THE ASYMPTOTIC DISTRIBUTION OF A HYBRID ARITHMETIC FUNCTION

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Abstract

We investigate the average order of artithmetic functions of the form $d^a(n)\sigma^b(n)\phi^c(n)$ where a, b, and c are real numbers. In the case when $2^a \in \mathbb{N}$ with $2^a \geq 4$, we use analytic methods to obtain the asymptotic estimate

$$\sum_{n\leq x} d^a(n)\sigma^b(n)\phi^c(n) = x^{b+c+1}P(\log x) + O\left(x^{b+c+r_a+\varepsilon}\right)$$

for explicit $\frac{1}{2} \leq r_a < 1$ and where P is a polynomial. Using elementary techniques, we establish a similar but slightly weaker result for $2^a \notin \mathbb{N}$.

1. Introduction

An *arithmetic function* is a sequence of complex numbers. These functions frequently help reveal information about the integers. For example, the divisor function, $d(n)$, counts the number of divisors of n. Other arithmetic functions include Euler's totient function, $\phi(n)$, which counts the number of positive integers less than or equal to n that are relatively prime to n, and the sum-of-divisors function, $\sigma(n)$, which is the sum of all positive divisors of n.

In 1849, Dirichlet [3] proved that

$$\sum_{n\leq x} d(n) = x\log x + (2\gamma - 1)x + O(x^{\theta})$$

where $\theta = \frac{1}{2}$ and γ is the Euler-Mascheroni constant. Over the following 154 years, the upper bound for θ slowly improved. Theta has been shown to be as low as $\frac{131}{416}$, as established by Huxley [5] in 2003. Furthermore, Hardy and Landau [4] showed in 1916 that $\theta \geq \frac{1}{4}$.

A related problem is of estimating the partial sums of powers of the divisor function. In 1916, Ramanujan [9] provided (without proof) an estimate for positive integer powers, contingent on the truth of the Riemann hypothesis, as well as an unconditional estimate for noninteger powers. Seven years later, Wilson [10] gave a proof of Ramanujan's results in addition to establishing a unconditional estimate for integer powers $a \geq 2$,

$$\sum_{n\leq x} d^a(n) = xP(\log x) + O\left(x^{\frac{2^a-1}{2^a+2}+\epsilon}\right),$$

where P is a polynomial of degree $2^a - 1$. Asymptotic estimates for the partial sums of powers of Euler's totient function and the sum of divisors function have been given in [2] and [8].

Although the average orders of single arithmetic functions are well-studied, the aim of this paper is to look into combinations of arithmetic functions, specifically $d^a(n)\sigma^b(n)\phi^c(n)$ for arbitrary a, b, and c. We will use both elementary and analytic methods in order to estimate $d^a(n)\sigma^b(n)\phi^c(n)$ including estimating the Dirichlet series and utilizing the zeta function. We begin with some preliminary results in analytic number theory.

2. Preliminaries

Perron's formula is critical for using complex analytic techniques to bound the growth of multiplicative functions, by turning the partial summation of a multiplicative function into a line integral in the complex plane plus an error formula. It is given in section 1.2.1 of [7] as

Lemma 1. *Suppose $F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ converges absolutely for $\sigma > 1$ and $|a(n)| \leq A(n)$ where $A(n)$ is monotonically increasing and $\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} = O\left(\frac{1}{(\sigma-1)^\alpha}\right)$ with $\alpha > 0$ as $\sigma \to 1^+$. If $b > 1$ and $x = N + \frac{1}{2}$ where $N \in \mathbb{N}$, then for $T \geq 2$,*

$$\sum_{n\leq x} a(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} F(s)\frac{x^s}{s}ds + O\left(\frac{x^b}{T(b-1)^\alpha} + \frac{xA(2x)\log x}{T}\right).$$

We must now discuss general divisor problems. If we allow σ, the real part of s, to be fixed such that $1/2 < \sigma < 1$, then we define $m(\sigma)$ to be the supremum of all numbers m such that

$$\int_1^T |\zeta(\sigma + it)|^m \, dt \ll T^{1+\varepsilon} \tag{1}$$

holds. We give the following result (theorem 8.4 from [6])

Lemma 2. *Given* (1), *the following relations between* σ *and* $m(\sigma)$ *hold:*

$$
\begin{array}{ll}
m(\sigma) \geq 4/(3-4\sigma) & \text{for} \quad 1/2 < \sigma \leq 5/8, \\
m(\sigma) \geq 10/(5-6\sigma) & \text{for} \quad 5/8 \leq \sigma \leq 35/54, \\
m(\sigma) \geq 19/(6-6\sigma) & \text{for} \quad 35/54 \leq \sigma \leq 41/60, \\
m(\sigma) \geq 2112/(859-948\sigma) & \text{for} \quad 41/60 \leq \sigma \leq 3/4, \\
m(\sigma) \geq 12408/(4537-4890\sigma) & \text{for} \quad 3/4 \leq \sigma \leq 5/6, \\
m(\sigma) \geq 4324/(1031-1044\sigma) & \text{for} \quad 5/6 \leq \sigma \leq 7/8, \\
m(\sigma) \geq 98/(31-32\sigma) & \text{for} \quad 7/8 \leq \sigma \leq 0.91591\ldots, \\
m(\sigma) \geq (24\sigma-9)/(4\sigma-1)(1-\sigma) & \text{for} \quad 0.91591\ldots \leq \sigma \leq 1-\varepsilon.
\end{array}
$$

This theorem gives us ability to choose a power of zeta first, and then derive the interval for σ for which our integral bound holds. We can use this relationship to choose the best contour that will minimize the error term within Perron's formula.

