ON A CUBIC MOMENT FOR SUMS OF HECKE EIGENVALUES

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Abstract

Let $\lambda_f(n)$ be the n-th normalized Fourier coefficient of the Fourier series associated with a holomorphic cusp form f for the full modular group of even weight k and let $A_f(x) := \sum_{n\leq x}\lambda_f(n)$. During the ELAZ 2014 conference in Hildesheim, Germany, K.-L. Kong (University of Hong Kong) presented his result, proved in his Master thesis, that

$$\int_2^X \Delta^2(t)\Delta(\alpha t)dt = C(\alpha)X^{7/4} + O_\varepsilon\left(X^{7/4-\delta}\right),$$

for some explicit $\delta > 0$, $C(\alpha)$, where $\alpha > 0$ is fixed and $\Delta(x)$ is the error term in the Dirichlet divisor problem. A problem posed by Professor Ivić at this conference was to obtain a formula analogous to the above formula for the sum $A_f(x)$ and especially to discuss the sign of $C(\alpha)$ in the new setting. In this paper, we will solve Ivić's problem and prove that for any $\varepsilon > 0$, we have

$$\int_2^X A_f^2(t)A_f(\alpha t)dt = C_f(\alpha)X^{7/4} + O_{\alpha,\varepsilon}\left(X^{\frac{41}{24}+\varepsilon}\right),$$

for some constant $C_f(\alpha)$ depending on only f, α and defined by

$$C_f(\alpha) = \frac{\alpha^{1/4}}{28\pi^3}\sum_{(i_0,i_1)\in\{0,1\}^2}\ \sum_{\substack{n,m,l=1\\ \sqrt{n}+(-1)^{i_0}\sqrt{m}+(-1)^{i_1}\sqrt{\alpha l}=0}}^{+\infty}\frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}},$$

where $\alpha > 0$ is a fixed constant. Our result is new and throws light on the behavior of the classical function $A_f(x)$.

1. Introduction

Let $k \geq 2$ be an even integer and H_k^* be the set of all primitive cusp forms of weight k for the full modular group $SL_2(\mathbb{Z})$. If $f \in H_k^*$, then it has the following Fourier expansion at the cusp ∞ :

$$f(z) = \sum_{n=1}^{+\infty}\lambda_f(n)n^{(k-1)/2}e^{2\pi inz} \qquad (\Im(z) > 0).$$

By the theory of Hecke operators, $\lambda_f(n)$ is real and satisfies the multiplicative property:

$$\lambda_f(n)\lambda_f(m) = \sum_{d|(n,m)} \lambda_f\left(\frac{mn}{d^2}\right) \tag{1}$$

for all integers $m \geq 1$ and $n \geq 1$. Besides, it is also known that $\lambda_f(n)$ satisfies the deep inequality:

$$|\lambda_f(n)| \leq d(n) \tag{2}$$

for all $n \geq 1$, where $d(n)$ denotes the number of positive divisors of n (this is the Ramanujan-Petersson conjecture proved by Deligne (see [1], [2])).

The sum of normalized Fourier coefficients over natural numbers occurs in the study of many important problems in number theory, such as the number of Hecke eigenvalues of same signs (see [5]).

The sequence $(\lambda_f(n))$ is of great arithmetic interest. The generating series

$$L(f,s) = \sum_{n=1}^{+\infty} \frac{\lambda_f(n)}{n^s}$$

has many analytic properties which offer tools to help achieve this goal. Moreover, Hecke (see [3], [4]) proved that $L(f,s)$ is an entire function that satisfies the following functional equation, which is a special case of the Langlands functoriality:

$$L(f,1-s) = (-1)^{k/2}\gamma(s)L(f,s) \qquad \text{if } \Re(s) > 1, \tag{3}$$

where

$$\gamma(s) = \pi^{(1-2s)} \prod_{j=1}^{2} \Gamma\left(\frac{s+\kappa_j}{2}\right)\Gamma\left(\frac{1-s+\kappa_j}{2}\right)^{-1}$$

and the parameters in the product are $\kappa_1 = \frac{k-1}{2}$ and $\kappa_2 = \frac{k+1}{2}$.

One of the basic goals of number theory is the foundation of asymptotic formula, as accurate as possible, for the sum $A_f(x)$. Rankin (see [7]) showed that for any $\varepsilon > 0$ and $x \geq 2$,

$$A_f(x) \ll_f x^{\frac{1}{3}}(\log x)^{-\delta+\varepsilon},$$

where $\delta = 0.0652$. Wu (see [8, Thereom 2.]) got a better bound for $A_f(x)$, that is

$$A_f(x) \ll_f x^{\frac{1}{3}}(\log x)^{-\delta},$$

where $\delta = 0.1185$.

In this paper, we shall evaluate the integral $\int_2^X A_f^2(x)A_f(\alpha x)dx$ basing on the truncated Voronoi formula for the sum $A_f(x)$ and we will investigate the sign of the constant $C_f(\alpha)$. The main result of this paper is the following.

Theorem 1. *Let $f \in H_k^*$ and $0 < \alpha \in \mathbb{Q}[\sqrt{N}]$, where $N > 1$ is a square-free natural number. If there exist integers t, r, s, a_2', b_2', d_1 and d_2 such that*

$$\begin{cases} t|N, t \quad odd \\ r \quad odd, \frac{N}{t}s^2 \quad even, \quad \gcd(r,s) = 1 \\ b_2' | \left(tr^2 + \frac{N}{t}s^2\right) \\ a_2'|2rs \\ \gcd(d_1, d_2) = 1 \\ \alpha = \frac{d_1}{a_2' b_2' d_2}\left[\left(tr^2 + \frac{N}{t}s^2\right) \pm 2rs\sqrt{N}\right] \end{cases} \tag{4}$$

or

$$\begin{cases} t|N, N \quad odd \\ r, s \quad odd, \quad \gcd(r,s) = 1 \\ b_2' | \frac{1}{2}\left(tr^2 + \frac{N}{t}s^2\right) \\ a_2'|rs \\ \gcd(d_1, d_2) = 1 \\ \alpha = \frac{d_1}{a_2' b_2' d_2}\left[\frac{1}{2}\left(tr^2 + \frac{N}{t}s^2\right) \pm rs\sqrt{N}\right], \end{cases} \tag{5}$$

then there exists a constant $C_f(\alpha)$ depending on only α and f and defined by

$$C_f(\alpha) = \frac{\alpha^{1/4}}{28\pi^3} \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n,m,l=1 \\ \sqrt{n}+(-1)^{i_0}\sqrt{m}+(-1)^{i_1}\sqrt{\alpha l}=0}}^{+\infty} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} \tag{6}$$

such that for any $\varepsilon > 0$, we have

$$\int_2^X A_f^2(t) A_f(\alpha t)\,dt = C_f(\alpha) X^{7/4} + O_{\alpha,\varepsilon}\left(X^{\frac{41}{24}+\varepsilon}\right).$$

