



CHARACTERIZING CONGRUENCE PRESERVING FUNCTIONS
 $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ VIA RATIONAL POLYNOMIALS

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Abstract

Using a simple basis of rational polynomial-like functions P_0, \dots, P_{n-1} for the free module of functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$, we characterize the subfamily of congruence preserving functions as the set of linear combinations of the products $\text{lcm}(k)P_k$ where $\text{lcm}(k)$ is the least common multiple of $2, \dots, k$ (viewed in $\mathbb{Z}/m\mathbb{Z}$). As a consequence, when $n \geq m$, the number of such functions is independent of n .

1. Introduction

The notion of a congruence preserving function on rings of residue classes was introduced in Chen [3] and studied in Bhargava [1].

Definition 1.1. Let $m, n \geq 1$. A function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is said to be *congruence preserving* if for all d dividing m

$$\text{for all } a, b \in \{0, \dots, n-1\} \quad a \equiv b \pmod{d} \text{ implies } f(a) \equiv f(b) \pmod{d}. \quad (1)$$

Remark 1.2. 1. If $n \in \{1, 2\}$ or $m = 1$ then every function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is trivially congruence preserving.

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2. Observe that since d is assumed to divide m , equivalence modulo d is a congruence on $(\mathbb{Z}/m\mathbb{Z}, +, \times)$. However, since d is not supposed to divide n , equivalence modulo d may not be a congruence on $(\mathbb{Z}/n\mathbb{Z}, +, \times)$.

Example 1.3. 1. For functions $\mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$, condition (1) reduces to the conditions $f(3) \equiv f(0) \pmod{3}$, $f(4) \equiv f(1) \pmod{3}$, $f(5) \equiv f(2) \pmod{3}$.
 2. For functions $\mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/8\mathbb{Z}$, condition (1) reduces to $f(2) \equiv f(0) \pmod{2}$, $f(3) \equiv f(1) \pmod{2}$, $f(4) \equiv f(0) \pmod{4}$, $f(5) \equiv f(1) \pmod{4}$.

In this paper, we characterize congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$.

We denote by \mathbb{Z} the set of integers and by \mathbb{N} the set of nonnegative integers (including zero).

Definition 1.4. The unary *lcm* function $\mathbb{N} \rightarrow \mathbb{N}$ maps 0 to 1 and $k \geq 1$ to the least common multiple of $1, 2, \dots, k$.

A natural way to associate with each map from \mathbb{N} to \mathbb{Z} a map from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ is to restrict F to $\{0, \dots, n-1\}$ and take its values modulo m .

Definition 1.5. With each map $F : \mathbb{N} \rightarrow \mathbb{Z}$, we associate the map $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ defined by $f = \pi_m \circ F \circ \iota_n$, where $\pi_m(x) = x \pmod{m}$, and $\iota_n(z)$ is the unique element of $\pi_n^{-1}(z) \cap \{0, \dots, n-1\}$.

Definition 1.5 is best pictured by the commutativity of diagram (2).

$$\begin{array}{ccc}
 \mathbb{N} & \xrightarrow{F} & \mathbb{Z} \\
 \iota_n \uparrow & & \downarrow \pi_m \\
 \mathbb{Z}/n\mathbb{Z} & \xrightarrow{f} & \mathbb{Z}/m\mathbb{Z}
 \end{array} \tag{2}$$

Applying Definition 1.5 to binomial coefficients, we obtain a basis of the $(\mathbb{Z}/m\mathbb{Z})$ -module of functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$.

Proposition 1.6. Let $P_k : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ be associated with the $\mathbb{N} \rightarrow \mathbb{N}$ binomial function $x \mapsto \binom{x}{k}$. For every function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ there is a unique sequence (a_0, \dots, a_{n-1}) of elements of $\mathbb{Z}/m\mathbb{Z}$ such that

$$f = \sum_{k=0}^{k=n-1} a_k P_k . \tag{3}$$

In other words, the family $\{P_0, \dots, P_{n-1}\}$ is a basis of the $(\mathbb{Z}/m\mathbb{Z})$ -module of functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$.

Our main result can be stated as

Theorem 1.7. *A function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is congruence preserving if and only if, for each $k = 0, \dots, n - 1$, in equation (3) the coefficient a_k is a multiple of the residue of $\text{lcm}(k)$ in $\mathbb{Z}/m\mathbb{Z}$.*

The paper is organized as follows.

Proposition 1.6 is proved in Section 2 where, after recalling Chen’s notion of a polynomial function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ (cf. [3]), we extend it to a notion of a rational polynomial function.

The proof of our main result, Theorem 1.7, is given in Section 3. We adapt the techniques of our paper [2], exploiting similarities between Definition 1.1 and the condition studied in [2] for functions $f : \mathbb{N} \rightarrow \mathbb{Z}$ (namely, $x - y$ divides $f(x) - f(y)$ for all $x, y \in \mathbb{N}$). As a consequence of Theorem 1.7, the number of congruence preserving functions is independent of n for $n \geq m$ and even for $n \geq gpp(m)$ (the greatest prime power dividing m). Also, every congruence preserving function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is a rational polynomial of degree strictly less than the minimum of n and $gpp(m)$.