The generalized divisor function, $d_k(n)$, counts the number of ways in which n can be written as a product of k nontrivial factors. This function has a number of interesting and important properties which can be used in bounding powers of $\zeta(s)$. This is because $\sum_{n=1}^{\infty} d_k(n)n^{-s} = \zeta^k(s)$. In order to understand how $\sum_{n\leq x} d_k(n)$ grows when k is a fixed integer, we will utilize the partial summation formula,

$$\sum_{n\leq x} d_k(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \zeta^k(w)x^w w^{-1}dw + O(x^{1+\varepsilon}T^{-1}) \qquad (T \leq x),$$

and then discuss the order of the error term, Δ_k, defined by,

$$\Delta_k(x) = \sum_{n\leq x} d_k(n) - \operatorname*{Res}_{s=1} \zeta^k(s)x^s s^{-1} = I_1 + I_2 + I_3 + O(x^{1+\varepsilon}T^{-1})$$

where,

$$I_1 = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \zeta^k(w)x^w w^{-1}dw \ll x^\sigma + x^\sigma \int_1^T |\zeta(\sigma+iv)|^k v^{-1}dv$$

$$I_2 + I_3 \ll \int_\sigma^{1+\varepsilon} x^\theta |\zeta(\theta+iT)|^k T^{-1}d\theta \ll \max_{\sigma\ll\theta\ll 1+\varepsilon} x^\theta T^{k\mu(\theta)-1+\varepsilon}.$$

Where $\mu(\theta)$ is a function defined by equation (1.65) of [6] and has the properties that $\mu(\theta) \leq \frac{1}{m(\theta)}$ and $\mu(\theta) = 0$ when $\theta > 1$. Thus, results on the general divisor problem are given by estimates for power moments of $\zeta(s)$. These results for $\Delta_k(n)$ are summarized by theorem 13.2 in [6]

Lemma 3. *Let* α_k *be the infimum of numbers* α_k *such that* $\Delta_k(x) \ll x^{\alpha_k+\varepsilon}$ *for any* $\varepsilon > 0$. *Then,*

$$\begin{array}{lll}
\alpha_k \le (3k-4)/4k & \text{for} & (4 \le k \le 8), \\
\alpha_9 \le 35/54,\ \alpha_{10} \le 41/60,\ \alpha_{11} \le 7/10, & & \\
\alpha_k \le (k-2)/(k+2) & \text{for} & (12 \le k \le 25), \\
\alpha_k \le (k-1)/(k+4) & \text{for} & (26 \le k \le 50), \\
\alpha_k \le (31k-98)/32k & \text{for} & (51 \le k \le 57), \\
\alpha_k \le (7k-34)/7k & \text{for} & (k \ge 58).
\end{array}$$

Thus far, the results have bounded d_k when k is an integer, but these can be extended to a general treatment of d_z where z is a complex number. We get the same formulation for the Dirichlet series of d_z:

$$\sum_{n=1}^{\infty} d_z(n)n^{-s} = \zeta^z(s).$$

Because we raised zeta to a noninteger power, we must specify a branch of zeta,

$$\zeta^z(s) = \exp\{z \log \zeta(s)\} = \exp\left(-z \sum_p \sum_{j=1}^{\infty} j^{-1} p^{-js}\right) \qquad (\sigma > 1).$$

With this we can now bound $\sum_{n \le x} d_z(n)$. We give the following result (theorem 14.9 from [6]).

Lemma 4. *Let $A > 0$ be arbitrary but fixed, and let $N \ge 1$ be an arbitrary but fixed integer. If $|z| \le A$, then uniformly in z*

$$D_z(x) = \sum_{n \le x} d_z(n) = \sum_{k=1}^{N} (c_k(z) x \log^{z-k} x) + O(x \log^{\mathrm{Re}\ z-N-1} x),$$

where $c_k(z) = B_{k-1}(z)/\Gamma(z-k+1)$ and each $B_k(z)$ is regular for $|z| \le A$.

3. Main Results

Define $S(x) := \sum_{n \le x} d^a(n)\sigma^b(n)\phi^c(n)$. By analytic methods, we prove in Section 3.2 that

Theorem 1. *If $a, b, c \in \mathbb{R}$ such that $2^a \in \mathbb{N}$ and $b + c > -r_a$, then for every $\varepsilon > 0$,*

$$S(x) = x^{b+c+1} P_{2^a-1}(\log x) + O\left(x^{b+c+r_a+\varepsilon}\right)$$

where P_d is a degree d polynomial and r_a is given by

$$r_a = \begin{cases} \frac{3}{4} - 2^{-a} & \text{for } 4 \le 2^a \le 8 \\ \frac{35}{54} & \text{for } 2^a = 9 \\ \frac{41}{60} & \text{for } 2^a = 10 \\ \frac{859}{948} - \frac{176}{79} \cdot 2^{-a} & \text{for } 11 \le 2^a \le 14 \\ \frac{4537}{4890} - \frac{2068}{815} \cdot 2^{-a} & \text{for } 15 \le 2^a \le 26 \\ \frac{1031}{1044} - \frac{1081}{261} \cdot 2^{-a} & \text{for } 27 \le 2^a \le 36 \\ \frac{31}{32} - \frac{49}{16} \cdot 2^{-a} & \text{for } 37 \le 2^a \le 57 \\ \frac{5}{8} - 3 \cdot 2^{-a} + 2^{-3-a}\sqrt{576 - 3 \cdot 2^{5+a} + 9 \cdot 2^{2a}} & \text{for } 2^a \ge 58. \end{cases} \tag{2}$$

Note that when $2^a \ge 58$, r_a satisfies $.915... \le r_a < 1$.

