FINITE RECIPROCAL SUMS IN WHICH THE DENOMINATOR OF THE SUMMAND CONTAINS SQUARES OF GENERALIZED FIBONACCI NUMBERS

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Abstract

In this paper, we present closed forms for finite reciprocal sums that involve generalized Fibonacci numbers. In each case, the denominator of the summand is a product of generalized Fibonacci numbers with at least one squared term.

1. Introduction

In [2], [3], [4], and [5], we consider finite reciprocal sums involving generalized Fibonacci numbers. Indeed we give closed forms, in terms of rational numbers, for these sums. In the aforementioned papers, the denominator of the summand of each finite sum consists of a product of generalized Fibonacci numbers. In this paper, the denominator of the summand of each finite sum that we consider is a product of generalized Fibonacci numbers with at least one squared term.

Closed forms for reciprocal sums in which the denominator of the summand is a product that contains squares of Fibonacci (Lucas) numbers seem to be rare in the literature. For instances of such sums in which the upper limit of summation is infinite, we refer the reader to Brousseau[1], formulas 6, 7, 12, and 15. Formula 20 of Brousseau does not make sense due to a vanishing denominator. At the end of Section 2, we indicate that formulas 6, 7, 12, and 15 of Brousseau follow from two of our results.

We begin by introducing the three pairs of integer sequences that feature in this paper. Let $a \geq 0$ and $b \geq 0$ be integers with $(a, b) \neq (0, 0)$. For p a positive integer, we define, for all integers n, the sequences $\{W_n\}$ and $\{\overline{W}_n\}$ by

$$W_n = pW_{n-1} + W_{n-2}, \quad W_0 = a, \quad W_1 = b,$$

and

$$\overline{W}_n = W_{n-1} + W_{n+1}.$$

For $(a, b, p) = (0, 1, 1)$, we have $\{W_n\}$=$\{F_n\}$, and $\{\overline{W}_n\}$=$\{L_n\}$, which are the Fibonacci and Lucas numbers, respectively. Retaining the parameter p, and taking $(a, b) = (0, 1)$, we write $\{W_n\}$=$\{U_n\}$, and $\{\overline{W}_n\}$=$\{V_n\}$, which are integer sequences that generalize the Fibonacci and Lucas numbers, respectively.

Let α and β denote the two distinct real roots of $x^2 - px - 1 = 0$. Set $A = b - a\beta$ and $B = b - a\alpha$. Then the closed forms (the Binet forms) for $\{W_n\}$ and $\{\overline{W}_n\}$ are, respectively,

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

and

$$\overline{W}_n = A\alpha^n + B\beta^n.$$

Set $\Delta = p^2 + 4$. We require also the constant $e_W = AB = b^2 - pab - a^2$. In this regard, note that $e_U = 1$, and $e_V = -\Delta$.

Throughout this paper, $d \geq 1$, $k \geq 1$, $m \geq 0$, and $n \geq 2$ are assumed to be integers. In Section 2, we give a closed form for the finite sum

$$S_2(d, k, m, n) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} U_{k(2i+d)+2m}}{U_{ki+m}^2 U_{k(i+d)+m}^2},$$

with an accompanying *dual* result.

All of the finite sums considered in this paper are evaluated in terms of rational numbers. Furthermore, each finite sum is categorized according to the number of distinct terms in the denominator of its summand. For instance, S_2 has two distinct terms in the denominator of its summand. The scope of this paper is limited as follows: each finite sum that is considered has at most five distinct factors in the denominator of its summand. Furthermore, as stated earlier, each finite sum has at least one squared term in the denominator of its summand. Not all the results that we have discovered are presented. Instead, we have chosen to present a selection that demonstrates the kind of results that are possible.

