ARITHMETIC OF 3^t-CORE PARTITION FUNCTIONS

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Abstract

Let $t \geq 1$ be an integer and $a_{3^t}(n)$ be the number of 3^t-cores of n. We prove a class of congruences for $a_{3^t}(n) \bmod 3$ by Hecke nilpotence.

1. Introduction

If t is a positive integer, let $a_t(n)$ be the the number of t-cores of n, that is, the number of partitions of n with no hooks of length divisible by t. Then, as shown by Garvan, Kim and Stanton [4], the generating function for $a_t(n)$ is given by the following infinite product:

$$\sum_{n=0}^{\infty} a_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{nt})^t}{(1-q^n)}. \tag{1.1}$$

Some congruence properties were established for small t. See for example [5, 6, 9].

For t a power of 2, Hirschhorn and Sellers [5] made the following conjecture:

Conjecture 1. *If t and n are positive integers with $t \geq 2$, then for $k = 0, 2$,*

$$a_{2^t}\left(\frac{3^{2^{t-1}-1}(24n+8k+7) - \frac{4^t-1}{3}}{8}\right) \equiv 0 (\bmod 2).$$

Using Hecke nilpotence [13], Boylan [2] made some progress on the above conjecture. In 2012, Nicolas and Serre [11] determined the structure of Hecke rings modulo 2 and sharpened the degree of nilpotence of the mod 2 Hecke algebras. In [3], Chen confirmed Hirschhorn and Sellers's conjecture with the help of Nicolas and Serre's result.

Recently, motivated by the work of Nicolas and Serre [11], Bellaïche and Khare [1] studied the structure of the Hecke algebras of modular forms modulo p for all primes p, extending the results of Nicolas and Serre for $p = 2$. In particular, in the appendix of [1], Bellaïche and Khare explicitly determined the upper bound of the degree of nilpotence of Hecke algebras modulo 3. It is obvious that we can use their results to study 3^t-core partition functions.

Theorem 1. *Let l be an integer such that $l \geq \frac{4^t-1}{3}$. Then for any distinct primes $\ell_1, \ldots, \ell_l$ which are congruent to $2 \pmod 3$, we have*

$$a_{3^t}\left(\frac{\ell_1 \cdots \ell_l n - \frac{9^t-1}{8}}{3}\right) \equiv 0 \pmod 3$$

for all n coprime to $\ell_1 \cdots \ell_l$.

The paper is laid out as follows. In Section 2, we recall the results on nilpotence of Hecke algebras for $p = 3$. In Section 3, we prove Theorem 1. In Section 4, we give a result on $a_{3^t}(n)$ modulo powers of 3. Throughout the paper, we put $a_t(\alpha) = 0$ if $\alpha \notin \mathbb{N}$.

2. Hecke Nilpotence

The proof of Theorem 1 relies on Hecke nilpotence of modular forms. We recall some facts on modular forms (see [8] for more). For integers $k > 0$, we denote by $\mathcal{S}_k$ the space of cusp forms of weight k with integer coefficients on $SL_2(\mathbb{Z})$. Moreover, let $\mathcal{S}_k \pmod p$ denote the modular forms in $\mathcal{S}_k$ with integer coefficients, reduced modulo p, where p is a prime number. Throughout this paper, $p = 3$. As usual, Ramanujan's Δ function is

$$\Delta(z) = \eta(z)^{24} = q \prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n \in \mathcal{S}_{12},$$

where $q = e^{2\pi i z}$, and z is on the upper half of the complex plane. Let ℓ be a prime $\neq p$. If $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in \mathcal{S}_k$, then the action of the Hecke operator $T_{\ell,k}$ on $f(z) \pmod p$ is defined by

$$f(z)|T_{\ell,k} = \sum_{n=1}^{\infty} c(n) q^n,$$

where

$$c(n) = \begin{cases} a(\ell n), & \text{if } \ell \nmid n, \\ a(\ell n) + \ell^{k-1} a(n/\ell), & \text{if } \ell | n. \end{cases} \tag{1}$$

Based on the work Bellaïche and Khare [1], it is known that the action of Hecke algebras on the spaces of modular forms modulo 3 is locally nilpotent. Let

$$T'_\ell = \begin{cases} T_\ell, & \text{if } \ell \equiv 2 \pmod 3, \\ 1 + T_\ell, & \text{if } \ell \equiv 1 \pmod 3. \end{cases}$$

If $f(z) \in \mathcal{S}_k$, then there is a positive integer i with the property that

$$f(z)|T'_{\ell_1}|T'_{\ell_2} \cdots |T'_{\ell_i} \equiv 0 \pmod 3$$

for every collection of primes $\ell_1, \ldots, \ell_i$, where $\ell_j \neq p$ for $j = 1, \ldots, i$. Suppose that $f(z) \not\equiv 0 (\bmod\, p)$. We say that $f(z)$ has degree of nilpotence i if there exist primes $\ell_1, \ldots, \ell_{i-1}$ such that $\ell_j \neq 3$ for $j = 1, \ldots, i-1$ for which

$$f(z)|T'_{\ell_1}|T'_{\ell_2} \cdots |T'_{\ell_{i-1}} \not\equiv 0 (\bmod\, 3)$$

and every collection of primes $p_1, \ldots, p_i$, such that $p_j \neq 3$ for $j = 1, \ldots, i$ for which

$$f(z)|T'_{p_1}|T'_{p_2} \cdots |T'_{p_i} \equiv 0 (\bmod\, 3). \tag{2.2}$$

We denote by $g_k(3)$ the degree of nilpotnece of $\Delta^k(z) (\bmod\, 3)$. To obtain congruences for $a_{3^t}(n)$, we need the upper bound for $g_k(3)$.