Our main tool in the proof of Theorem 1 is the Voronoi summation formula. Lau and Wu (see [5, Lemma 3.1]) established the truncated Voronoi formula for $A_f(x)$. They obtained the following result.

Lemma 1. *Let $f \in H_k^*$. Then for any $A > 0$ and $\varepsilon > 0$, uniformly for $1 \le M \le x^A$ and $x \ge 1$, we have*

$$A_f(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \le M} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O_{A,\varepsilon,k}\left(x^{\varepsilon}\left\{1 + \left(\frac{x}{M}\right)^{1/2} + x^{-1/4}\right\}\right), \tag{7}$$

where the implied O-constant depends on A, ε and k only.

We will use (7) and write

$$A_f(x) = B_M(x) + O_{A,\varepsilon,k}(x^{\varepsilon}) + O_{A,\varepsilon,k}\left(x^{1/2+\varepsilon} M^{-1/2}\right), \tag{8}$$

where

$$B_M(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \le M} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right).$$

Now, we will study the sign of the coefficient $C_f(\alpha)$ in the following theorem.

Theorem 2. *Let $k \geq 2$ be an even integer and let $f \in H_k^*$. Set*

$$\mathcal{A}_\pm := \{a \in \mathbb{N}^*;\ \lambda_f(a) \gtrless 0\},$$

$$S_+ := \sum_{\substack{l''=1 \\ Nr^2s^2l'' \in \mathcal{A}_+}}^{+\infty} \left(\frac{\lambda_f\left(Nr^2s^2l''\right)\lambda_f\left(l''\right)^2}{l''^{9/4}} \right), \text{ and}$$

$$S_- := -\sum_{\substack{l''=1 \\ Nr^2s^2l'' \in \mathcal{A}_-}}^{+\infty} \left(\frac{\lambda_f\left(Nr^2s^2l''\right)\lambda_f\left(l''\right)^2}{l''^{9/4}} \right),$$

so that both S_+ and S_- are positive.

1. *Suppose that α satisfies (4). If $d_1 = a_2'b_2'd_2 = 1$, then*

$$\left\{ \begin{array}{ll} C_f(\alpha) > 0 & \text{if } S_+ > S_- \\ C_f(\alpha) < 0 & \text{if } S_+ < S_- \\ C_f(\alpha) = 0 & \text{if } S_+ = S_- \end{array} \right\}.$$

2. *Suppose that α satisfies (5). If $d_1 = 2$ and $a_2'b_2'd_2 = 1$, then*

$$\left\{ \begin{array}{ll} C_f(\alpha) > 0 & \text{if } S_+ > S_- \\ C_f(\alpha) < 0 & \text{if } S_+ < S_- \\ C_f(\alpha) = 0 & \text{if } S_+ = S_- \end{array} \right\}.$$

In particular, examples when $C_f(\alpha) = 0$ are given in the following assertion.
3. *If $\alpha \in \mathbb{Q}_+^*$ or $\alpha \in \sqrt{N}\mathbb{Q}_+^*$, then $C_f(\alpha) = 0$.*

2. Some Lemmas

The proof of Theorem 1 is based on the following lemmas.

Lemma 2. *Let α be a positive number in $\mathbb{Q}[\sqrt{N}]$, where N is a square-free positive integer and let $i_0, i_1 \in \{0,1\}$. We put*

$$\alpha_1 := \alpha_1(i_0, i_1) := \sqrt{n} + (-1)^{i_0}\sqrt{m} + (-1)^{i_1}\sqrt{\alpha l}, \tag{9}$$

$$c_1(\alpha, y) := \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n\le y, m\le y\\ l\le y\\ \alpha_1=0}} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} \tag{10}$$

and

$$c_1(\alpha) := \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n,m,l=1\\ \alpha_1=0}}^{+\infty} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}}. \tag{11}$$

Then, we have the following assertions.
(a) The equation $\alpha_1 = 0$ *is solvable for* n, m, $l \in \mathbb{N}^*$ *if and only if there exist integers* t, r, s, d_1, d_2, a_2' *and* b_2' *such that*

$$\begin{cases} t|N, t \quad \textit{odd} \\ r \quad \textit{odd}, \frac{N}{t}s^2 \quad \textit{even}, \quad \gcd(r,s) = 1 \\ b_2'|\left(tr^2 + \frac{N}{t}s^2\right) \\ a_2'|2rs \\ \gcd(d_1, d_2) = 1 \\ \alpha = \frac{d_1}{d_2a_2'b_2'}\left[\left(tr^2 + \frac{N}{t}s^2\right) \pm 2rs\sqrt{N}\right] \end{cases}$$

or

$$\begin{cases} t|N, N \quad \textit{odd} \\ r, s \quad \textit{odd}, \quad \gcd(r,s) = 1 \\ b_2'|\frac{1}{2}\left(tr^2 + \frac{N}{t}s^2\right) \\ a_2'|rs \\ \gcd(d_1, d_2) = 1 \\ \alpha = \frac{d_1}{d_2a_2'b_2'}\left[\frac{1}{2}\left(tr^2 + \frac{N}{t}s^2\right) \pm rs\sqrt{N}\right]. \end{cases}$$

(b) On the hypotheses of (4) and (5), the series $c_1(\alpha)$ *converges.*
(c) $|c_1(\alpha, y) - c_1(\alpha)| \ll_{\alpha,\varepsilon} y^{-5/4+\varepsilon}$.

