_{2,j}(x) + (-1)^{j}\left(\prod_{i=2}^{j+1} b_i\right)\Delta_{j+2,k}(x)} & \text{if} \quad \varphi_k(x)-c > 2|d|\,. \end{cases}$$

Using Lemma 1 and (16) we obtain (20). □

Corollary 2. *If* $|\Delta_k - c| > 2|d|$ *then for* $0 \le j \le k-1$

$$\lim_{n\to+\infty} \frac{Q_{(n+1)k+j+1}}{Q_{nk+j+1}} = \frac{(\Delta_k - c) + sgn\,(\Delta_k - c)\sqrt{(\Delta_k - c)^2 - 4d^2}}{2}\,.$$

Furthermore, for $0 \le j \le k-1$

$$\lim_{n\to+\infty}\frac{Q_{nk+j+1}}{Q_{nk+j}} = \begin{cases} \frac{Q_{k+j+1}-\beta Q_{j+1}}{Q_{k+j}-\beta Q_j} & \text{if} \quad \Delta_k - c < -2|d| \\ \frac{Q_{k+j+1}-\alpha Q_{j+1}}{Q_{k+j}-\alpha Q_j} & \text{if} \quad \Delta_k - c > 2|d| \ . \end{cases}$$

5. Some Examples

5.1. The Generating Function of the Generalized Fibonacci Sequence for the Case $k = 3$

In this case we obtain $\Delta_{2,0} = 0$, $\Delta_{2,1} = \Delta_{4,3} = 1$, $\Delta_{2,2} = a_2$, $\Delta_{2,3} = a_0a_2 + b_0$, $\Delta_{3,3} = a_0$, $\Delta_3 = a_0a_1a_2 + a_0b_2 + a_1b_0 + a_2b_1 + b_0b_1 - b_2$, $b = d^2 = -b_0b_1b_2$ and $c = -b_2 + b_0b_1$. Then, by (13), we have

$$G(t) = \frac{b_0b_2t^5 - a_0b_2t^4 + (a_0a_2 + b_0)t^3 + a_2t^2 + t}{1 - (a_0a_1a_2 + a_0b_2 + a_1b_0 + a_2b_1)t^3 - b_0b_1b_2t^6}\ .$$

By setting $b_0 = b_1 = b_2 = 1$, we obtain the generating function deduced in [11, Example 1].

5.2. The Generating Function for the $k-$periodic Fibonacci Sequence

If $b_0 = b_1 = \cdots = b_{k-1} = 1$, then, using (4), the sequence $\{Q_n\}_{n=0}^{\infty}$ becomes the $k-$periodic Fibonacci sequence $\{q_n\}_{n=0}^{\infty}$ defined in [1].
In this case, using Remark 2 we deduce that the generating function is given by

$$G(t) = \frac{\sum_{j=0}^{k-1} t^j q_j + \sum_{j=0}^{k-1}\left(q_{k+j} - (\Delta_k - c)q_j\right)t^{k+j}}{1 - (\Delta_k - c)t^k + (-1)^k t^{2k}}\ ,$$

recovering Theorem 13 in [1]. Indeed, by Remark 1, we have $\Delta_k - c = \Delta_{1,k} + \Delta_{2,k-1} = q^0_{k+1} + q_{k-1} := A$, where the sequence $\{q^0_n\}_{n=0}^{\infty}$ is defined in [1]. Thus,
if $|A| > 2$ then, by Corollary 2, we have

$$\lim_{n\to+\infty}\frac{q_{(n+1)k+j+1}}{q_{nk+j+1}} = \frac{A + sgn\,(A)\sqrt{A^2 - 4(-1)^k}}{2}$$

and

$$\lim_{n\to+\infty}\frac{q_{nk+j+1}}{q_{nk+j}} = \begin{cases} \frac{q_{k+j+1}-\beta q_{j+1}}{q_{k+j}-\beta q_j} & \text{if} \quad A < -2 \\ \frac{q_{k+j+1}-\alpha q_{j+1}}{q_{k+j}-\alpha q_j} & \text{if} \quad A > 2\ , \end{cases}$$

recovering Theorem 17 in [1].

5.3. The Generating Function for the Fibonacci Numbers

If in (4) we take $a_0 = a_1 = \cdots = a_{k-1} = b_0 = b_1 = \cdots = b_{k-1} = 1$, then the sequence $\{Q_n\}_{n=0}^{\infty}$ becomes the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$. In this situation, taking into account (2), we deduce

$$\begin{gathered} b = (-1)^k \ , \quad c = 1 + (-1)^k \ , \\ \Delta_k = F_{k+1} + F_{k-1} + 1 + (-1)^k = L_k + 1 + (-1)^k \ , \\ A_j = F_{j+1} \ , \quad B_j = (-1)^{j+1} F_{k-j-1} \quad (0 \le j \le k-1) \ , \end{gathered}$$

where $L_k = F_{k+1} + F_{k-1}$ is the k^{th} Lucas number.
Furthermore, (13) reduces to (3). Before justifying this assertion we consider the following lemma.