In the second case of Section 3.3, we give an elementary proof of the above result, though with a slightly weaker error term. In the first case of Section 3.3, we provide an elementary proof of the following result.

Theorem 2. *If $a, b, c \in \mathbb{R}$ such that $b + c > -1$ and $2^a \notin \mathbb{N}$, then*

$$S(x) = x^{b+c+1} \sum_{j=1}^{N} b_j (\log x)^{2^a - j} + O\left(x^{b+c+1} (\log x)^{2^a - N - 1}\right)$$

where $N \in \mathbb{N}$ may be chosen arbitrarily.

In the special case of Section 3.3, we prove

Theorem 3. *If $a = 1$ and $b, c \in \mathbb{R}$ with $b + c > -\theta$ where θ is of the Dirichlet divisor problem, then*

$$S(x) = \frac{x^{b+c+1}}{b+c+1}\left[\left(\sum_{m=1}^{\infty} \frac{g(m)}{m^{b+c+1}}\right)\left(\log x + 2\gamma - \frac{1}{b+c+1}\right) - \sum_{m=1}^{\infty} \frac{g(m)\log m}{m^{b+c+1}}\right] + O(x^{b+c+\theta}).$$

Note that as of the time of this writing, θ has been shown to be as low as $\frac{131}{416}$ by Huxley [5].

3.1. Dirichlet Series

We begin by computing the Dirichlet series of $d^a \sigma^b \phi^c$, which we will need in both the analytic and elementary proofs.

Lemma 5. *The Dirichlet series of $d^a \sigma^b \phi^c$ is analytic for $\mathrm{Re}(s) > b + c + 1$ and is given by*

$$\sum_{n=1}^{\infty} \frac{d^a(n)\sigma^b(n)\phi^c(n)}{n^s} = \zeta^{2^a}(s - b - c)H(s)$$

where H is analytic and bounded for $\mathrm{Re}(s) > b + c + \frac{1}{2}$.

Proof. Let $F(s) := \sum_{n=1}^{\infty} \frac{d^a(n)\sigma^b(n)\phi^c(n)}{n^s}$ be the Dirichlet series of $d^a\sigma^b\phi^c$. Each of d, σ, and ϕ are multiplicative, so $d^a\sigma^b\phi^c$ is multiplicative as well. Thus we may write F as a product over primes, $F(s) = \prod_p f_p(s)$. Then if $s = \beta + it$,

$$\begin{aligned}
f_p(s) &= \sum_{k=0}^{\infty} \frac{d^a(p^k)\sigma^b(p^k)\phi^c(p^k)}{p^{ks}} \\
&= 1 + \frac{2^a(p+1)^b(p-1)^c}{p^s} + \frac{3^a(p^2+p+1)^b(p^2-p)^c}{p^{2s}} + \dots \\
&= 1 + \frac{2^a}{p^{s-b-c}} \left(1+\frac{1}{p}\right)^b \left(1-\frac{1}{p}\right)^c + O\left(p^{2(b+c)-2\beta}\right) \\
&= 1 + \frac{2^a}{p^{s-b-c}} \left(1 + O\left(\frac{1}{p}\right)\right) + O\left(p^{2(b+c)-2\beta}\right) \\
&= 1 + \frac{2^a}{p^{s-b-c}} + O\left(p^{b+c-\beta-1} + p^{2(b+c)-2\beta}\right).
\end{aligned}$$

Consequently F is analytic when $\beta > b + c + 1$.

Now we write F as a power of zeta times a function that is analytic and bounded on an extended half-plane. Observe that

$$\begin{aligned}
&\zeta^{-2^a}(s-b-c)F(s) \\
&= \prod_p \left(1 - \frac{1}{p^{s-b-c}}\right)^{2^a} \left(1 + \frac{2^a}{p^{s-b-c}} + O\left(p^{b+c-\beta-1} + p^{2(b+c)-2\beta}\right)\right) \\
&= \prod_p \left(1 - \frac{2^a}{p^{s-b-c}} + O\left(p^{2(b+c-\beta)}\right)\right)\left(1 + \frac{2^a}{p^{s-b-c}} + O\left(p^{b+c-\beta-1} + p^{2(b+c)-2\beta}\right)\right) \\
&= \prod_p \left(1 + O\left(p^{2(b+c-\beta)}\right) + O\left(p^{b+c-\beta-1}\right)\right).
\end{aligned}$$

Thus $H(s) := \zeta^{-2^a}(s-b-c)F(s)$ is analytic and bounded whenever $\beta > b+c+\frac{1}{2}$. Note that by the definition of H, $F(s) = \zeta^{2^a}(s-b-c)H(s)$. $\square$

Remark 1. A more careful calculation shows that the Dirichlet series of $d^a\sigma^b\phi^c$ is given by

$$F(s) = \zeta^{2^a}(s-b-c)\zeta^{\delta_2}(2(s-b-c))\zeta^{\delta_3}(3(s-b-c))\cdots\zeta^{\delta_k}(k(s-b-c))G(s)$$

where $G(s)$ is analytic and bounded for $\mathrm{Re}(s) > b+c+\frac{1}{k+1}$ and each δ_j is computable with

$$\begin{aligned}
\delta_2 &= 3^a - \frac{1}{2}\left(2^a + 4^a\right), \\
\delta_3 &= \frac{1}{3}\left(8^a - 2^a\right) + 4^a - 6^a, \\
&\vdots
\end{aligned}$$

We will neither use nor prove this extended result, but note that its proof follows very similarly to the proof of Lemma 5.