Proof. (a) Let $\alpha = \frac{a_1}{a_2} + \frac{b_1}{b_2}\sqrt{N}$, where a_1, $b_1 \in \mathbb{Z}$, a_2, $b_2 \in \mathbb{N}^*$ and $\gcd(a_1, a_2) = \gcd(b_1, b_2) = 1$ and let $d_i = \gcd(a_i, b_i)$. So, we have $a_i = d_ia_i'$ and $b_i = d_ib_i'$, $i \in \{1, 2\}$. Note that at most one of the equations $\alpha_1(0, 1) = 0$ and $\alpha_1(1, 1) = 0$ is solvable in n, m and $l \in \mathbb{N}^*$.

If the equation $\alpha_1(0, 1) = 0$ is solvable for n, m, $l \in \mathbb{N}^*$, then by squaring both sides, we obtain

$$a_2b_2(n + m) - b_2a_1l = a_2b_1l\sqrt{N} - 2a_2b_2\sqrt{nm}. \tag{12}$$

By using the same procedure, we get

$$4a_2^2b_1b_2l\sqrt{Nnm} = (a_2b_1l)^2N + (2a_2b_2)^2nm - (a_2b_2(n + m) - b_2a_1l)^2.$$

This shows that $\sqrt{Nnm}$ is necessarily an integer. Also, it is easy to show that $\sqrt{\frac{mn}{N}}$ is an integer. Hence, Formula (12) leads to

$$a_2b_2(n+m) - b_2a_1l = \sqrt{N}\left(a_2b_1l - 2a_2b_2\sqrt{\frac{mn}{N}}\right).$$

Since N is a square-free integer and $\sqrt{\frac{mn}{N}}$ is an integer, then we have $\sqrt{N}$ is an irrational and $\sqrt{N}\left(a_2b_1l - 2a_2b_2\sqrt{\frac{mn}{N}}\right)$ is an irrational or zero. So, we get

$$a_2(n+m) = a_1l \qquad \text{and} \qquad b_1l = 2b_2\sqrt{\frac{mn}{N}}. \tag{13}$$

This implies that $a_1 > 0$ and $b_1 > 0$. Moreover, we know that a_2 divides a_1l and b_2 divides b_1l. Since $\gcd(a_1, a_2) = \gcd(b_1, b_2) = 1$, it follows that a_2 and b_2 divide l. Therefore, there exists a positive integer l' such that $l = a_2l'$ and b_2 divides a_2l'. Then $l' = \frac{b_2}{d_2}l$", where $l'' = \frac{l}{a_2'b_2} \in \mathbb{N}$. Thus

$$l = a_2l' = \frac{a_2b_2}{d_2}l'' = a_2'b_2'd_2l''. \tag{14}$$

By considering (13) and (14), we get

$$(n+m) = a_1b_2'l'' = a_1'd_1b_2'l'' \qquad \text{and} \qquad \sqrt{\frac{mn}{N}} = \frac{b_1a_2'l''}{2} = \frac{a_2'b_1'd_1l''}{2},$$

which is equivalent to

$$(n+m) = a_1b_2'l'' = a_1'd_1b_2'l'' \qquad \text{and} \qquad mn = N\frac{(b_1'd_1a_2'l'')^2}{4}.$$

By solving the last two equations for n, we obtain

$$n = d_1l''\frac{\left(a_1'b_2' \pm \sqrt{(a_1'b_2')^2 - N(a_2'b_1')^2}\right)}{2}.$$

Since $\gcd(a_1', a_2') = \gcd(b_1', b_2') = 1$, we get $\gcd(a_1'b_2', a_2'b_1') = 1$.

Notice that n is well-defined i.e the equation

$$\sqrt{(a_1'b_2')^2 - N(a_2'b_1')^2} = e \in \mathbb{N}$$

has solutions in $\mathbb{N}^3$ provided we have the following cases:
If N is odd, there exist integers t, r and s such that

$$\begin{cases} t|N, t \quad \text{odd} \\ r \quad \text{odd}, s \quad \text{even}, \quad \gcd(r,s) = 1 \\ a_1'b_2' = tr^2 + \frac{N}{t}s^2 \\ a_2'b_1' = 2rs \end{cases} \tag{15}$$

or

$$\begin{cases} t|N, t \quad \text{odd} \\ r, s \quad \text{odd} \quad \gcd(r,s) = 1 \\ a'_1 b'_2 = \frac{1}{2}\left(tr^2 + \frac{N}{t}s^2\right) \\ a'_2 b'_1 = rs. \end{cases} \tag{16}$$

However, if N is even, then

$$\begin{cases} t|N, r \quad \text{odd} \quad \gcd(r,s) = 1 \\ a'_1 b'_2 = tr^2 + \frac{N}{t}s^2 \\ a'_2 b'_1 = 2rs. \end{cases} \tag{17}$$

Now, if (15) is valid, then

$$n \qquad = d_1 l'' \frac{\left(tr^2 + \frac{N}{t}s^2 \pm (tr^2 - \frac{N}{t}s^2)\right)}{2}.$$

So, we have $n = d_1 l'' tr^2$ and $m = d_1 l'' \frac{N}{t}s^2$, or $n = d_1 l'' \frac{N}{t}s^2$ and $m = d_1 l'' tr^2$. Therefore $b_1 = d_1 b'_1 = d_1 \frac{2rs}{a'_2}$ and $a_1 = d_1 a'_1 = d_1 \frac{\left(tr^2 + \frac{N}{t}s^2\right)}{b'_2}$. It follows that

$$\alpha = \frac{d_1 \frac{\left(tr^2 + \frac{N}{t}s^2\right)}{b'_2}}{d_2 a'_2} + \frac{d_1 \frac{2rs}{a'_2}}{d_2 b'_2}\sqrt{N} = \frac{d_1}{d_2 a'_2 b'_2}\left[tr^2 + \frac{N}{t}s^2 + 2rs\sqrt{N}\right].$$