Lemma 2. *For every positive integer $k \ge 3$, we have*

$$F_k t^k + \sum_{j=0}^{k-1} F_j \left(t^j + (-1)^{k-j} t^{2k-j}\right) = \frac{t}{1-t-t^2}\left(1 - L_k t^k + (-1)^k t^{2k}\right) \ . \tag{22}$$

Proof. We proceed by induction on k. For $k = 3$, we have

$$\begin{aligned} F_3 t^3 + \textstyle\sum_{j=0}^{2} F_j \left(t^j + (-1)^{3-j} t^{6-j}\right) &= F_3 t^3 + F_1 \left(t + t^5\right) + F_2 \left(t^2 - t^4\right) = \\ = t^5 - t^4 + 2t^3 + t^2 + t &= \frac{t - 4t^4 - t^7}{1-t-t^2} = \frac{t}{1-t-t^2}\left(1 - L_3 t^3 - t^6\right) \ . \end{aligned}$$

Now, by assuming that our claim is true for an integer $k \ge 3$, we will prove that it is true for $k+1$. Indeed,

$$\begin{aligned} & F_{k+1} t^{k+1} + \textstyle\sum_{j=0}^{k} F_j \left(t^j + (-1)^{k+1-j} t^{2k+2-j}\right) \\ & = \left(F_k + F_{k-1}\right) t^{k+1} + \textstyle\sum_{j=0}^{k-1} F_j \left(t^j + (-1)^{k+1-j} t^{2k+2-j}\right) + F_k \left(t^k - t^{k+2}\right) \\ & = t \left(F_k t^k + \textstyle\sum_{j=2}^{k-1} F_{j-1} \left(t^{j-1} + (-1)^{k+1-j} t^{2k+1-j}\right)\right) + t + (-1)^k t^{2k+1} \\ & \quad + t^2 \left(F_{k-1} t^{k-1} + \textstyle\sum_{j=2}^{k-1} F_{j-2} \left(t^{j-2} + (-1)^{k+1-j} t^{2k-j}\right)\right) + F_k \left(t^k - t^{k+2}\right) \end{aligned}$$

$$\begin{aligned} & = t \left(F_k t^k + \textstyle\sum_{j=0}^{k-1} F_j \left(t^j + (-1)^{k-j} t^{2k-j}\right)\right) - t F_{k-1} \left(t^{k-1} - t^{k+1}\right) \\ & \quad + t^2 \left(F_{k-1} t^{k-1} + \textstyle\sum_{j=0}^{k-2} F_j \left(t^j + (-1)^{k+1-j} t^{2k-j-2}\right)\right) - \\ & \quad - t^2 F_{k-2} \left(t^{k-2} - t^k\right) + F_k \left(t^k - t^{k+2}\right) + t + (-1)^k t^{2k+1} \\ & = \tfrac{t^2}{1-t-t^2} \left(1 - L_k t^k + (-1)^k t^{2k}\right) + \tfrac{t^3}{1-t-t^2} \left(1 - L_{k-1} t^{k-1} + (-1)^{k-1} t^{2k-2}\right) + \\ & \quad + t \left(F_{k-1} \left(t^{k+1} - t^{k-1}\right) + t F_{k-2} \left(t^k - t^{k-2}\right) + F_k \left(t^{k-1} - t^{k+1}\right) + 1 + (-1)^k t^{2k}\right) \\ & = \tfrac{t}{1-t-t^2} \left(1 - \left(L_{k-1} + L_k\right) t^{k+1} + (-1)^{k+1} t^{2(k+1)}\right) \\ & = \tfrac{t}{1-t-t^2} \left(1 - L_{k+1} t^{k+1} + (-1)^{k+1} t^{2(k+1)}\right) \ . \end{aligned}$$

□

By (13) and taking into account Lemma 2, we obtain

$$\begin{aligned} F(t) &= \sum_{j=0}^{k-1} t^j \left\{F_j + (-1)^j F_{k-j} t^k\right\} \frac{1}{1 - L_k t^k + (-1)^k t^{2k}} \\ &= \left(F_k t^k + \sum_{j=0}^{k-1} F_j \left(t^j + (-1)^{k-j} t^{2k-j}\right)\right) \frac{1}{1 - L_k t^k + (-1)^k t^{2k}} \\ &= \frac{t}{1 - t - t^2} \,. \end{aligned}$$

By Corollary 2 and using the well known identity $L_k^2 = 5F_k^2 + 4(-1)^k$, we have

$$\lim_{n\to+\infty} \frac{F_{(n+1)k+j+1}}{F_{nk+j+1}} = \frac{L_k + \sqrt{5}F_k}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^k$$

and

$$\lim_{n\to+\infty} \frac{F_{nk+j+1}}{F_{nk+j}} = \frac{2^k F_{k+j+1} - (1-\sqrt{5})^k F_{j+1}}{2^k F_{k+j} - (1-\sqrt{5})^k F_j} \,.$$

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