There are two finite sums that feature throughout. For integers $0 \leq l_1 < l_2$, these finite sums are

$$\Phi_W(k, m, n, l_1, l_2) = \frac{(-1)^{k+m}}{e_W U_k} \sum_{i=l_1}^{l_2-1} \left(\frac{W_{k(i+n-1)+m}^2}{W_{k(i+n)+m}^2} - \frac{W_{k(i+1)+m}^2}{W_{k(i+2)+m}^2} \right),$$

and

$$\Psi_W(k, m, n, l_1, l_2) = \sum_{i=l_1}^{l_2-1} \frac{(-1)^{ki}}{W_{k(i+2)+m} W_{k(i+n)+m}}.$$

If, for instance, in the definition of $\Phi_W(k,m,n,l_1,l_2)$ we replace each occurrence of W by U, we denote the resulting sum by $\Phi_U(k,m,n,l_1,l_2)$.

We suppress certain arguments from quantities when there is no danger of confusion. For instance, $S_2(n)$ will denote $S_2(d,k,m,n)$ when we want n to vary, and the other parameters to remain fixed.

We now give two identities, involving Φ_W and Ψ_W, that are required for the proofs of all the theorems in this paper. We state these identities as lemmas in which, for instance (with the notation agreed upon in the previous paragraph), $\Phi_W(n)$ means $\Phi_W(k,m,n,l_1,l_2)$.

Lemma 1. *With Φ_W as defined above,*

$$\Phi_W(n+1)-\Phi_W(n)=\frac{(-1)^{k+m}}{e_W U_k}\left(\frac{W^2_{k(n+l_2-1)+m}}{W^2_{k(n+l_2)+m}}-\frac{W^2_{k(n+l_1-1)+m}}{W^2_{k(n+l_1)+m}}\right).$$

Lemma 2. *With Ψ_W as defined above,*

$$U_{k(n-1)}\Psi_W(n+1)-U_{k(n-2)}\Psi_W(n)=\frac{(-1)^{k(n+l_1)}U_{k(l_2-l_1)}}{W_{k(n+l_1)+m}W_{k(n+l_2)+m}}.$$

In Lemma 1, the right side arises from a simple telescoping sum. Lemma 2 is proved in [3, Sec. 9], where it occurs as Lemma 1.

The method of proof that can be employed to prove each of our results is mechanical and transparent. We demonstrate this method of proof in Section 3, where we use it to prove Theorem 2.

2. A Closed Form for S_2 and a Dual Result

We remind the reader that, throughout this paper, $d\geq 1$, $k\geq 1$, $m\geq 0$, and $n\geq 2$ are taken to be integers.

Theorem 1. *With S_2 as defined in Section 1,*

$$\begin{aligned}U_kU_{dk}\left(S_2(n)-S_2(2)\right)&=(-1)^{k+1}\Phi_U(k,m,n,0,d)\\&\quad+V_kU_{k(n-2)}\Psi_U(k,m,n,0,d).\end{aligned}\tag{1}$$

Theorem 1 has a dual result that we conveniently state as follows: redefine S_2 by replacing each occurrence of U in the denominator of the summand by V. Then (1) remains valid provided we multiply the left side by Δ, and replace Φ_U and Ψ_U by Φ_V and Ψ_V, respectively.

Formula 6 of Brousseau [1], to which we refer in the introduction, is an infinite sum. Before proceeding, we remark that, in this formula, the numerator of the summand should be $(-1)^{n+r-1}F_{2n-1}$. Shifting the range of summation, and expressing

Brousseau's formula in the notation of the present paper, we see that Brousseau essentially obtains a closed form for

$$\sum_{i=1}^{\infty} \frac{(-1)^i F_{2i+2r+1}}{F_i^2 F_{i+2r+1}^2}. \tag{2}$$

But (2) is obtained by considering $\{U_n\}$=$\{F_n\}$, and letting $n \to \infty$ in the finite sum $S_2(2r+1, 1, 0, n)$.

Likewise, Brousseau's formulas 12 and 15 follow easily from the closed form for $S_2(d, k, m, n)$ that we give in Theorem 1. We leave the reader the simple task of verifying that Brousseau's formula 7 follows from the dual result of Theorem 1.