If (16) is valid, then

$$n \qquad = d_1 l'' \frac{\frac{1}{2}\left(tr^2 + \frac{N}{t}s^2\right) \pm \frac{1}{2}\left(tr^2 - \frac{N}{t}s^2\right)}{2}.$$

Assuming that l'' and d_1 are not both odd, then this implies that $n = \frac{d_1 l'' tr^2}{2}$ and that $m = \frac{d_1 l'' \frac{N}{t} s^2}{2}$, or $n = \frac{d_1 l'' \frac{N}{t} s^2}{2}$ and $m = \frac{d_1 l'' tr^2}{2}$. Thus $b_1 = d_1 \frac{rs}{a'_2}$ and $a_1 = \frac{d_1\left(tr^2 + \frac{N}{t}s^2\right)}{2b'_2}$. This leads to

$$\alpha = \frac{d_1\left(tr^2 + \frac{N}{t}s^2\right)}{2b'_2 d_2 a'_2} + \frac{d_1 \frac{rs}{a'_2}}{d_2 b'_2}\sqrt{N} = \frac{d_1}{d_2 a'_2 b'_2}\left[\frac{1}{2}\left(tr^2 + \frac{N}{t}s^2\right) + rs\sqrt{N}\right].$$

If (17) is true, we will obtain the same values of n and m as in (15), and therefore

$$\alpha = \frac{d_1}{d_2 a'_2 b'_2}\left[tr^2 + \frac{N}{t}s^2 + 2rs\sqrt{N}\right].$$

Subsequently, the numbers n, m and l are well-defined so that the equation $\alpha_1(0,1) = 0$ is solvable for n, m and $l \in \mathbb{N}^*$.

Now, if the equation $\alpha_1(1,1)=0$ is solvable then, by following the aforementioned steps, we get

$$a_2(n+m)=a_1 l \qquad \text{and} \qquad 2b_2\sqrt{\frac{mn}{N}}=-b_1 l.$$

Let $\tilde{b}_1=-b_1>0$. We replace $\tilde{b}_1$ by b_1 and we will obtain

$$\alpha=\frac{d_1}{d_2a_2'b_2'}\left[tr^2+\frac{N}{t}s^2-2rs\sqrt{N}\right]$$

or

$$\alpha=\frac{d_1}{d_2a_2'b_2'}\left[\frac{1}{2}\left(tr^2+\frac{N}{t}s^2\right)-rs\sqrt{N}\right].$$

(b) Now, assume that α satisfies (4); then

$$\left\{\begin{array}{l} n=d_1l''tr^2 \\ m=d_1l''\frac{N}{t}s^2 \\ l=a_2'b_2'd_2l'' \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} n=d_1l''\frac{N}{t}s^2 \\ m=d_1l''tr^2 \\ l=a_2'b_2'd_2l''. \end{array}\right. \tag{18}$$

This is valid provided that

$$\alpha=\frac{d_1}{d_2a_2'b_2'}\left[tr^2+\frac{N}{t}s^2+2rs\sqrt{N}\right]$$

or

$$\alpha=\frac{d_1}{d_2a_2'b_2'}\left[tr^2+\frac{N}{t}s^2-2rs\sqrt{N}\right].$$

Formula (18) implies that

$$\begin{aligned} c_1(\alpha) \qquad &\ll \sum_{l''=1}^{+\infty}\frac{|\lambda_f(d_1l''\frac{N}{t}s^2)||\lambda_f(d_1l''tr^2)||\lambda_f(a_2'b_2'd_2l'')|}{(d_1^2Ns^2r^2a_2'b_2'd_2l''^3)^{3/4}} \\ &\ll \frac{|\lambda_f(d_1\frac{N}{t}s^2)||\lambda_f(d_1tr^2)||\lambda_f(a_2'b_2'd_2)|}{(d_1^2Ns^2r^2a_2'b_2'd_2)^{3/4}}\sum_{l''=1}^{+\infty}\frac{|\lambda_f(l'')|^3}{l''^{9/4}}. \end{aligned}$$

From (2), we get $|\lambda_f(l'')|\ll_\varepsilon l''^\varepsilon$ for any $\varepsilon>0$. Thus, we have $c_1(\alpha)\ll_\varepsilon 1$ and therefore, the series $c_1(\alpha)$ converges.

If α satisfies (5), then

$$\left\{\begin{array}{l} n=\frac{1}{2}(d_1tr^2)l'' \\ m=\frac{1}{2}(d_1\frac{N}{t}s^2)l'' \\ l=a_2'b_2'd_2l'' \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} n=\frac{1}{2}(d_1\frac{N}{t}s^2)l'' \\ m=\frac{1}{2}(d_1tr^2)l'' \\ l=a_2'b_2'd_2l'', \end{array}\right. \tag{19}$$

this is valid provided that

$$\alpha = \frac{d_1}{d_2 a_2' b_2'} \left[\frac{1}{2} \left(tr^2 + \frac{N}{t} s^2 \right) + rs\sqrt{N} \right]$$

or

$$\alpha = \frac{d_1}{d_2 a_2' b_2'} \left[\frac{1}{2} \left(tr^2 + \frac{N}{t} s^2 \right) - rs\sqrt{N} \right].$$

Formula (19) implies that

$$\begin{aligned} c_1(\alpha) \quad & \ll \sum_{l''=1}^{+\infty} \frac{\left|\lambda_f\left(\frac{1}{2}(d_1 tr^2)l''\right)\right| \left|\lambda_f\left(\frac{1}{2}(d_1 \frac{N}{t} s^2)l''\right)\right| |\lambda_f(a_2' b_2' d_2 l'')|}{(d_1^2 N s^2 r^2 a_2' b_2' d_2 l''^3)^{3/4}} \\ & \ll \sum_{l''=1}^{+\infty} \frac{\left|\lambda_f\left(d_1 tr^2 l''\right)\right| \left|\lambda_f\left(d_1 \frac{N}{t} s^2 l''\right)\right| |\lambda_f(a_2' b_2' d_2 (2l''))|}{(d_1^2 N s^2 r^2 a_2' b_2' d_2 (2l)"^3)^{3/4}} \\ & \ll \frac{|\lambda_f(d_1 \frac{N}{t} s^2)||\lambda_f(d_1 tr^2)||\lambda_f(2a_2' b_2' d_2)|}{(d_1^2 N s^2 r^2 a_2' b_2' d_2)^{3/4} 2^{9/4}} \sum_{l''=1}^{+\infty} \frac{|\lambda_f(l'')|^3}{l''^{9/4}} \ll_\varepsilon 1. \end{aligned}$$

Hence, by Formula (2), the series $c_1(\alpha)$ converges.