In Section 4 we use our main theorem to count the congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. We thus get an expression equivalent to that obtained by Bhargava in [1] and which makes apparent the fact that, for $n \geq gpp(m)$ (hence for $n \geq m$), this number depends only on m and is independent of n .

2. Representing Functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ by Rational Polynomials

In [3, 1], congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ are introduced and studied together with an original notion of polynomial function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$.

Definition 2.1 (Chen [3]). A function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is *polynomial* if it is associated (in the sense of Definition 1.5) with a function $F : \mathbb{N} \rightarrow \mathbb{Z}$ given by a polynomial in $\mathbb{Z}[X]$.

Polynomial functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ are obviously congruence preserving. Are all congruence preserving functions polynomial? Chen [3] observed that this is not the case for some values of n, m , for instance $n = 6, m = 8$. He also proves that a stronger identity holds for infinitely many ordered pairs $\langle n, m \rangle$: *every function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is polynomial if and only n is not greater than the first prime factor of m* (in particular, this is the case when $n = m$ and m is prime, cf. Kempner [4]). Using counting arguments, Bhargava [1] characterizes the ordered pairs $\langle n, m \rangle$ such that every congruence preserving function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is polynomial.

Some polynomials in $\mathbb{Q}[X]$ (i.e., polynomials with rational coefficients) happen to map integers into integers.

Definition 2.2. For $k \in \mathbb{N}$, let $P_k \in \mathbb{Q}[X]$ be the following polynomial:

$$P_k(x) = \binom{x}{k} = \frac{\prod_{i=0}^{k-1} (x-i)}{k!}.$$

We will use the following examples later on:

$$P_0(x) = 1, P_1(x) = x, P_2(x) = x(x-1)/2, P_3(x) = x(x-1)(x-2)/6, P_4(x) = x(x-1)(x-2)(x-3)/24, P_5(x) = x(x-1)(x-2)(x-3)(x-4)/120.$$

In [5], Pólya used the P_k 's to give the following very elegant and elementary characterization of polynomials in $\mathbb{Q}[X]$ mapping integers to integers.

Theorem 2.3 (Pólya). *A polynomial in $\mathbb{Q}[X]$ is integer-valued on \mathbb{Z} if and only if it can be written as a \mathbb{Z} -linear combination of the polynomials P_k , $k = 0, 1, 2, \dots$*

It turns out that the representation of functions $\mathbb{N} \rightarrow \mathbb{Z}$ as \mathbb{Z} -linear combinations of the P_k 's used in [2] also fits in the case of functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$: every such function is a $(\mathbb{Z}/m\mathbb{Z})$ -linear combination of the P_k 's.

Definition 2.4. 1. A function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is *rat-polynomial* if is associated in the sense of Definition 1.5 with some polynomial in $\mathbb{Q}[X]$.

2. The *degree* of a rat-polynomial function is the smallest degree of an associated polynomial in $\mathbb{Q}[X]$.

3. We denote by $P_k^{n,m}$ the rat-polynomial function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ associated with the polynomial P_k of Definition 2.2 in the sense of Definition 1.5. When there is no ambiguity, $P_k^{n,m}$ will be denoted simply as P_k .

Remark 2.5. In Definition 2.4, the polynomial *crucially depends* on the choice of representatives of elements of $\mathbb{Z}/n\mathbb{Z}$: e.g., for $n = m = 6$, $0 \equiv 6 \pmod{6}$ but $0 = P_2(0) \not\equiv P_2(6) = 3 \pmod{6}$. The chosen representatives for elements of $\mathbb{Z}/n\mathbb{Z}$ will always be $0, 1, \dots, n-1$.

We now prove the representation result by the P_k 's.

Proof of Proposition 1.6. Let us start with uniqueness. We have $f(0) = a_0$, and hence a_0 is $f(0)$. We have $f(1) = a_0 + a_1$, and hence $a_1 = f(1) - f(0)$. By induction, letting $Q_k = \sum_{\ell=0}^{k-1} a_\ell P_\ell$, and noting that $P_k(k) = 1$, we have $f(k) = Q_k(k) + a_k P_k(k) = Q_k(k) + a_k$, and hence $a_k = f(k) - Q_k(k)$. We thus are able to determine a_k in $\mathbb{Z}/m\mathbb{Z}$.