3. The Summand Has Three Distinct Factors in the Denominator

In this section, we present two theorems. Furthermore, as stated in the introduction, we demonstrate our method of proof by proving the first of these theorems.

Define

$$S_3^0(d, k, m, n) = \sum_{i=1}^{n-1} \frac{\overline{W}_{k(i+d)+m}}{W_{ki+m}^2 W_{k(i+d)+m} W_{k(i+2d)+m}^2},$$

and

$$S_3^1(d, k, m, n) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} V_{k(i+d)+m} V_{2k(i+d)+2m}}{U_{ki+m}^2 U_{k(i+d)+m} U_{k(i+2d)+m}^2}.$$

In this section, we give the closed forms for S_3^0 and S_3^1. We remark that, when conducting our research, our preference was to discover closed forms for finite sums involving the more general sequences $\{W_n\}$ and $\left\{\overline{W}_n\right\}$. When such closed forms were not forthcoming, we turned our attention to discovering closed forms for finite sums involving the sequences $\{U_n\}$ and $\{V_n\}$.

To assist us in presenting the closed form for S_3^0 succinctly, we define, for $0 \leq i \leq 3$, the quantities $a_i = a_i(d, k, m)$ as

$$\begin{aligned}
a_0 &= e_W U_k U_{dk}^3 U_{2dk}, \\
a_1 &= (-1)^{k+m+1} U_{dk}, \\
a_2 &= U_{(d-1)k}, \\
a_3 &= (-1)^k U_{(d+1)k}.
\end{aligned}$$

We can now state our next theorem.

Theorem 2. *With S_3^0 as defined above,*

$$a_0\left(S_3^0(n)-S_3^0(2)\right)=a_1\sum_{i=0}^{1}\Phi_W(k,m,n,id,(i+1)d)$$
$$+2(-1)^{k+m}U_{k(n-2)}\sum_{i=0}^{1}a_{i+2}\Psi_W(k,m,n,id,(i+1)d). \quad (3)$$

Proof. Since $\alpha\beta=-1$, we write the Binet forms for the various sequences thus:

$$\begin{aligned}U_n&=\left(\alpha^n+(-1)^{n+1}\alpha^{-n}\right)/\sqrt{\Delta},\\ V_n&=\alpha^n+(-1)^n\alpha^{-n},\\ W_n&=\left(\left(b+a\alpha^{-1}\right)\alpha^n+(-1)^{n+1}\left(b-a\alpha\right)\alpha^{-n}\right)/\sqrt{\Delta},\\ \overline{W}_n&=\left(b+a\alpha^{-1}\right)\alpha^n+(-1)^n\left(b-a\alpha\right)\alpha^{-n}.\end{aligned}$$

These closed forms are valid for all integers n. We note also that $p=\alpha-\alpha^{-1}$, and recall that $e_W=b^2-pab-a^2$.

We require the following two identities:

$$W_{k(n+2d)+m}W_{k(n-1)+m}-W_{k(n+2d-1)+m}W_{kn+m}=e_W(-1)^{k(n+1)+m+1}U_kU_{2dk}, \quad (4)$$

and

$$U_{(d-1)k}W_{k(n+2d)+m}+(-1)^{k(d+1)}U_{(d+1)k}W_{kn+m}=U_{2dk}W_{k(n+d-1)+m}. \quad (5)$$

Since (4) and (5) can be proved routinely with the use of the Binet forms, we leave this task to the interested reader.