(c) Recall that (see (10))

$$c_1(\alpha, y) = \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n\le y, m\le y \\ l\le y \\ \alpha_1=0}} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}}.$$

Note that if α does not satisfy (4) and (5), then, based on the findings of (a), the equation $\alpha_1 = 0$ is not solvable in n, m and $l \in \mathbb{N}^*$. Thus, we obtain

$$\sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n,m,l=1 \\ \alpha_1=0}}^{+\infty} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} = \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n\le y, m\le y \\ l\le y \\ \alpha_1=0}} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} = 0,$$

which is equivalent to $c_1(\alpha) = c_1(\alpha, y) = 0$. Suppose now that α satisfies (4) or (5).

For instance, if α satisfies (4); then we have

$$\begin{aligned}
|c_1(\alpha)-c_1(\alpha,y)| &= \left|\sum_{(i_0,i_1)\in\{0,1\}^2}\ \sum_{\substack{n>y\\ \text{or } m>y\\ \text{or } l>y\\ \alpha_1=0}}^{+\infty}\frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}}\right|\\
&\ll \sum_{(i_0,i_1)\in\{0,1\}^2}\ \sum_{\substack{n>y\\ \text{or } m>y\\ \text{or } l>y\\ \alpha_1=0}}^{+\infty}\frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}}\\
&= \sum_{\substack{d_1l''tr^2>y\\ \text{or } d_1l''\frac{N}{t}s^2>y\\ \text{or } a_2'b_2'd_2l''>y}}\frac{|\lambda_f(d_1l''tr^2)||\lambda_f(d_1l''\frac{N}{t}s^2)||\lambda_f(a_2'b_2'd_2l'')|}{(d_1^2r^2Ns^2a_2'b_2'd_2l''^3)^{3/4}}\\
&\ll \frac{|\lambda_f(d_1tr^2)||\lambda_f(d_1\frac{N}{t}s^2)||\lambda_f(a_2'b_2'd_2)|}{(d_1^2r^2Ns^2a_2'b_2'd_2)^{3/4}}\sum_{\substack{d_1l''tr^2>y\\ \text{or } d_1l''\frac{N}{t}s^2>y\\ \text{or } a_2'b_2'd_2l''>y}}\frac{|\lambda_f(l'')|^3}{l''^{9/4}}.
\end{aligned}$$

By using (2) and since we have $d_1l''tr^2>y$ or $d_1l''\frac{N}{t}s^2>y$ or $a_2'b_2'd_2l''>y$, then the last formula implies that

$$|c_1(\alpha)-c_1(\alpha,y)| \ll_{\alpha,\varepsilon} y^{-5/4+\varepsilon}.$$

The other case is similar. □

Lemma 3. *Let α be a positive number in $\mathbb{Q}[\sqrt{N}]$, where $N>1$ is a square-free integer. Let U_i and V_i for $i=1,2,3$, be positive real numbers such that $U_i<V_i\ll T^b$ and at least two of the U_i's are $\gg T^a$ for some positive real numbers a and b with $a\le b$. Then we have*

(a)

$$\sum_{\substack{U_1<n\le V_1,U_2<m\le V_2,\\ U_3<l\le V_3,\alpha_1\neq 0}}\frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}}\left|\int_T^{2T}t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|$$

$$\ll_{\alpha,\varepsilon} T^{5/4+b/4+\varepsilon}+T^{7/4-a/4+\varepsilon}$$

and

(b)

$$\sum_{\substack{n\le V_1, m\le V_2, l\le V_3\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left| \int_T^{2T} t^{3/4} \cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right) dt \right|$$

$$\ll_{\alpha,\varepsilon} T^{5/4+11b/4+\varepsilon},$$

where α_1 *is defined by (9).*

Proof. (a) Note that $\alpha_1(0,0) \gg (\alpha nml)^{-1/6}$. Also, by using the fact that for any $t \in [T, 2T]$, we have $t^{3/4} \ll T^{3/4}$ and that $|\left[\cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right)\right]'| \gg \frac{4\pi|\alpha_1|}{\sqrt{T}}$, we get

$$\left| \int_T^{2T} t^{3/4} \cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right) dt \right| \ll \frac{T^{5/4}}{|\alpha_1|}. \tag{20}$$

Thus,

$$\sum_{\substack{U_1<n\le V_1, U_2<m\le V_2,\\ U_3<l\le V_3, \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left| \int_T^{2T} t^{3/4} \cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right) dt \right|$$

$$\ll_{\alpha,\varepsilon} T^{5/4} \sum_{\substack{U_1<n\le V_1, U_2<m\le V_2, U_3<l\le V_3\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{|\alpha_1|(nml)^{3/4}}$$

$$\ll_{\alpha,\varepsilon} T^{5/4} \sum_{\substack{U_1<n\le V_1, U_2<m\le V_2, U_3<l\le V_3\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{11/12}}.$$

Now, since $U_i < V_i \ll T^b$ for all $i = 1, 2, 3$, it follows that n, m and l are $\ll T^b$ and hence, by Formula (2), we obtain: $|\lambda_f(n)|$, $|\lambda_f(m)|$ and $|\lambda_f(l)|$ are $\ll_\varepsilon T^{b+\varepsilon}$. Therefore,

$$T^{5/4} \sum_{\substack{U_1<n\le V_1, U_2<m\le V_2, U_3<l\le V_3\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{11/12}} \ll T^{5/4+b/4+\varepsilon}.$$