For existence, argue backwards to see that this sequence suits. □

Remark 2.6. The evaluation of $a_k P_k(x)$ in $\mathbb{Z}/m\mathbb{Z}$ has to be done as follows: for x an element of $\mathbb{Z}/n\mathbb{Z}$, we consider it as an element of $\{0, \dots, n-1\} \subseteq \mathbb{N}$ and we evaluate $P_k(x) = \frac{1}{k!} \prod_{i=0}^{k-1} (x-i)$ as an element of \mathbb{Z} , then we consider the remainder modulo m , and finally we multiply the result by a_k in $\mathbb{Z}/m\mathbb{Z}$. For instance, for

$n = m = 8$, we have $4P_2(3) = 4 \times \frac{3 \times 2}{2} = 4 \times 3 = 4$, but we might be tempted to evaluate it as $4P_2(3) = \frac{4 \times 3 \times 2}{2} = \frac{0}{2} = 0$, which does *not* correspond to our definition. However, dividing a_k by a factor of the denominator is allowed.

Corollary 2.7. 1. Every function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is rat-polynomial with degree less than n .

2. The family of rat-polynomial functions $\{P_k \mid k = 0, 1, \dots, n - 1\}$ is a basis of the $(\mathbb{Z}/m\mathbb{Z})$ -module of functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$.

Example 2.8. The function $f : \mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z}$ such that $f(0) = 0, f(1) = 3, f(2) = 4, f(3) = 3, f(4) = 0, f(5) = 1$, is represented by the rational polynomial $P_f(x) = 3x + 4 \frac{x(x-1)}{2}$ which can be simplified to $P_f(x) = 3x - x(x-1)$ on $\mathbb{Z}/6\mathbb{Z}$.

Example 2.9. The function $f : \mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/8\mathbb{Z}$ given by Chen [3] as a non-polynomial congruence preserving function, namely the function such that $f(0) = 0, f(1) = 3, f(2) = 4, f(3) = 1, f(4) = 4, f(5) = 7$, is represented by the rational polynomial with coefficients $a_0 = 0, a_1 = 3, a_2 = 6, a_3 = 2, a_4 = 4, a_5 = 4$. Thus,

$$\begin{aligned} f(x) &= 3x + 6 \frac{x(x-1)}{2} + 2 \frac{x(x-1)(x-2)}{2} + 4 \frac{x(x-1)(x-2)(x-3)}{8} \\ &\quad + 4 \frac{x(x-1)(x-2)(x-3)(x-4)}{8} \\ &= 3x + 3x(x-1) + x(x-1)(x-2) + \frac{x(x-1)(x-2)(x-3)}{2} \\ &\quad + \frac{x(x-1)(x-2)(x-3)(x-4)}{2}. \end{aligned}$$

3. Characterizing Congruence Preserving Functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$

Congruence preserving functions $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ can be characterized by a simple condition on the coefficients of the rat-polynomial representation of f given in Proposition 1.6.

3.1. Proof of Theorem 1.7

For proving Theorem 1.7 we will need some relations involving binomial coefficients and the unary lcm function; these relations are stated in the next three lemmata. The proofs are elementary but technical and can be found in our paper [2].

Lemma 3.1. If $0 \leq n - k < p \leq n$ then p divides $\text{lcm}(k) \binom{n}{k}$ in \mathbb{N} .

Lemma 3.2. If $k \leq b$ then n divides $A_{k,b}^n = \text{lcm}(k) \left(\binom{b+n}{k} - \binom{b}{k} \right)$ in \mathbb{N} .

The following is an immediate consequence of Lemma 3.2 (set $a = b + n$).

Lemma 3.3. *If $a \geq b$ and $k \leq b$, then $a - b$ divides $\text{lcm}(k) \left(\binom{a}{k} - \binom{b}{k} \right)$ in \mathbb{N} .*

Besides these lemmata which deal with divisibility on integers, we shall use a classical result in $\mathbb{Z}/m\mathbb{Z}$. For $x, y \in \mathbb{Z}$ we say x divides y in $\mathbb{Z}/m\mathbb{Z}$ if and only if the residue class of x divides the residue class of y in $\mathbb{Z}/m\mathbb{Z}$.

Lemma 3.4. *Let $1 \leq c_1, \dots, c_k \leq m$ and let c be their least common multiple in \mathbb{N} . If c_1, \dots, c_k all divide a in $\mathbb{Z}/m\mathbb{Z}$ then so does c .*

Proof. It suffices to consider the case $k = 2$ since the passage to any k is done via a straightforward induction. Let $c = c_1b_1 = c_2b_2$ with b_1, b_2 coprime. Let t, u be such that $a = c_1t = c_2u$ in $\mathbb{Z}/m\mathbb{Z}$. Then $a \equiv c_1t \equiv c_2u \pmod{m}$. Using Bézout’s identity, let $\alpha, \beta \in \mathbb{Z}$ be such that $\alpha b_1 + \beta b_2 = 1$. Then $c(t\alpha + u\beta) = c_1b_1t\alpha + c_2b_2u\beta \equiv \alpha a + \beta a \pmod{m}$, and hence $c(t\alpha + u\beta) \equiv a \pmod{m}$, proving that c divides a in $\mathbb{Z}/m\mathbb{Z}$. \square

Proof of the “only if” part of Theorem 1.7. Assume $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is congruence preserving and consider its decomposition $f(x) = \sum_{k=0}^{n-1} a_k P_k(x)$ given by Proposition 1.6. We show that $\text{lcm}(k)$ divides a_k in $\mathbb{Z}/m\mathbb{Z}$ for all $k < n$. The cases $k = 0$ and $k = 1$ are trivial since $\text{lcm}(0) = \text{lcm}(1) = 1$.