Theorem 2 (Bellaïche and Khare [1]). *For any positive integer $k = \sum_{i=0}^{r} a_i 3^i$, with $a_i \in \{0, 1, 2\}, a_r \neq 0$, we have $g_k(3) \leq \sum_{i=0}^{r} a_i 2^i$.*

Corollary 1. *If $k = \frac{9^t-1}{8}$, we have $g_{\frac{9^t-1}{8}}(3) \leq \frac{4^t-1}{3}$.*

Proof. This is a consequence of Theorem 2 since $k = \frac{9^t-1}{8} = 1 + 3^2 + \cdots + 3^{2t-2}$. □

3. Proof of Theorem 1

We now prove Theorem 1.

Proof of Theorem 1. By the definition of $a_t(n)$, we have

$$\sum_{n=0}^{\infty} a_{3^t}(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{3^t n})^{3^t}}{(1-q^n)} \equiv \prod_{n=1}^{\infty} (1-q^n)^{9^t-1} \, (\bmod\, 3).$$

From the definition of $\Delta(z)$, it follows that:

$$\sum_{n=0}^{\infty} a_{3^t}\left(\frac{n - \frac{9^t-1}{8}}{3}\right) q^n \equiv \sum_{n=0}^{\infty} a_{3^t}(n) q^{3n + \frac{9^t-1}{8}} \equiv \Delta^{\frac{9^t-1}{8}}(z) \, (\bmod\, 3). \tag{3.1}$$

Now by (2.1), the definition of T'_ℓ and Corollary 1, the proof of Theorem 1 is immediate. □

Example 1. If $t = 1$, then $g_1(3) = 1$. Theorem 1 asserts that for primes $\ell \equiv 2 (\bmod\, 3)$, then

$$a_3\left(\frac{\ell n - 1}{3}\right) \equiv 0 \, (\bmod\, 3)$$

for all n coprime to ℓ. In particular, if $\ell \neq 2$, we choose $n = 3\ell m + 2$, then

$$a_3\left(\ell^2 m + \frac{2\ell - 1}{3}\right) \equiv 0 \, (\bmod\, 3) \tag{3.2}$$

for all $m \geq 0$. For $\ell = 2$, we can choose $n = 6m+5$, then

$$a_3(4m+3) \equiv 0 \pmod 3 \tag{3.3}$$

for all $m \geq 0$. In fact, in [7, Cor. 9], we know that $a_3(n)$ are zero in (3.2) and (3.3). So our results are trivial.

Example 2. If $t = 2$, then $g_{10}(3) = 5$. If $\ell_i \equiv 2 \pmod 3$ for distinct primes ℓ_i, then we have

$$a_9\left(\frac{\ell_1\ell_2\ell_3\ell_4\ell_5 n - 10}{3}\right) \equiv 0 \pmod 3$$

for all n coprime to $\ell_1\ell_2\ell_3\ell_4\ell_5$. Suppose we choose $\ell_1 = 2, \ell_2 = 5, \ell_3 = 11, \ell_4 = 17, \ell_5 = 23$, then we know

$$a_9\left(\frac{43010n - 10}{3}\right) \equiv 0 \pmod 3$$

for all n such that $(43010, n) = 1$. Actually, since $\ell_4 = 17 \equiv 8 \pmod 9$, from Theorem 24 and Lemma 35 in [1] we know that $g_3(T'_{17}f) \leq g_3(f) - 2$. So we have

$$a_9\left(\frac{1870n - 10}{3}\right) \equiv 0 \pmod 3$$

for all n coprime to 1870.

4. Further Remarks

In [10], Moon and Taguchi proved the following theorem.

Theorem 3. *Let $k \geq 1$ be a positive integer. Let $\varepsilon : (\mathbb{Z}/3^a \cdot 4\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Then there exist integers $c \geq 0$ and $e \geq 1$, depending on k, a and ε such that for any modular form $f(z) = \sum_{n=0}^\infty a(n)q^n \in \mathcal{M}_k(\Gamma_0(3^a \cdot 4), \varepsilon; \mathbb{Z})$, any integer $j \geq 1$, and any $c + ej$ primes $p_1, p_2, \ldots, p_{c+ej} \equiv -1 \pmod{12}$, we have*

$$f(z)|T_{p_1}|T_{p_2}|\cdots|T_{p_{c+ej}} \equiv 0 \pmod{3^j}.$$

Furthermore, if the primes $p_1, p_2, \ldots, p_{c+ej}$ are distinct, then for any positive integer m coprime to $p_1, p_2, \ldots, p_{c+ej}$, we have

$$a(p_1p_2\cdots p_{c+ej}m) \equiv 0 \pmod{3^j}.$$

Since it is easy to see that [12, Theorem 1.64]

$$\sum_{n=0}^{\infty} a_{3^t}(n) q^{3n+\frac{9^t-1}{8}} = \frac{\eta(3^{t+1}z)^{3^t}}{\eta(3z)}$$

belongs to $\mathcal{M}_{\frac{3^t-1}{2}}(\Gamma_0(3^{t+1}),\chi)$ where $\chi(d)=\left(\frac{(-1)^{\frac{3^t-1}{2}}3^t}{d}\right)$, we have the following theorem.

Theorem 4. *There exist integers $c\geq 0$ and $e\geq 1$, depending on t such that for any positive integer j and any distinct $c+ej$ primes $p_1,p_2,\ldots,p_{c+ej}\equiv -1(\bmod 12)$, we have*

$$a_{3^t}\left(\frac{p_1p_2\cdots p_{c+ej}n-\frac{9^t-1}{8}}{3}\right)\equiv 0(\bmod 3^j)$$

for any positive integer n coprime to $p_1,p_2,\ldots,p_{c+ej}$.

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