Suppose now that $(i_0, i_1) = (0, 1)$ as the other case is similar. There exist integers N, N', M, M', L and L' such that $U_1 < N < n \le N' \le \min(V_1, 2N)$, $U_2 < M < m \le M' \le \min(V_2, 2M)$ and $U_3 < L < l \le L' \le \min(V_3, 2L)$. Let $D = \max(N, M, L)$ and $d = \min(N, M, L)$. For $\delta \ll \sqrt{D}$, we have

$$|\{(n, m, l) : n \sim N, \quad m \sim M, \quad l \sim L, \quad 0 < |\alpha_1| < \delta\}|$$

$$\ll_{\alpha,\varepsilon} D^{-1/2}\delta NML + \left(\frac{L}{d}\right)^{1/4} D^{\varepsilon}\sqrt{NML}. \tag{21}$$

So, we obtain

$$\sum_{\substack{U_1<n\le V_1, U_2<m\le V_2, U_3<l\le V_3\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left|\int_T^{2T} t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|$$

$$\ll \sum_{\substack{N<n\le N', M<m\le M',\\ L<l\le L', \alpha_1\neq 0}} \frac{T^{\varepsilon}|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left|\int_T^{2T} t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|$$

$$\ll \left(\sum_{\substack{N<n\le N', M<m\le M',\\ L<l\le L', |\alpha_1|\ge\frac{1}{10}}} + \sum_{\substack{N<n\le N', M<m\le M',\\ L<l\le L', T^{-1/2}\le|\alpha_1|<\frac{1}{10}}} + \sum_{\substack{N<n\le N', M<m\le M',\\ L<l\le L', 0<|\alpha_1|<T^{-1/2}}}\right)$$

$$\frac{T^{\varepsilon}|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left|\int_T^{2T} t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|$$

$$\ll C_1 + C_2 + C_3.$$

Since $\alpha_1(0,1) < 2\sqrt{\alpha D}$ and by using Formulas (20) and (21) with $\frac{1}{10} \le \delta < 2\sqrt{\alpha D}$, we get

$$C_1 \ll_{\alpha,\varepsilon} T^{5/4+b/4+\varepsilon}.$$

Moreover, we have $L \asymp \max(N, M) \asymp D$ and $d \asymp \min(N, M)$, in case $\alpha_1(0,1)$ is between $T^{\frac{-1}{2}}$ and $1/10$. By using the same equations with $T^{-1/2} \le \delta < \frac{1}{10}$ in Formula (21), we obtain

$$C_2 \ll_{\alpha,\varepsilon} T^{5/4+b/4+\varepsilon} + T^{7/4-a/4+\varepsilon}.$$

Now, by trivial estimation and by Formula (21), we have

$$C_3 \ll_{\alpha,\varepsilon} T^{5/4+b/4+\varepsilon} + T^{7/4-a/4+\varepsilon}.$$

Consequently, we get

$$\sum_{\substack{U_1<n\le V_1, U_2<m\le V_2, U_3<l\le V_3\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left|\int_T^{2T} t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|$$

$$\ll_{\alpha,\varepsilon} T^{5/4+b/4+\varepsilon} + T^{7/4-a/4+\varepsilon}.$$

(b) We use the same arguments as above for $(i_0, i_1) = (0,0)$, and we get

$$\sum_{\substack{n\le V_1, m\le V_2, l\le V_3\\ \alpha_1(0,0)\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left| \int_T^{2T} t^{3/4} \cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right) dt \right| \ll_{\alpha,\varepsilon} T^{\frac{5}{4}+\frac{b}{4}+\varepsilon}.$$

For $(i_0, i_1) = (0,1)$, we have

$$\begin{aligned}
&\sum_{\substack{n\le V_1, m\le V_2, l\le V_3\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left| \int_T^{2T} t^{3/4} \cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right) dt \right| \\
&\ll \left(\sum_{\substack{N<n\le N', M<m\le M', L<l\le L'\\ |\alpha_1|\ge \frac{1}{10}}} + \sum_{\substack{N<n\le N', M<m\le M', L<l\le L'\\ |\alpha_1|< \frac{1}{10}}} \right) \\
&\frac{T^{\varepsilon}|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left| \int_T^{2T} t^{3/4} \cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right) dt \right| \\
&\ll C_1' + C_2'.
\end{aligned}$$

By the same token, we get $C_1' \ll_{\alpha,\varepsilon} T^{5/4+b/4+\varepsilon}$. Since $\alpha_1 \gg_{\alpha} D^{-3}d^{-1/2}$ and by using Formula (21) with $D^{-3}d^{-1/2} \le \delta < \frac{1}{10}$, we obtain

$$C_2' \ll_{\alpha,\varepsilon} T^{5/4+11b/4+\varepsilon}.$$

Therefore, we have

$$\sum_{\substack{n\le V_1, m\le V_2, l\le V_3\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left| \int_T^{2T} t^{3/4} \cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right) dt \right| \ll_{\alpha,\varepsilon} T^{\frac{5}{4}+\frac{11b}{4}+\varepsilon}.$$

□

3. Proof of Theorem 1

In this section, we will use the results of the previous lemmas to prove Theorem 1. Let $M = T$. From Formula (8), we have

$$\begin{aligned}
\int_T^{2T} A_f^2(t)A_f(\alpha t)dt &= \int_T^{2T} (B_M(t))^2 B_M(\alpha t)dt + O_{A,\varepsilon,k}\left(T^{\varepsilon}\int_T^{2T} |B_M(t)|\,|A_f(\alpha t)|dt\right) \\
&\quad + O_{A,\varepsilon,k}\left(T^{2\varepsilon}\int_T^{2T} |A_f(\alpha t)|dt\right) + O_{A,\varepsilon,k}\left(T^{\varepsilon}\int_T^{2T} A_f^2(t)dt\right) \\
&= \int_T^{2T} (B_M(t))^2 B_M(\alpha t)dt + O_{\alpha,A,\varepsilon,k}\left(T^{3/2+\varepsilon}\right).
\end{aligned}$$