Claim 1. *For all $2 \leq k < n$, k divides a_k in $\mathbb{Z}/m\mathbb{Z}$.*

Proof. Recall that $f(k) = \sum_{i=0}^{n-1} a_i \binom{k}{i} = \sum_{i=0}^k a_i \binom{k}{i}$ since $\binom{k}{i} = 0$ for $i > k$. We argue by induction on $k \geq 2$.

Base case $k = 2$. If 2 does not divide m then 2 and m are coprime, and hence 2 is invertible and divides a_2 in $\mathbb{Z}/m\mathbb{Z}$. Assume 2 divides m . As 2 divides $2 - 0$ and f is congruence preserving, 2 also divides $f(2) - f(0) = 2a_1 + a_2$, and hence 2 divides a_2 .

Inductive step. Let $2 < k < n - 1$. The inductive hypothesis ensures that ℓ divides a_ℓ in $\mathbb{Z}/m\mathbb{Z}$ for every $\ell \leq k$. Let $a_\ell \equiv \ell q_\ell \pmod{m}$ for $0 \leq \ell \leq k$. We prove that $k + 1$ divides a_{k+1} in $\mathbb{Z}/m\mathbb{Z}$. First, observe that

$$\begin{aligned} f(k+1) - f(0) &= (k+1)a_1 + \left(\sum_{i=2}^k \binom{k+1}{i} a_i \right) + a_{k+1} \\ &\equiv (k+1)a_1 + \left(\sum_{i=2}^k \binom{k+1}{i} i q_i \right) + a_{k+1} \pmod{m} \\ f(k+1) - f(0) &= (k+1)a_1 + \left(\sum_{i=2}^k (k+1) \binom{k}{i-1} q_i \right) + \alpha m + a_{k+1} \end{aligned} \tag{4}$$

for some α . Let $d = \gcd(k + 1, m)$. Since d divides m and $k + 1 - 0$ and f is congruence preserving, d also divides $f(k + 1) - f(0)$. Using equality (4), we see that d divides the last term a_{k+1} of the sum. Using Bézout's identity, let u, v be such that $u(k + 1) + vm = d$. Then $u(k + 1) \equiv d \pmod{m}$, and hence $k + 1$ divides d in $\mathbb{Z}/m\mathbb{Z}$. Since d divides a_{k+1} , we conclude that $k + 1$ divides a_{k+1} in $\mathbb{Z}/m\mathbb{Z}$. \square

Claim 2. (i) For all $2 \leq p \leq k < n$, p divides a_k in $\mathbb{Z}/m\mathbb{Z}$.
(ii) For all $2 \leq k < n$, $\text{lcm}(k)$ divides a_k in $\mathbb{Z}/m\mathbb{Z}$.

Proof. Assertion (ii) is a direct application of Lemma 3.4 and assertion (i). We prove (i) by induction on $p \geq 2$. Both the base case and the inductive step of this induction are proved by induction on k .

Base case $p = 2$. We have to prove that 2 divides a_k for all $k \geq 2$. If 2 does not divide m , then 2 is invertible and divides all numbers in $\mathbb{Z}/m\mathbb{Z}$. Assume now that 2 divides m . We argue by induction on $k \geq 2$.

Base case. Apply Claim 1: 2 divides a_2 .

Inductive step. Let $k < n - 1$. Assuming that 2 divides a_i for all $2 \leq i \leq k$, we prove that 2 divides a_{k+1} . Two cases can occur.

Subcase 1: $k + 1$ is odd. Then 2 divides k and hence, by congruence preservation, 2 divides $f(k + 1) - f(1)$. As $f(k + 1) - f(1) = ka_1 + \left(\sum_{i=2}^k a_i \binom{k+1}{i}\right) + a_{k+1}$, and 2 divides k and also, by the induction hypothesis, 2 divides a_i for $2 \leq i \leq k$, we see that 2 divides a_{k+1} .

Subcase 2: $k + 1$ is even. By congruence preservation, 2 divides $f(k + 1) - f(0) = (k + 1)a_1 + \left(\sum_{i=2}^k a_i \binom{k+1}{i}\right) + a_{k+1}$. Since 2 divides $k + 1$ and a_i for $2 \leq i \leq k$ (induction hypothesis), we infer that 2 divides a_{k+1} .

Inductive step. Let $2 \leq p < n - 1$ and assume that

$$\text{for all } q \leq p \text{ and all } \ell \text{ such that } q \leq \ell < n, q \text{ divides } a_\ell \text{ in } \mathbb{Z}/m\mathbb{Z}. \quad (5)$$

By induction on $k \geq p + 1$, we prove that $p + 1$ divides a_k for all k such that $p + 1 \leq k < n$.