As we have already stated, $n\geq 2$ is taken to be an integer. In the statement of Theorem 2, denote the quantities on the left and right sides of (3) by $L(n)$ and $R(n)$, respectively. The expression for $L(n+1)-L(n)$ is

$$L(n+1)-L(n)=\frac{e_WU_kU_{dk}^3U_{2dk}\overline{W}_{k(n+d)+m}}{W_{kn+m}^2W_{k(n+d)+m}W_{k(n+2d)+m}^2}. \quad (6)$$

With the use of Lemmas 1 and 2, we find, after some simplification, that

$$R(n+1)-R(n)=\frac{U_{dk}}{e_WU_k}\left(\frac{W_{k(n-1)+m}^2}{W_{kn+m}^2}-\frac{W_{k(n+2d-1)+m}^2}{W_{k(n+2d)+m}^2}\right)$$
$$+\frac{2(-1)^{k(n+1)+m}U_{dk}}{W_{k(n+d)+m}}\left(\frac{U_{(d-1)k}}{W_{kn+m}}+\frac{(-1)^{k(d+1)}U_{(d+1)k}}{W_{k(n+2d)+m}}\right). \quad (7)$$

Next, we express the right side of (7) with the same denominator as the right side of (6), while making use of (4) and (5) to simplify the resulting expressions. We then see that

$$R(n+1)-R(n)=\frac{c_1c_2+c_3}{W_{kn+m}^2W_{k(n+d)+m}W_{k(n+2d)+m}^2},$$

where the $c_i = c_i(d, k, m, n)$ are defined as

$$\begin{aligned}
c_1 &= (-1)^{k(n+1)+m+1} U_{dk} U_{2dk} W_{k(n+d)+m},\\
c_2 &= W_{k(n+2d)+m} W_{k(n-1)+m} + W_{k(n+2d-1)+m} W_{kn+m},\\
c_3 &= 2(-1)^{k(n+1)+m} U_{dk} U_{2dk} W_{k(n+d-1)+m} W_{kn+m} W_{k(n+2d)+m}.
\end{aligned}$$

Our aim is to prove that

$$R(n+1) - R(n) = L(n+1) - L(n), \tag{8}$$

which is equivalent to proving that

$$c_1 c_2 + c_3 - e_W U_k U_{dk}^3 U_{2dk} \overline{W}_{k(n+d)+m} = 0. \tag{9}$$

Upon substituting the Binet forms, we find that the left side of (9) can be expressed in factored form, and that one of the factors is $(-1)^{2(kn+m)} - 1$. This establishes (8). Furthermore, $R(2) = L(2) = 0$, and this completes the proof of Theorem 2. $\square$

In the proof above, the key identity is (9). Likewise, the proof of each theorem in this paper hinges around the proof of a key identity that is analogous to (9), and each such identity can be proved by substitution of the appropriate closed forms. The method is mechanical, and is not dependent upon and special identities. However, the use of a computer algebra system (in our case *Mathematica 8*) is essential.

To assist us in presenting the closed form for S_3^1 succinctly, we define, for $0 \le i \le 3$, the quantities $b_i = b_i(d, k)$ as

$$\begin{aligned}
b_0 &= U_k U_{dk}^4 V_{dk},\\
b_1 &= (-1)^{k+1} U_{dk} V_{2dk},\\
b_2 &= (-1)^{dk}\left((-1)^{dk} U_{3dk} V_k - 3U_k V_{dk} - 2(-1)^k U_{(d-1)k}\right),\\
b_3 &= (-1)^{dk}\left((-1)^{dk} U_{3dk} V_k + U_k V_{dk} - 2(-1)^k U_{(d-1)k}\right).
\end{aligned}$$

We then have the theorem that follows.

Theorem 3. *With S_3^1 as defined above,*

$$\begin{aligned}
b_0\left(S_3^1(n) - S_3^1(2)\right) = b_1 &\sum_{i=0}^{1} \Phi_U(k, m, n, id, (i+1)d)\\
&+ U_{k(n-2)} \sum_{i=0}^{1} b_{i+2} \Psi_U(k, m, n, id, (i+1)d).
\end{aligned}$$

The only so-called dual result that we could find for Theorem 3 is Theorem 1. Specifically, in the summand of S_3^1, replace each occurrence of V by U. Then the summand that results is the summand of S_2 with d replaced by $2d$.