The last identity is due to Cauchy-Schwarz's inequality and the well-known mean-square results (see [6, Theorem 1.1])

$$\int_T^{2T} A_f^2(t)dt \asymp T^{\frac{3}{2}},$$

where $f(t) \asymp g(t)$ means that $f(t) \ll g(t)$ and $g(t) \ll f(t)$. This can be established also by using (8). Let

$$B_{M,y}(t) = \frac{t^{1/4}}{\pi\sqrt{2}} \sum_{M<n\leq y} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right)$$

and let $M_0 = T^{1/6}$. We have

$$B_M(t) = B_{M_0}(t) + B_{M_0,M}(t).$$

Thus, we get

$$\begin{aligned}
\int_T^{2T} (B_M(t))^2 B_M(\alpha t)dt \ &= \int_T^{2T} (B_{M_0}(t))^2 B_{M_0}(\alpha t)dt + \int_T^{2T} (B_{M_0}(t))^2 B_{M_0,M}(\alpha t)dt \\
&\quad +2\int_T^{2T} B_{M_0}(t)B_{M_0,M}(t)B_{M_0}(\alpha t)dt \\
&\quad +2\int_T^{2T} B_{M_0}(t)B_{M_0,M}(t)B_{M_0,M}(\alpha t)dt \\
&\quad +\int_T^{2T} (B_{M_0,M}(t))^2 B_{M_0}(\alpha t)dt \\
&\quad +\int_T^{2T} (B_{M_0,M}(t))^2 B_{M_0,M}(\alpha t)dt \\
&= S_1 + S_2 + S_3 + S_4 + S_5 + S_6.
\end{aligned}$$

The formula $\cos(a)\cos(b)\cos(c) = \frac{1}{4}[\cos(a+b+c) + \cos(a+b-c) + \cos(a-b+c) + \cos(a-b-c)]$ implies that

$$\begin{aligned}
&\cos\left(4\pi\sqrt{nt} - \frac{\pi}{4}\right)\cos\left(4\pi\sqrt{mt} - \frac{\pi}{4}\right)\cos\left(4\pi\sqrt{\alpha lt} - \frac{\pi}{4}\right) \\
&\qquad = \frac{1}{4} \sum_{(i_0,i_1)\in\{0,1\}^2} \cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right),
\end{aligned} \tag{22}$$

where α_1 is defined by (9). By using Formula (22), the sum S_1 is written as the following:

$$\begin{aligned} S_1 &= \frac{\alpha^{1/4}}{8\pi^3\sqrt{2}} \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n\le M_0, m\le M_0\\ l\le M_0\\ \alpha_1=0}} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} \int_T^{2T} t^{3/4}\cos\left(\frac{\pi}{4}\right) dt \\ &+ \frac{\alpha^{1/4}}{8\pi^3\sqrt{2}} \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n\le M_0, m\le M_0\\ l\le M_0\\ \alpha_1\neq 0}} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} \int_T^{2T} t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right) dt. \end{aligned}$$

It follows that

$$\begin{aligned} S_1 &\ll \frac{\alpha^{1/4}}{8\pi^3\sqrt{2}} \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n\le M_0, m\le M_0\\ l\le M_0\\ \alpha_1=0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left|\int_T^{2T} t^{3/4}\cos\left(\frac{\pi}{4}\right) dt\right| \\ &+ \frac{\alpha^{1/4}}{8\pi^3\sqrt{2}} \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n\le M_0\\ m\le M_0\\ l\le M_0\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left|\int_T^{2T} t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)\right| dt. \end{aligned}$$

By using Lemma 3, we get

$$\sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n\le M_0\\ m\le M_0\\ l\le M_0\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}} \left|\int_T^{2T} t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right) dt\right| \ll_{\alpha,\varepsilon} T^{41/24+\varepsilon}.$$

Recall that based on the hypotheses of (4) and (5), the series $c_1(\alpha)$ converges absolutely. Then, we obtain

$$\begin{aligned} \frac{\alpha^{1/4}}{8\pi^3\sqrt{2}} & \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n\le M_0, m\le M_0\\ l\le M_0\\ \alpha_1=0}} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} \int_T^{2T} t^{3/4}\cos\left(\frac{\pi}{4}\right) dt \\ &= \frac{\alpha^{1/4}}{28\pi^3} c_1(\alpha, M_0)\left((2T)^{7/4}-T^{7/4}\right) \\ &= \frac{\alpha^{1/4}}{28\pi^3}\left[c_1(\alpha)+c_1(\alpha,M_0)-c_1(\alpha)\right]\left((2T)^{7/4}-T^{7/4}\right) \\ &= \frac{\alpha^{1/4}}{28\pi^3} c_1(\alpha)\left((2T)^{7/4}-T^{7/4}\right) + O_{\alpha,\varepsilon}\left(T^{7/4}\left|c_1(\alpha)-c_1(\alpha,M_0)\right|\right). \end{aligned}$$

Lemma 2 (c) leads to

$$|c_1(\alpha)-c_1(\alpha,M_0)| \ll_{\alpha,\varepsilon} M_0^{-5/4+\varepsilon} \ll_{\alpha,\varepsilon} T^{-5/24+\varepsilon}.$$

Hence, we get

$$\frac{\alpha^{1/4}}{28\pi^3} c_1(\alpha)\left((2T)^{7/4}-T^{7/4}\right) + O_{\alpha,\varepsilon}\left(T^{7/4}\left|c_1(\alpha)-c_1(\alpha,M_0)\right|\right)$$

$$= \frac{\alpha^{1/4}}{28\pi^3} c_1(\alpha) \left((2T)^{7/4} - T^{7/4} \right) + O_{\alpha,\varepsilon}\left(T^{37/24+\varepsilon} \right)$$

and therefore,

$$S_1 = \frac{\alpha^{1/4}}{28\pi^3} c_1(\alpha) \left((2T)^{7/4} - T^{7/4} \right) + O_{\alpha,\varepsilon}\left(T^{37/24+\varepsilon} \right).$$