Base case $k = p + 1$. Apply Claim 1: $p + 1$ divides a_{p+1} .

Inductive step. Let $k < n - 1$. Assuming that $p + 1$ divides a_i in $\mathbb{Z}/m\mathbb{Z}$ for all i

such that $p + 1 \leq i \leq k$, we prove that $p + 1$ divides a_{k+1} in $\mathbb{Z}/m\mathbb{Z}$. We have

$$f(k + 1) - f(k - p) = \sum_{i=1}^{k-p} a_i \left(\binom{k + 1}{i} - \binom{k - p}{i} \right) + \left(\sum_{i=k+1-p}^k a_i \binom{k + 1}{i} \right) + a_{k+1} \quad (6)$$

We first look at the terms of the first sum on the right side of (6) corresponding to $1 \leq i \leq p$. Applying (5) with $\ell = i$, we see that q divides a_i in $\mathbb{Z}/m\mathbb{Z}$ for all $q \leq \min(p, i) = i$. Using Lemma 3.4, we conclude that $\text{lcm}(i)$ divides a_i in $\mathbb{Z}/m\mathbb{Z}$. Observing that $(k + 1) = (k - p) + (p + 1)$, we can apply Lemma 3.2 (with $k - p, p + 1$ and i in place of b, n and k) and conclude that $p + 1$ divides $\text{lcm}(i) \left(\binom{k+1}{i} - \binom{k-p}{i} \right)$ in \mathbb{N} . Thus, $p + 1$ divides $a_i \left(\binom{k+1}{i} - \binom{k-p}{i} \right)$ in $\mathbb{Z}/m\mathbb{Z}$.

We now turn to the terms of the first sum on the right side of (6) corresponding to $p + 1 \leq i \leq k - p$ (if there are any). Each of these terms is divisible by $p + 1$ in $\mathbb{Z}/m\mathbb{Z}$, because the induction hypothesis on k ensures that $p + 1$ divides a_i in $\mathbb{Z}/m\mathbb{Z}$ whenever $p + 1 \leq i \leq k$.

Consider next the terms of the second sum on the right side of (6). For those terms corresponding to values of i such that $p + 1 \leq i \leq k$, divisibility by $p + 1$ in $\mathbb{Z}/m\mathbb{Z}$ follows from the fact that, by the induction hypothesis on k , $p + 1$ divides a_i . It remains to look at the terms associated with the i 's such that $k + 1 - p \leq i \leq p$ (there are such i 's in case $k + 1 - p < p + 1$). For such i 's we have $0 \leq (k + 1) - i \leq (k + 1) - p < p + 1 \leq k + 1$ and Lemma 3.1 (used with $k + 1, i$ and $p + 1$ in place of n, k and p) implies that $p + 1$ divides $\text{lcm}(i) \binom{k+1}{i}$. Now, for such i 's, the induction hypothesis (5) on p shows that $\text{lcm}(i)$ divides a_i in $\mathbb{Z}/m\mathbb{Z}$. A fortiori, $p + 1$ divides $a_i \binom{k+1}{i}$ in $\mathbb{Z}/m\mathbb{Z}$.

Let $d = \text{gcd}(p + 1, m)$. As $p + 1$ divides in $\mathbb{Z}/m\mathbb{Z}$ all terms of the two sums on the right side of (6) so does d . Since d divides m and $k + 1 - (k - p) = p + 1$ and f is congruence preserving, d also divides $f(k + 1) - f(k - p)$. Using equality (6), we conclude that d divides in $\mathbb{Z}/m\mathbb{Z}$ the last term a_{k+1} . Using Bézout's identity, let u, v be such that $u(p + 1) + vm = d$. Then $u(p + 1) \equiv d \pmod{m}$, and hence $p + 1$ divides d in $\mathbb{Z}/m\mathbb{Z}$. As d divides a_{k+1} in $\mathbb{Z}/m\mathbb{Z}$, we conclude that $p + 1$ divides a_{k+1} in $\mathbb{Z}/m\mathbb{Z}$.

This ends the proof of the induction in the inductive step, and hence also the proof of Claim 2 and of the "only if" part of the Theorem. \square

Proof of the "if" part of Theorem 1.7. Assume $f = \sum_{k=0}^{k=n-1} a_k P_k$ and that all of the a_k 's are divisible by $\text{lcm}(k)$ in $\mathbb{Z}/m\mathbb{Z}$. We can write f in the form $f(n) = \sum_{k=0}^n c_k \text{lcm}(k) \binom{n}{k}$. We prove that f is congruence preserving, i.e., if $0 \leq b < a \leq$

$n - 1$ and d divides both m and $a - b$ then d also divides $f(a) - f(b)$. Observe that

$$f(a) - f(b) = \left(\sum_{k=0}^b c_k \text{lcm}(k) \left(\binom{a}{k} - \binom{b}{k} \right) \right) + \sum_{k=b+1}^a c_k \text{lcm}(k) \binom{a}{k}.$$

By Lemma 3.3, $a - b$ divides each term of the first sum. Consider the terms of the second sum. For $b + 1 \leq k \leq a$, we have $0 \leq a - k < a - b \leq a$ and Lemma 3.1 (used with a, k and $a - b$ in place of n, k and p) shows that $a - b$ divides $\text{lcm}(k) \binom{a}{k}$. Thus, $a - b$ divides $f(a) - f(b)$. \square

3.2. On a Family of Generators

We now sharpen the degree of the rat-polynomial representing a congruence preserving function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. We first state some properties of the lcm function in \mathbb{N} .