3.2. Analytic Proof

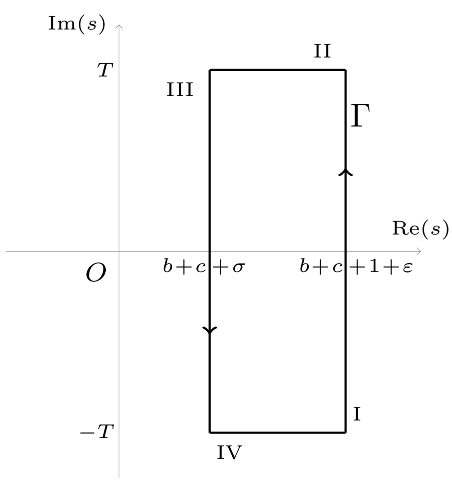
Let $f(n) := d^a(n)\sigma^b(n)\phi^c(n)$ where $2^a \in \mathbb{N}$ with $2^a \geq 4$. Assume also that $b + c > -r_a$ where r_a will be given later. Note that r_a is dependent on a and satisfies $\frac{1}{2} \leq r_a < 1$. Let $S(x) := \sum_{n=1}^{x} f(n)$. We shall proceed using Lemma 1 (Perron's Formula).

By Lemma 5, the Dirichlet series of f, given by $F(s) := \zeta^{2^a}(s - b - c)H(s)$, converges absolutely for $\mathrm{Re}(s) > b + c + 1$. Thus if $\nu = s - b - c$, we have that $F(\nu) = \zeta^{2^a}(\nu)H(\nu + b + c)$ is also the Dirichlet series of f and converges absolutely for $\mathrm{Re}(\nu) > 1$. Using well known bounds, we have $f(n) \leq Bn^{b+c+\varepsilon}$ where B is real constant depending on ε. By the Laurent series for ζ about its pole, we have $\zeta(\nu) = O\left((\nu - 1)^{-1}\right)$ as $\nu \to 1$ so $\zeta^{2^a}(\nu) = O\left((\nu - 1)^{-2^a}\right)$ as $\nu \to 1$. The function $H(\nu + b + c)$ is bounded about $\nu = 1$, so $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\mathrm{Re}(\nu)}} = \zeta^{2^a}(\nu)H(\nu + b + c) = O\left((\mathrm{Re}(\nu) - 1)^{-2^a}\right)$ as $\mathrm{Re}(\nu) \to 1^+$.

By Lemma 1,

$$
\begin{aligned}
S(x) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} &\zeta^{2^a}(\nu)H(\nu + b + c)\frac{x^{\nu+b+c}}{\nu + b + c}d\nu \\
&+ O\left(\frac{x^{1+\varepsilon}}{T(\nu - 1)^{2^a}} + \frac{xB(2x)^{b+c+\varepsilon}\log x}{T}\right) \\
= \frac{1}{2\pi i}\int_{b+c+1+\varepsilon-iT}^{b+c+1+\varepsilon+iT} &\zeta^{2^a}(s - b - c)H(s)\frac{x^s}{s}ds \\
&+ O\left(\frac{x^{1+\varepsilon}}{T} + \frac{x^{b+c+1+\varepsilon}}{T}\right).
\end{aligned} \tag{3}
$$

We now estimate the integral in (3). To do so, let σ satisfy $m(\sigma) \geq 2^a$ where the m function is defined in (1). Then consider the closed contour Γ:



$$
\begin{aligned}
\mathrm{I} &= [b+c+1+\varepsilon-iT, b+c+1+\varepsilon+iT],\\
\mathrm{II} &= [b+c+1+\varepsilon+iT, b+c+\sigma+iT],\\
\mathrm{III} &= [b+c+\sigma+iT, b+c+\sigma-iT],\\
\mathrm{IV} &= [b+c+\sigma-iT, b+c+1+\varepsilon-iT].
\end{aligned}
$$

Where $\varepsilon > 0$ and $T \in \mathbb{R}$ satisfying $T \geq 2$. Note that by its definition, σ satisfies $\frac{1}{2} < \sigma < 1$. Define

$$
\begin{aligned}
I_1 &= \int_{\mathrm{I}} \zeta^{2^a}(s-b-c)H(s)\frac{x^s}{s}ds,\\
I_2 &= \int_{\mathrm{II}} \zeta^{2^a}(s-b-c)H(s)\frac{x^s}{s}ds,\\
I_3 &= \int_{\mathrm{III}} \zeta^{2^a}(s-b-c)H(s)\frac{x^s}{s}ds,\\
I_4 &= \int_{\mathrm{IV}} \zeta^{2^a}(s-b-c)H(s)\frac{x^s}{s}ds.
\end{aligned}
$$

Then we are interested in estimating I_1. To do so, we proceed by the residue theorem. The function $\zeta^{2^a}(s-b-c)H(s)\frac{x^s}{s}$ is analytic in Γ except at $s = b+c+1$. The residue of $\zeta^{2^a}(s-b-c)H(s)\frac{x^s}{s}$ at $s = b+c+1$ is $x^{b+c+1}P_{2^a-1}(\log x)$ where P_d denotes a polynomial of degree d. Thus by the residue theorem,

$$
\begin{aligned}
\frac{1}{2\pi i}I_1 &= \frac{1}{2\pi i}\int_{\Gamma} \zeta^{2^a}(s-b-c)H(s)\frac{x^s}{s}ds - \frac{1}{2\pi i}\left(I_2+I_3+I_4\right)\\
&= x^{b+c+1}P_{2^a-1}(\log x) - \frac{1}{2\pi i}\left(I_2+I_3+I_4\right). \qquad (4)
\end{aligned}
$$

In order to obtain an error term for I_1, we now bound the integrals I_2, I_3, and I_4.