At this point we give an example. Let $d = 2$, $k = 1$, and $m = 0$. Then, for $W_n = L_n$, (3) becomes

$$375\sum_{i=1}^{n-1}\frac{F_{i+2}}{L_i^2L_{i+2}L_{i+4}^2} = \frac{229331}{77616} + 5F_{n-2}\left(\frac{2}{3L_n} - \frac{1}{2L_{n+1}} - \frac{4}{7L_{n+2}} + \frac{4}{11L_{n+3}}\right) - \left(\frac{L_{n-1}^2}{L_n^2} + \frac{L_n^2}{L_{n+1}^2} + \frac{L_{n+1}^2}{L_{n+2}^2} + \frac{L_{n+2}^2}{L_{n+3}^2}\right).$$

4. The Summand Has Four Distinct Factors in the Denominator

We have discovered closed forms for each of the following:

$$S_4^0(d,k,m,n) = \sum_{i=1}^{n-1}\frac{(-1)^{ki}U_{k(2i+3d)+2m}}{U_{ki+m}^2U_{k(i+d)+m}^2U_{k(i+2d)+m}^2U_{k(i+3d)+m}^2},$$

$$S_4^1(d,k,m,n) = \sum_{i=1}^{n-1}\frac{(-1)^{ki}U_{k(2i+3d)+2m}^3}{U_{ki+m}^2U_{k(i+d)+m}^2U_{k(i+2d)+m}^2U_{k(i+3d)+m}^2},$$

$$S_4^2(d,k,m,n) = \sum_{i=1}^{n-1}\frac{U_{k(4i+6d)+4m}}{U_{ki+m}^2U_{k(i+d)+m}^2U_{k(i+2d)+m}^2U_{k(i+3d)+m}^2},$$

and

$$S_4^3(d,k,m,n) = \sum_{i=1}^{n-1}\frac{(-1)^{ki}U_{k(6i+9d)+6m}}{U_{ki+m}^2U_{k(i+d)+m}^2U_{k(i+2d)+m}^2U_{k(i+3d)+m}^2}.$$

In order to keep the presentation to a reasonable length, we present only the closed forms for S_4^0 and S_4^1. Also, to keep our notation simple, we redefine the a_i of Section 3. Accordingly, for $0 \le i \le 6$, define the quantities $a_i = a_i(d,k)$ by

$$\begin{aligned}
a_0 &= U_kU_{dk}U_{2dk}^3U_{3dk}^2,\\
a_1 &= (-1)^{k+1}U_{3dk}V_{dk},\\
a_2 &= (-1)^{(d+1)k+1}U_{3dk}V_{dk}^3,\\
a_3 &= a_1,\\
a_4 &= 3(-1)^kU_{(d-1)k}V_{dk}\left(V_{2dk} + (-1)^{dk}\right) - U_{3dk}V_{(d+1)k},\\
a_5 &= (-1)^{dk}U_{3dk}V_kV_{dk}^3,\\
a_6 &= U_{(d+1)k}V_{dk}\left(3V_{2dk} + 2(-1)^{dk}\right) + (-1)^{k+1}U_{(2d-1)k}V_{2dk}.
\end{aligned}$$

We then have

Theorem 4. *With S_4^0 as defined above,*

$$a_0\left(S_4^0(n) - S_4^0(2)\right) = \sum_{i=0}^{2} a_{i+1}\Phi_U(k,m,n,id,(i+1)d) + U_{k(n-2)}\sum_{i=0}^{2} a_{i+4}\Psi_U(k,m,n,id,(i+1)d). \qquad (10)$$

Theorem 4 has the following dual result: redefine S_4^0 by replacing each occurrence of U in the denominator of the summand by V. Then (10) remains valid provided we multiply the left side by Δ^3, and replace Φ_U and Ψ_U by Φ_V and Ψ_V, respectively.