Lemma 3.5. *Let $m \geq 1$ be an integer with prime factorization $m = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$. Then $\text{lcm}(k) = u \prod_{i=1}^\ell p_i^{\alpha_{i,k}}$, where u is coprime with m and $\alpha_{i,k} = \max\{\beta_i \mid p_i^{\beta_i} \leq k\}$.*

Definition 3.6. Let $m \geq 1$ be an integer with prime factorization $m = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$. We let $\text{gpp}(m) = \max\{p_i^{\alpha_i} \mid i \in \{1, \dots, \ell\}\}$ be the greatest power of prime dividing m in \mathbb{N} .

Lemma 3.7. *The number $\text{gpp}(m)$ is the least integer k such that m divides $\text{lcm}(k)$.*

Example 3.8. We have $\text{gpp}(8) = 8$, $\text{gpp}(12) = 4$ and $\text{gpp}(14) = 7$. The successive values of the residues in $\mathbb{Z}/m\mathbb{Z}$ of $\text{lcm}(k)$ are

k	1	2	3	4	5	6	7	8
$\text{lcm}(k)$ in $\mathbb{Z}/8\mathbb{Z}$	1	2	2	4	4	4	4	0
$\text{lcm}(k)$ in $\mathbb{Z}/12\mathbb{Z}$	1	2	6	0	0	0	0	0
$\text{lcm}(k)$ in $\mathbb{Z}/14\mathbb{Z}$	1	2	6	12	4	4	0	0

For all $\ell \geq \text{gpp}(m)$, $\text{lcm}(\ell)$ is zero in $\mathbb{Z}/m\mathbb{Z}$.

Remark 3.9. 1. Either $\text{gpp}(m) = m$ or $\text{gpp}(m) \leq m/2$.
 2. In general, $\text{gpp}(m)$ is greater than $\lambda(m)$, the least k such that m divides $k!$ (a function considered in [3]): for $m = 8$, $\text{gpp}(m) = 8$ whereas $\lambda(m) = 4$.

Using Lemma 3.7, we can get a better version of Theorem 1.7.

Theorem 3.10. *A function $f: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is congruence preserving if and only if it is associated in the sense of Definition 1.5 with a rational polynomial $P = \sum_{k=0}^{d-1} a_k \binom{x}{k}$ where $d = \min(n, \text{gpp}(m))$ and such that $\text{lcm}(k)$ divides a_k in $\mathbb{Z}/m\mathbb{Z}$ for all $k < d$.*

Proof. For $k \geq gpp(m)$, m divides $lcm(k)$ hence the coefficient a_k is 0. \square

Theorem 3.11. (i) Every congruence preserving function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is rat-polynomial with degree less than $gpp(m)$.

(ii) The family of rat-polynomial functions

$$\mathcal{F} = \{lcm(k)P_k \mid 0 \leq k < \min(n, gpp(m))\}$$

generates the set of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$.

(iii) \mathcal{F} is a basis of the set of congruence preserving functions if and only if m has no prime divisor $p < \min(n, m)$ (in case $n \geq m$ this means that m is prime).

Proof. Assertions (i) and (ii) are restatements of Theorem 3.10. Let us prove (iii).

“Only IF” part. Assume m has a prime divisor $p < \min(n, m)$ and let p be the least one. Then $lcm(p) = pa$ with a coprime with m , and hence $lcm(p) \neq 0$ in $\mathbb{Z}/m\mathbb{Z}$. Since $P_p(p) = 1$ this shows that $lcm(p)P_p$ is not the null function. However $(m/p)lcm(p) = 0$ in $\mathbb{Z}/m\mathbb{Z}$, and hence $(m/p)lcm(p)P_p$ is the null function. As $(m/p) \neq 0$ in $\mathbb{Z}/m\mathbb{Z}$, this proves that \mathcal{F} cannot be a basis.

“IF” part. Assume that m has no prime divisor $p < \min(n, m)$. We prove that \mathcal{F} is $(\mathbb{Z}/m\mathbb{Z})$ -linearly independent. Suppose that the $(\mathbb{Z}/m\mathbb{Z})$ -linear combination $L = \sum_{k=0}^{\min(n, gpp(m))-1} a_k lcm(k)P_k$ is the null function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. By induction on $k = 0, \dots, \min(n, gpp(m)) - 1$ we prove that $a_k = 0$.