We'll start with I_2 and I_4. Recall that σ was defined so that $m(\sigma) \geq 2^a$. By the the remarks made in Section 2, we have $\mu(\sigma) \leq \frac{1}{m(\sigma)} \leq 2^{-a}$ where μ is defined in Section 2. Thus

$$
\begin{aligned}
I_2 + I_4 &\ll \int_{\sigma}^{1+\varepsilon} \left|\zeta^{2^a}(\beta+iT)H(b+c+\beta+iT)\right| \frac{x^{b+c+\beta}}{|b+c+\beta+iT|}d\beta\\
&\ll x^{b+c}\int_{\sigma}^{1+\varepsilon} \left|\zeta^{2^a}(\beta+iT)\right| \frac{x^{\beta}}{T}d\beta\\
&\ll x^{b+c} \max_{\sigma\leq\beta\leq 1+\varepsilon} x^{\beta}T^{2^a\mu(\beta)-1+\varepsilon}\\
&\ll x^{b+c}\left(x^{\sigma}T^{2^a\mu(\sigma)-1+\varepsilon} + x^{1+\varepsilon}T^{2^a\mu(1+\varepsilon)-1+\varepsilon}\right)\\
&\ll x^{b+c}\left(x^{\sigma}T^{\varepsilon} + x^{1+\varepsilon}T^{-1+\varepsilon}\right)\\
&\ll \frac{x^{b+c+1+\varepsilon}}{T}. \qquad (5)
\end{aligned}
$$

Now, we bound I_3. We first split this integral to avoid issues of convergence near $s=0$. Then by integration by parts and Lemma 2, we have

$$\begin{aligned}
I_3 &\ll \int_1^T \left|\zeta^{2^a}(\sigma+it)\right| \frac{x^{b+c+\sigma}}{|b+c+\sigma+it|}dt + \int_0^1 \left|\zeta^{2^a}(\sigma+it)\right| \frac{x^{b+c+\sigma}}{|b+c+\sigma+it|}dt \\
&\ll x^{b+c+\sigma} \int_1^T \left|\zeta^{2^a}(\sigma+it)\right| \frac{1}{t}dt \\
&\ll \frac{x^{b+c+\sigma}}{T} \int_1^T \left|\zeta^{2^a}(\sigma+it)\right| dt \\
&\ll x^{b+c+\sigma}T^{\varepsilon} \\
&\ll x^{b+c+\sigma+\varepsilon}. \qquad (6)
\end{aligned}$$

Substituting (5) and (6) into (4),

$$\frac{1}{2\pi i}I_1 = x^{b+c+1}P_{2^a-1}(\log x) + O\left(\frac{x^{b+c+1+\varepsilon}}{T}\right) + O\left(x^{b+c+\sigma+\varepsilon}\right). \qquad (7)$$

Now substituting (7) into (3),

$$S(x) = x^{b+c+1}P_{2^a-1}(\log x) + O\left(x^{b+c+\sigma+\varepsilon}\right) + O\left(\frac{x^{1+\varepsilon}}{T} + \frac{x^{b+c+1+\varepsilon}}{T}\right).$$

And letting $T = x$,

$$S(x) = x^{b+c+1}P_{2^a-1}(\log x) + O\left(x^{b+c+\sigma+\varepsilon}\right).$$

Recall that we chose σ so that $m(\sigma) \geq 2^a$. In order to get the least error term, we want to choose the least σ such that $m(\sigma) \geq 2^a$. For this, Lemma 2 gives

$$\sigma = \begin{cases}
\frac{1}{2}+\delta & \text{for } a=2 \\
\frac{3}{4}-2^{-a} & \text{for } 5 \leq 2^a \leq 8 \\
\frac{35}{54} & \text{for } 2^a = 9 \\
\frac{41}{60} & \text{for } 2^a = 10 \\
\frac{859}{948} - \frac{176}{79}\cdot 2^{-a} & \text{for } 11 \leq 2^a \leq 14 \\
\frac{4537}{4890} - \frac{2068}{815}\cdot 2^{-a} & \text{for } 15 \leq 2^a \leq 26 \\
\frac{1031}{1044} - \frac{1081}{261}\cdot 2^{-a} & \text{for } 27 \leq 2^a \leq 36 \\
\frac{31}{32} - \frac{49}{16}\cdot 2^{-a} & \text{for } 37 \leq 2^a \leq 57 \\
\frac{5}{8} - 3\cdot 2^{-a} + 2^{-3-a}\sqrt{576 - 3\cdot 2^{5+a} + 9\cdot 2^{2a}} & \text{for } 2^a \geq 58.
\end{cases} \qquad (8)$$

Where $\delta > 0$ may be arbitrarily small for $a = 2$. Now r_a may be chosen in accordance with σ to give Theorem 1.