To present the closed form for S_4^1, we define, for $0 \le i \le 6$, the quantities $b_i = b_i(d,k)$ as

$$\begin{aligned}
b_0 &= U_k U_{dk} U_{2dk}^3,\\
b_1 &= (-1)^{k+1} U_{3dk} V_{dk},\\
b_2 &= (-1)^{k+1} U_{2dk} V_{dk}^2,\\
b_3 &= b_1,\\
b_4 &= U_{dk} V_k V_{3dk} + 4(-1)^{(d+1)k} U_{dk} V_{(d-1)k} - 10U_k,\\
b_5 &= U_{2dk} V_k V_{dk}^2,\\
b_6 &= U_{dk} V_k V_{3dk} + 4(-1)^{dk} U_{dk} V_{(d+1)k} + 10U_k.
\end{aligned}$$

We then have the following theorem.

Theorem 5. *With S_4^1 as defined above,*

$$b_0\left(S_4^1(n) - S_4^1(2)\right) = \sum_{i=0}^{2} b_{i+1}\Phi_U(k,m,n,id,(i+1)d) + U_{k(n-2)}\sum_{i=0}^{2} b_{i+4}\Psi_U(k,m,n,id,(i+1)d).$$

Theorem 5 has a dual result that is obtained by making precisely the same changes as those described for the dual result of Theorem 4. Interestingly, the same can be said for the dual result that arises from the closed form for S_4^3. However, in the case of S_4^2 we multiply by $-\Delta^3$ instead of Δ^3.

5. The Summand Has Five Distinct Factors in the Denominator

In this section, the finite sums for which we have managed to find closed forms fall into three categories. The first category consists of the four sums

$$S_5^0(d,k,m,n)=\sum_{i=1}^{n-1}\frac{(-1)^{ki}\overline{W}^7_{k(i+2d)+m}}{W^2_{ki+m}W^2_{k(i+d)+m}W_{k(i+2d)+m}W^2_{k(i+3d)+m}W^2_{k(i+4d)+m}},$$

$$S_5^1(d,k,m,n)=\sum_{i=1}^{n-1}\frac{\overline{W}^5_{k(i+2d)+m}}{W^2_{ki+m}W^2_{k(i+d)+m}W_{k(i+2d)+m}W^2_{k(i+3d)+m}W^2_{k(i+4d)+m}},$$

$$S_5^2(d,k,m,n)=\sum_{i=1}^{n-1}\frac{(-1)^{ki}\overline{W}^3_{k(i+2d)+m}}{W^2_{ki+m}W^2_{k(i+d)+m}W_{k(i+2d)+m}W^2_{k(i+3d)+m}W^2_{k(i+4d)+m}},$$

and

$$S_5^3(d,k,m,n)=\sum_{i=1}^{n-1}\frac{\overline{W}_{k(i+2d)+m}}{W^2_{ki+m}W^2_{k(i+d)+m}W_{k(i+2d)+m}W^2_{k(i+3d)+m}W^2_{k(i+4d)+m}}.$$

The second category consists of the three sums

$$S_5^4(d,k,m,n)=\sum_{i=1}^{n-1}\frac{(-1)^{ki}V_{7k(i+2d)+7m}}{U^2_{ki+m}U^2_{k(i+d)+m}U_{k(i+2d)+m}U^2_{k(i+3d)+m}U^2_{k(i+4d)+m}},$$

$$S_5^5(d,k,m,n)=\sum_{i=1}^{n-1}\frac{V_{5k(i+2d)+5m}}{U^2_{ki+m}U^2_{k(i+d)+m}U_{k(i+2d)+m}U^2_{k(i+3d)+m}U^2_{k(i+4d)+m}},$$

and

$$S_5^6(d,k,m,n)=\sum_{i=1}^{n-1}\frac{(-1)^{ki}V_{3k(i+2d)+3m}}{U^2_{ki+m}U^2_{k(i+d)+m}U_{k(i+2d)+m}U^2_{k(i+3d)+m}U^2_{k(i+4d)+m}}.$$

Furthermore, for each of S_5^4, S_5^5, and S_5^6, we have found the corresponding dual results where, in each summand, U is replaced by V, and V is replaced by U. We were unable to find closed forms for the analogous sums that involve the sequences W_n and $\overline{W}_n$.