- *Basic cases* $k = 0, 1$. From $L(0) = a_0$ and $L(1) = a_0 + a_1$ we deduce $a_0 = a_1 = 0$.
- *Induction step.* Assuming $k \geq 2$ and $a_i = 0$ for $i = 0, \dots, k - 1$, we prove that $a_k = 0$. Observe that $P_\ell(k) = \binom{k}{\ell} = 0$ for $k < \ell < n$. Since $a_i = 0$ for $i = 0, \dots, k - 1$, and $P_k(k) = 1$ we get $L(k) = a_k lcm(k)$. As $k < \min(n, gpp(m)) \leq \min(n, m)$ and m has no prime divisor $p < \min(n, m)$, the numbers $lcm(k)$ and m are coprime. Thus, $lcm(k)$ is invertible in $\mathbb{Z}/m\mathbb{Z}$ and equality $L(k) = a_k lcm(k) = 0$ implies $a_k = 0$. \square

4. Counting Congruence Preserving Functions

We now compute the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. As two different rational polynomials correspond to different functions by Proposition 1.6 (uniqueness of the representation by a rational polynomial), the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is equal to the number of polynomials representing them.

Proposition 4.1. Let $CP(n, m)$ be the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. Let $m = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$ be the decomposition of m in powers of

primes. Let $\mathcal{I} = \{i \mid p_i^{e_i} < gpp(m)\}$ and $\mathcal{J} = \{i \mid p_i^{e_i} \geq gpp(m)\}$. Then

$$CP(n, m) = \begin{cases} p_1^{p_1+p_1^2+\dots+p_1^{e_1}} \times \dots \times p_\ell^{p_\ell+p_\ell^2+\dots+p_\ell^{e_\ell}} & \text{if } n \geq gpp(m), \\ \prod_{i \in \mathcal{I}} p_i^{p_i+p_i^2+\dots+p_i^{e_i}} \times \prod_{i \in \mathcal{J}} p_i^{p_i+p_i^2+\dots+p_i^{\lfloor \log_p n \rfloor + n(e - \lfloor \log_p n \rfloor)}} & \text{if } n < gpp(m). \end{cases}$$

Equivalently, writing $E(p, \alpha)$ instead of p^α for better readability, we have

$$CP(n, m) = \begin{cases} \prod_{i=1}^{\ell} E(p_i, \sum_{k=1}^{e_i} p_i^k) & \text{if } n \geq gpp(m), \\ \prod_{i \in \mathcal{I}} E(p_i, \sum_{k=1}^{e_i} p_i^k) \times \prod_{i \in \mathcal{J}} E(p_i, (\sum_{k=1}^{\lfloor \log_p n \rfloor} p_i^k) + n(e - \lfloor \log_p n \rfloor)) & \text{if } n < gpp(m). \end{cases}$$

Corollary 4.2. For $n \geq gpp(m)$, $CP(n, m)$ does not depend on n .

Proof of Proposition 4.1. By Theorem 3.10, we must count the number of n -tuples of coefficients (a_0, \dots, a_{n-1}) , with, for $k = 0, \dots, n - 1$, a_k being a multiple of $lcm(k)$ in $\mathbb{Z}/m\mathbb{Z}$.

Claim 1. For $m = p_1^{e_1} p_2^{e_2} \dots p_\ell^{e_\ell}$, for all n , $CP(n, m) = \prod_{i=1}^{\ell} CP(n, p_i^{e_i})$.

Proof of Claim 1. Let $E(r, k)$ be the set of multiples in $\mathbb{Z}/r\mathbb{Z}$ of $lcm(k)$ and $\lambda(r, k)$ be the cardinal of $E(r, k)$. The Chinese remainder theorem shows that the map $\rho : z \mapsto (z \pmod{p_i^{e_i}})_{i=1, \dots, \ell}$ is an isomorphism and also that ρ maps the set $E(m, k)$ onto the Cartesian product $P = \prod_{i=1}^{\ell} E(p_i^{e_i}, k)$. Indeed, let $(t_i)_{i=1, \dots, \ell} \in P$. For each $i = 1, \dots, \ell$, there is $0 \leq q_i < p_i^{e_i}$ such that $t_i \equiv q_i lcm(k) \pmod{p_i^{e_i}}$. Applying the Chinese remainder theorem, there are $0 \leq t, q < m$ such that $t \equiv t_i \pmod{p_i^{e_i}}$ and $q \equiv q_i \pmod{p_i^{e_i}}$. Then $t \equiv q lcm(k) \pmod{m}$, and hence $\rho(t) = (t_i)_{i=1, \dots, \ell}$. This proves that $\lambda(m, k) = \prod_{i=1}^{\ell} \lambda(p_i^{e_i}, k)$ for each k . Thus, the number $CP(n, m)$ of n -tuples (a_0, \dots, a_{n-1}) such that $lcm(k)$ divides a_k is equal to

$$CP(n, m) = \prod_{k < n} \lambda(m, k) = \prod_{k < n} \prod_{i=1}^{\ell} \lambda(p_i^{e_i}, k) = \prod_{i=1}^{\ell} \prod_{k < n} \lambda(p_i^{e_i}, k) = \prod_{i=1}^{\ell} CP(n, p_i^{e_i}). \quad \square$$

Claim 1 reduces the problem to that of counting the congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/p_i^{e_i}\mathbb{Z}$. We will use Theorem 3.10 to this end.