3.3. Elementary Proof for General Case

Recall that by Lemma 5 the Dirichlet series of $d^a(n)\sigma^b(n)\phi^c(n)$ is

$$F(s) = \zeta^{2^a}(s-b-c)H(s)$$

where $H(s)$ is analytic and bounded when $\mathrm{Re}(s) > b+c+\frac{1}{2}$ and $b, c \in \mathbb{R}$.

Let

$$H(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

when $\mathrm{Re}(s) > b+c+\frac{1}{2}$ and

$$\zeta^z(s) = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^s}$$

when $\mathrm{Re}(s) > 1$. Therefore,

$$\zeta^{2^a}(s-b-c) = \sum_{n=1}^{\infty} \frac{d_{2^a}(n)}{n^{s-b-c}} = \sum_{n=1}^{\infty} \frac{d_{2^a}(n)n^{b+c}}{n^s}.$$

Letting $k(n) := d_{2^a}(n)n^{b+c}$, we have

$$d^a(n)\sigma^b(n)\phi^c(n) = (g*k)(n).$$

Let

$$S(x) := \sum_{n\le x} d^a(n)\sigma^b(n)\phi^c(n) = \sum_{n\le x}(g*k)(n).$$

Then,

$$\begin{aligned} S(x) = \sum_{n\le x}\sum_{m|n} g(m)k\left(\frac{n}{m}\right) = \sum_{qm\le x} g(m)k(q) &= \sum_{m\le x} g(m) \sum_{q\le \frac{x}{m}} k(q) \\ &= \sum_{m\le x} g(m)K\left(\frac{x}{m}\right). \end{aligned}$$

`Case 1.` $b+c > -1, 2^a \notin \mathbb{N}$.

We begin by splitting $S(x)$ into $S_1(x)$ and $S_2(x)$ where N is an arbitrary but fixed integer.

$$\begin{aligned} S(x) &= \sum_{m\le x} g(m)K\left(\frac{x}{m}\right) \\ &= \sum_{m\le (\log x)^{4N}} g(m)K\left(\frac{x}{m}\right) + \sum_{(\log x)^{4N} < m\le x} g(m)K\left(\frac{x}{m}\right) \\ &= S_1(x) + S_2(x). \end{aligned}$$

From Lemma 4 and by partial summation, we have

$$
\begin{aligned}
K(x) = \sum_{n\le x} k(n) &= \int_1^x t^{b+c} d\left(\sum_{n\le t} d_{2^a}(n)\right) \\
&= t^{b+c} \sum_{n\le t} d_{2^a}(n)\Big|_1^x - \int_1^x \sum_{n\le t} d_{2^a}(n)(b+c)t^{b+c-1}dt \\
&= \left(\sum_{j=1}^{N} c_j (\log x)^{2^a-j}\right) x^{b+c+1} + O\left(x^{b+c+1}(\log x)^{2^a-N-1}\right). \qquad (9)
\end{aligned}
$$

Substituting (9) into $S_1(x)$,

$$
\begin{aligned}
S_1(x) = &\sum_{m\le(\log x)^{4N}} g(m)\left(\sum_{j=1}^{N} c_j\left(\log\frac{x}{m}\right)^{2^a-j}\right)\left(\frac{x}{m}\right)^{b+c+1} \\
&+ O\left(\sum_{m\le(\log x)^{4N}} |g(m)|\left(\frac{x}{m}\right)^{b+c+1}(\log x)^{2^a-N-1}\right) \\
= &\, x^{b+c+1}\sum_{j=1}^{N} c_j \sum_{m\le(\log x)^{4N}} \frac{g(m)}{m^{b+c+1}}\left(\log\frac{x}{m}\right)^{2^a-j} + O\left(x^{b+c+1}(\log x)^{2^a-N-1}\right).
\end{aligned}
$$

Note that

$$
\sum_{m\le(\log x)^{4N}} \frac{g(m)}{m^{b+c+1}}\left(\log\frac{x}{m}\right)^{2^a-j} = \sum_{m\le(\log x)^{4N}} \frac{g(m)}{m^{b+c+1}}(\log x)^{2^a-j}\left(1-\frac{\log m}{\log x}\right)^{2^a-j}.
$$

Using the Taylor expansion of $\left(1-\frac{\log m}{\log x}\right)^{2^a-j}$ we have

$$
\begin{aligned}
&\sum_{m\le(\log x)^{4N}} \frac{g(m)}{m^{b+c+1}}(\log x)^{2^a-j}\left(1-\frac{\log m}{\log x}\right)^{2^a-j} \\
&\qquad = (\log x)^{2^a-j}\sum_{m\le(\log x)^{4N}} \frac{g(m)}{m^{b+c+1}}\sum_{h=0}^{\infty} a_{j,h}\left(\frac{\log m}{\log x}\right)^h \\
&\qquad = (\log x)^{2^a-j}(b_{j,0}+b_{j,1}(\log x)^{-1}+b_{j,2}(\log x)^{-2}+\ldots).
\end{aligned}
$$

Because $H(s)$ is analytic and bounded for $\mathrm{Re}(s) > b+c+\frac{1}{2}$, the error term of $S_1(x)$ becomes $O(x^{b+c+1}(\log x)^{2^a-N-1})$.
Therefore,

$$
S_1(x) = x^{b+c+1}\sum_{j=1}^{N} b_j(\log x)^{2^a-j} + O\left(x^{b+c+1}(\log x)^{2^a-N-1}\right)
$$

where b_j is a combination of $\log m$ and the constants $a_{j,h}$ and c_j.