The third category consists of the two sums

$$S_5^7(d,k,m,n)=\sum_{i=1}^{n-1}\frac{\overline{W}_{k(i+d)+m}\overline{W}_{k(i+2d)+m}\overline{W}_{k(i+3d)+m}}{W^2_{ki+m}W_{k(i+d)+m}W_{k(i+2d)+m}W_{k(i+3d)+m}W^2_{k(i+4d)+m}},$$

and

$$S_5^8(d,k,m,n)=\sum_{i=1}^{n-1}\frac{\overline{W}_{ki+m}\overline{W}_{k(i+2d)+m}\overline{W}_{k(i+4d)+m}}{W_{ki+m}W^2_{k(i+d)+m}W_{k(i+2d)+m}W^2_{k(i+3d)+m}W_{k(i+4d)+m}}.$$

Since the closed forms for the above sums are rather lengthy, we present only the closed form for S_5^0. Accordingly, for $0 \le i \le 8$, define the quantities $a_i = a_i(d,k)$ as

$$\begin{aligned}
a_0 &= U_k U_{dk} U_{2dk}^4 U_{3dk}^3,\\
a_1 &= (-1)^{k+1} U_{4dk} V_{2dk}^4 \left(V_{2dk} + (-1)^{dk}\right),\\
a_2 &= a_1 + (-1)^{(d+1)k+1} U_{3dk} V_{dk}^9,\\
a_3 &= a_2,\\
a_4 &= a_1,\\
a_5 &= (-1)^{k+1} V_k a_1 - 2(-1)^{dk} U_k V_{2dk}^4 \left(\left(V_{2dk} + 3(-1)^{dk}\right)^2 - 3\right),\\
a_6 &= (-1)^{k+1} V_k a_2 - 64 U_k \left(V_{2dk} + (-1)^{dk}\right)^3,\\
a_7 &= (-1)^{k+1} V_k a_2 + 64 U_k \left(V_{2dk} + (-1)^{dk}\right)^3,\\
a_8 &= (-1)^{k+1} V_k a_1 + 2(-1)^{dk} U_k V_{2dk}^4 \left(\left(V_{2dk} + 3(-1)^{dk}\right)^2 - 3\right).
\end{aligned}$$

We then have

Theorem 6. *With S_5^0 as defined above,*

$$\begin{aligned}
a_0\left(S_5^0(n) - S_5^0(2)\right) = {}& \sum_{i=0}^{3} a_{i+1} \Phi_W(k,m,n,id,(i+1)d)\\
& + U_{k(n-2)} \sum_{i=0}^{3} a_{i+5} \Psi_W(k,m,n,id,(i+1)d). \qquad (11)
\end{aligned}$$

We conclude this section with an example. Let $d = 1$, $k = 1$, and $m = 0$. Then, for $W_n = F_n$, (11) becomes

$$\begin{aligned}
\sum_{i=1}^{n-1} \frac{40(-1)^i L_{i+2}^7}{F_i^2 F_{i+1}^2 F_{i+2} F_{i+3}^2 F_{i+4}^2} = {}& 3529 + 2F_{n-2}\left(\frac{35}{F_{n+1}} + \frac{830}{F_{n+2}} - \frac{486}{F_{n+3}}\right)\\
& - 10\left(\frac{243F_{n-1}^2}{F_n^2} + \frac{242F_n^2}{F_{n+1}^2} + \frac{242F_{n+1}^2}{F_{n+2}^2} + \frac{243F_{n+2}^2}{F_{n+3}^2}\right).
\end{aligned}$$

6. Concluding Comments

Numerical evidence suggests that results, analogous to those presented here, exist for finite sums in which the denominator of the summand has six or more distinct factors. Such results become more unwieldy as the number of factors in the denominator of the summand increases. It is for this reason that we have limited the scope of this paper. We trust that our presentation has given the reader an appreciation of the kinds of results that are possible.

Finally, the author would like to thank the referee for a constructive and carefully considered report. The input of this referee has significantly enhanced the presentation of this paper.

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