Claim 2. Letting $\ell = \lfloor \log_p n \rfloor$ (and using the $E(p, \alpha)$ notation for p^α), we have

$$CP(n, p^\ell) = \begin{cases} E(p, p + p^2 + \dots + p^\ell) & \text{if } n \geq p^\ell, \\ E(p, p + p^2 + \dots + p^\ell + (e - \ell)n) & \text{if } p^\ell \leq n < p^e. \end{cases}$$

Proof of Claim 2. By Theorem 3.10, as $gpp(p^\ell) = p^\ell$, letting $\nu = \inf(n, p^e)$, we have $CP(n, p^\ell) = CP(\nu, p^\ell) = \prod_{k=0}^{\nu-1} \lambda(p^\ell, k)$. As we noted in the proof of Claim 1, for

$p^j \leq k < p^{j+1}$, the order $\lambda(p^e, k)$ of the subgroup generated by $\text{lcm}(k)$ in $\mathbb{Z}/p^e\mathbb{Z}$ is p^{e-j} , and there are $p^{j+1} - p^j$ such k 's. For $k = 0$, $\text{lcm}(0) = 1$ yields $\lambda(p^e, 0) = p^e$.

- If $n \geq p^e$ then $CP(n, p^e) = CP(p^e, p^e) = p^e \prod_{j=0}^{e-1} \prod_{k=p^j}^{p^{j+1}-1} p^{e-j} = p^M$ with

$$M = e + \sum_{j=0}^{e-1} (e-j)(p^{j+1} - p^j) = p + p^2 + \dots + p^e$$

- If $n < p^e$ then $p^\ell \leq n < p^e$ and

$$\begin{aligned} CP(n, p^e) &= \prod_{k=0}^{n-1} \lambda(p^e, k) \\ &= p^e \left(\prod_{j=0}^{\ell-1} \prod_{k=p^j}^{p^{j+1}-1} p^{e-j} \right) \left(\prod_{k=p^\ell}^{n-1} p^{e-\ell} \right) = p^M \text{ with} \\ M &= e + \sum_{j=0}^{\ell-1} (e-j)(p^{j+1} - p^j) + \sum_{k=p^\ell}^{n-1} (e-\ell) \\ &= (e-\ell)p^\ell + (p + p^2 + \dots + p^\ell) + (n - p^\ell)(e-\ell) \\ &= (p + p^2 + \dots + p^\ell) + n(e-\ell) \quad \square \end{aligned}$$

This finishes the proof of Proposition 4.1. □

Remark 4.3. In [1] the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/p^e\mathbb{Z}$ is shown to be equal to $E(p, en - \sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor\})$. For $p^i \leq k < p^{i+1}$, we have $\lfloor \log_p k \rfloor = i$, and hence $\min\{e, \lfloor \log_p k \rfloor\} = \lfloor \log_p k \rfloor$ for $k \leq p^e$, and $\min\{e, \lfloor \log_p k \rfloor\} = e$ for $k \geq p^e$. Thus, we have

- if $n \geq p^e$, then $\sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor\} = \sum_{k=1}^{p^e-1} \lfloor \log_p k \rfloor + \sum_{k=p^e}^{n-1} e = \sum_{j=0}^{e-1} j(p^{j+1} - p^j) + e(n - p^e) = -(p + \dots + p^e) + ep^e + e(n - p^e)$, and hence $en - \sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor\} = p + \dots + p^e$. This coincides with our counting in Claim 2.

- if $n < p^e$, and $l = \lfloor \log_p n \rfloor$, then, similarly, $\sum_{k=1}^{n-1} \lfloor \log_p k \rfloor = \sum_{k=1}^{l-1} \lfloor \log_p k \rfloor + \sum_{k=l}^{n-1} \lfloor \log_p k \rfloor = \sum_{j=0}^{l-1} j(p^{j+1} - p^j) + l(n - p^l) = -(p + \dots + p^l) + nl$, and hence $en - \sum_{k=1}^{n-1} \lfloor \log_p k \rfloor = p + \dots + p^l + (e - l)n$. Again, this coincides with our counting in Claim 2.

5. Conclusion

We proved that the rational polynomials $\text{lcm}(k) P_k$ generate the $\mathbb{Z}/m\mathbb{Z}$ submodule of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$. When n is larger than the greatest prime power dividing m , the number of functions in this submodule is independent of n . An open problem is the existence of a basis of this submodule.

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