For $S_2(x)$ we trivially have

$$S_2(x) \ll \sum_{(\log x)^{4N} < m \leq x} |g(m)| \left(\frac{x}{m}\right)^{b+c+1} (\log x)^{2^a - 1}.$$

Thus by partial summation we have,

$$\begin{aligned}
S_2(x) &\ll x^{b+c+1} (\log x)^{2^a-1} \sum_{(\log x)^{4N} < m \leq x} \frac{|g(m)|}{m^{b+c+\frac{3}{4}}} \frac{1}{m^{\frac{1}{4}}} \\
&\ll x^{b+c+1} (\log x)^{2^a-N-1} \int_{(\log x)^{4N}}^{x} t^{-\frac{1}{4}} d \left(\sum_{(\log x)^{4N} < m \leq x} \frac{|g(m)|}{m^{b+c+\frac{3}{4}}} \right) \\
&\ll x^{b+c+1} (\log x)^{2^a-N-1} \underbrace{\sum_{m=1}^{\infty} \frac{|g(m)|}{m^{b+c+\frac{3}{4}}}}_{\ll 1} \\
&\ll x^{b+c+1} (\log x)^{2^a-N-1}.
\end{aligned}$$

Therefore, when $2^a \notin \mathbb{N}$,

$$S(x) = x^{b+c+1} \sum_{j=1}^{N} b_j (\log x)^{2^a-j} + O\left(x^{b+c+1} (\log x)^{2^a-N-1}\right).$$

`Case 2.` $b+c > -\alpha_k + \varepsilon > -1, 2^a = k \in \mathbb{N}, k \neq 2, 3$ where α_k is defined by Lemma 3

Once again,

$$S(x) = \sum_{m \leq x} g(m) K\left(\frac{x}{m}\right).$$

From Lemma 3, we have

$$K(x) = x^{b+c+1} P_{k-1}(\log x) + O(x^{b+c+\alpha_k+\varepsilon}).$$

We can then say

$$\begin{aligned}
S(x) &= \sum_{m \leq x} g(m) \left[\left(\frac{x}{m}\right)^{b+c+1} P_{k-1}\left(\log \frac{x}{m}\right) + O\left(\left(\frac{x}{m}\right)^{b+c+\alpha_k+\varepsilon}\right) \right] \\
&= x^{b+c+1} \sum_{m \leq x} \frac{g(m)}{m^b + c + 1} P_{k-1}(\log x - \log m) \\
&\qquad + O\left(x^{b+c+\alpha_k+\varepsilon} \sum_{m \leq x} \frac{|g(m)|}{m^{b+c+\alpha_k+\varepsilon}} \right).
\end{aligned}$$

Then rearranging $P_{k-1}(\log x - \log m)$ into a summation of $\log x$ and a polynomial of $\log m$ of degree j we have,

$$\begin{aligned} S(x) &= x^{b+c+1} \sum_{m \le x} \frac{g(m)}{m^{b+c+1}} \sum_{j=0}^{k-1} (\log x)^j Q_j(\log m) + O(x^{b+c+\alpha_k+\varepsilon}) \\ &= x^{b+c+1} T_{k-1}(\log x) + O(x^{b+c+\alpha_k+\varepsilon}) \end{aligned}$$

where $T_{k-1}(t)$ is a polynomial of degree $k-1$.

`Special Case:  a = 1.`

Let $f(s) := \sum_{n=1}^{\infty} \frac{d(n)\sigma^b(n)\phi^c(n)}{n^s} = \prod_p f_p(s)$ for $\mathrm{Re}(s) > b+c+1$.

Expanding the product term by term and plugging in the values for d, σ, and ϕ at prime powers we have

$$\begin{aligned} f_p(s) &= 1 + \frac{2(p+1)^b(p-1)^c}{p^s} + \frac{3(p^2+p+1)^b(p^2-p)^c}{p^{2s}} + \dots \\ &= 1 + \frac{2}{p^{s-b-c}} \left(1+\frac{1}{p}\right)^b \left(1-\frac{1}{p}\right)^c + \frac{3}{p^{2(s-b-c)}} \left(1+\frac{1}{p}+\frac{1}{p^2}\right)^b \left(1-\frac{1}{p}\right)^c + \cdots \\ &= 1 + \frac{2}{p^{s-b-c}} + \frac{3}{p^{2(s-b-c)}} + \frac{4}{p^{3(s-b-c)}} + \dots + O(p^{b+c-\beta-1}) \\ &= \left(1 + \frac{2}{p^{s-b-c}} + \frac{3}{p^{2(s-b-c)}} + \dots\right) \left(1 + O(p^{b+c-\beta-1}\right). \end{aligned}$$

Therefore,

$$f(s) = \zeta^2(s-b-c)H(s)$$

where $H(s)$ is analytic and absolutely bounded when $\mathrm{Re}(s) > b+c$.

Now as before let $d(n)\sigma^b(n)\phi^c(n) = (k*g)(n)$ where $k(n) = d(n)n^{b+c}$, and notice that $\sum_{n=1}^{\infty} \frac{|g(n)|}{n^\sigma}$ converges when $\sigma > b+c$.

By Abel summation, we have

$$\begin{aligned} K(x) &:= \sum_{n \le x} k(n) = \sum_{n \le x} d(n) n^{b+c} \\ &= t^{b+c} \sum_{n \le x} d(n) \Big|_1^x - \int_1^x \left(\sum_{n \le t} d(n) \right) (b+c) t^{b+c-1} dt \end{aligned}$$

Recall $\sum_{n \le x} d(n) = x(\log x + 2\gamma - 1) + O(x^\theta)$ for some $\theta < \frac{1}{3}$. Using this and integration by parts K becomes

$$K(x) = \frac{x^{b+c+1}}{b+c+1} \left(\log x + 2\gamma - \frac{1}{b+c+1} \right) + O(x^{b+c+\theta} + 1). \tag{10}$$

In particular, the error is $O(x^{b+c+\theta})$ if $b+c > -\theta$.

Letting $S(x) := \sum_{n\le x} d(n)\sigma^b(n)\phi^c(n) = \sum_{n\le x}(k*g)(n)$ and using convolution as before, we have $S(x) = \sum_{m\le x} g(m)K\left(\frac{x}{m}\right)$. Substituting (10) into $S(x)$ gives us

$$S(x) = \sum_{m\le x} g(m)\left[\frac{\left(\frac{x}{m}\right)^{b+c+1}}{b+c+1}\left(\log\frac{x}{m}+2\gamma-\frac{1}{b+c+1}\right)+O\left(\left(\frac{x}{m}\right)^{b+c+\theta}+1\right)\right].$$

Extending the summations to infinity we have,

$$S(x) = \frac{x^{b+c+1}}{b+c+1}\left[\left(\sum_{m=1}^{\infty}\frac{g(m)}{m^{b+c+1}}\right)\left(\log x+2\gamma-\frac{1}{b+c+1}\right)-\sum_{m=1}^{\infty}\frac{g(m)\log m}{m^{b+c+1}}\right]+\Delta(x)$$

where

$$\Delta(x) \ll \sum_{m\le x}|g(m)|\left(\left(\frac{x}{m}\right)^{b+c+\theta}+1\right)+\sum_{m>x}|g(m)|\left(\frac{x}{m}\right)^{b+c+1}\log x$$

is the error term.

We can bound Δ as follows:

$$\begin{aligned}\Delta(x) &= \sum_{m\le x}|g(m)|\left(\left(\frac{x}{m}\right)^{b+c+\theta}+1\right)+\sum_{m>x}|g(m)|\underbrace{\left(\frac{x}{m}\right)^{b+c+1}}_{\ll\left(\frac{x}{m}\right)^{b+c+\varepsilon}\text{if } m>x}\log x\\ &\ll x^{b+c+\theta}\underbrace{\sum_{m\le x}\frac{|g(m)|}{m^{b+c+\theta}}}_{\ll 1}+\underbrace{\sum_{m\le x}|g(m)|}_{\ll x^{b+c+\varepsilon}\sum_{m\le x}\frac{|g(m)|}{m^{b+c+\varepsilon}}}+x^{b+c+\varepsilon}\underbrace{\sum_{m>x}\frac{|g(m)|}{m^{b+c+\varepsilon}}}_{\ll 1}\\ &\ll x^{b+c+\theta}.\end{aligned}$$

Thus, for $a=1$,

$$S(x) = \frac{x^{b+c+1}}{b+c+1}\left[\left(\sum_{m=1}^{\infty}\frac{g(m)}{m^{b+c+1}}\right)\left(\log x+2\gamma-\frac{1}{b+c+1}\right)-\sum_{m=1}^{\infty}\frac{g(m)\log m}{m^{b+c+1}}\right]+O(x^{b+c+\theta})$$

as desired.

We now conclude with some remarks regarding the elementary approach.

Remark 2. The case for $a=1$ is likely the only case that can be improved to an error below $O(x^{b+c+\frac{1}{2}})$ because it is the only case that can be factored into a power of zeta and a function $H(s)$ that is analytic for $\mathrm{Re}(s) > b+c$. In other cases, the zeta term contains a $\zeta(2(s-b-c))$ and therefore the $\mathrm{Re}(s) > b+c+\frac{1}{2}$ barrier cannot be broken.

Remark 3. We provide restrictions on $b+c$ for formally simpler proofs. There is little technical difficulty in extending the above results for other ranges of $b+c$. The following hold:

1. If $b+c<-1$ and $a \in \mathbb{R}$, then
$$S(x) = C + O\left(x^{b+c+1} (\log x)^{2^a-1}\right)$$
where $C \in \mathbb{R}$ is a constant.

2. If $2^a \in \mathbb{N}$ and $-1 < b+c \leq \alpha_k$, then
$$S(x) = x^{b+c+1} P_{2^a-1}(x) + C + O\left(x^{b+c+\alpha_k+\varepsilon}\right)$$
where P_d is a degree d polynomial and α_k's are defined by Lemma 3.

3. (a) If $b+c=-1$ and $a \in \mathbb{R}$ satisfies $a \geq 0$, then
$$S(x) = \sum_{j=0}^{N} A_j (\log x)^{2^a-j} + B + O\left((\log x)^{2^a-N-1}\right)$$
where $N = \lfloor 2^a \rfloor$ and $B, A_j \in \mathbb{R}$ are a constants.

   (b) If $b+c=-1$ and $2^a = k \in \mathbb{N}$, then
$$S(x) = \sum_{j=0}^{k} A_j (\log x)^{k-j} + O\left(x^{\alpha_k-1+\varepsilon}\right)$$
where $A_j \in \mathbb{R}$ are constants and α_k's are defined by Lemma 3.

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