For the sum S_2, we have

$$\begin{aligned} S_2 \quad &= \frac{\alpha^{1/4}}{(\pi\sqrt{2})^3} \sum_{n \le M_0} \sum_{m \le M_0} \sum_{M_0 < l \le M} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} \\ &\times \int_T^{2T} t^{3/4} \cos\left(4\pi\sqrt{nt} - \frac{\pi}{4}\right) \cos\left(4\pi\sqrt{mt} - \frac{\pi}{4}\right) \cos\left(4\pi\sqrt{\alpha l t} - \frac{\pi}{4}\right) dt. \end{aligned}$$

Formula (22) implies that

$$\begin{aligned} S_2 \; &= \frac{\alpha^{1/4}}{8\pi^3\sqrt{2}} \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{n \le M_0} \sum_{m \le M_0} \sum_{M_0 < l \le M} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} \\ &\qquad\qquad \times \int_T^{2T} t^{3/4} \cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right) dt \\ &\ll \frac{\alpha^{1/4}}{8\pi^3\sqrt{2}} \left| \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n \le M_0 \\ m \le M_0 \\ M_0 < l \le M}} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} \right| \left| \int_T^{2T} t^{3/4} \cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right) dt \right| \\ &\ll \frac{\alpha^{1/4}}{8\pi^3\sqrt{2}} \left| \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n \le M_0 \\ m \le M_0 \\ M_0 < l \le M \\ \alpha_1 = 0}} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} \right| \left| \int_T^{2T} t^{3/4} \cos\left(\frac{\pi}{4}\right) dt \right| \\ &+ \frac{\alpha^{1/4}}{8\pi^3\sqrt{2}} \left| \sum_{(i_0,i_1)\in\{0,1\}^2} \sum_{\substack{n \le M_0 \\ m \le M_0 \\ M_0 < l \le M \\ \alpha_1 \ne 0}} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}} \right| \left| \int_T^{2T} t^{3/4} \cos\left(4\pi\alpha_1\sqrt{t} - \frac{\pi}{4}\right) dt \right|. \end{aligned}$$

By virtue of Lemma 2 (c), we obtain

$$\begin{aligned}
&\left|\sum_{(i_0,i_1)\in\{0,1\}^2}\sum_{\substack{n\le M_0,m\le M_0\\ M_0<l\le M\\ \alpha_1=0}}\frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}}\right|\left|\int_T^{2T}t^{3/4}\cos\left(\frac{\pi}{4}\right)dt\right|\\
&\ll_{\alpha,\varepsilon} T^{7/4}\left|\left|\sum_{(i_0,i_1)\in\{0,1\}^2}\sum_{\substack{n,m,l=1\\ \alpha_1=0}}^{+\infty}\frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}}\right|\right.\\
&\qquad\qquad\left.-\left|\sum_{(i_0,i_1)\in\{0,1\}^2}\sum_{\substack{n\le M_0,m\le M_0,l\le M_0\\ \alpha_1=0}}\frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}}\right|\right|\\
&\ll_{\alpha,\varepsilon} T^{7/4}\left||c_1(\alpha)|-|c_1(\alpha,M_0)|\right|\\
&\ll_{\alpha,\varepsilon} T^{7/4}\left|c_1(\alpha)-c_1(\alpha,M_0)\right|\\
&\ll_{\alpha,\varepsilon} T^{7/4}M_0^{-5/4+\varepsilon}\ll_{\alpha,\varepsilon} T^{37/24+\varepsilon}.
\end{aligned}$$

Therefore, we get

$$\left|\sum_{(i_0,i_1)\in\{0,1\}^2}\sum_{\substack{n\le M_0,m\le M_0\\ M_0<l\le M\\ \alpha_1=0}}\frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}}\right|\left|\int_T^{2T}t^{3/4}\cos\left(\frac{\pi}{4}\right)dt\right|\ll_{\alpha,\varepsilon} T^{37/24+\varepsilon}. \tag{23}$$

In addition, we have

$$\begin{aligned}
&\frac{\alpha^{1/4}}{8\pi^3\sqrt{2}}\sum_{(i_0,i_1)\in\{0,1\}^2}\sum_{\substack{n\le M_0\\ m\le M_0\\ M_0<l\le M\\ \alpha_1\neq 0}}\frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}}\left|\int_T^{2T}t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|\\
&=\frac{\alpha^{1/4}}{8\pi^3\sqrt{2}}\sum_{(i_0,i_1)\in\{0,1\}^2}\sum_{\substack{n\le M_0,m\le M_0\\ M_0<l\le(50M_0/\alpha)\\ \alpha_1\neq 0}}\frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}}\left|\int_T^{2T}t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|\\
&\quad+\frac{\alpha^{1/4}}{8\pi^3\sqrt{2}}\sum_{(i_0,i_1)\in\{0,1\}^2}\sum_{\substack{n\le M_0,m\le M_0\\ (50M_0/\alpha)<l\le M\\ \alpha_1\neq 0}}\frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}}\left|\int_T^{2T}t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|.
\end{aligned}$$

Now, by applying Lemma 3 (b), we obtain

$$\sum_{(i_0,i_1)\in\{0,1\}^2}\ \sum_{\substack{n\le M_0, m\le M_0\\ M_0<l\le(50M_0/\alpha)\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}}\left|\int_T^{2T} t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|$$

$$\ll_{\alpha,\varepsilon} T^{41/24+\varepsilon}.$$

If $(50M_0/\alpha)<l$, then $\alpha_1 \gg_\alpha \sqrt{l}$. Hence, by Lemma 3 (a), we get

$$\sum_{(i_0,i_1)\in\{0,1\}^2}\ \sum_{\substack{n\le M_0, m\le M_0\\ (50M_0/\alpha)<l\le M\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}}\left|\int_T^{2T} t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|$$

$$\ll_{\alpha,\varepsilon} T^{41/24+\varepsilon}.$$

It follows that

$$S_2 \ll_{\alpha,\varepsilon} T^{41/24+\varepsilon}.$$

By using the same arguments as above, we obtain

$$S_3 \ll_{\alpha,\varepsilon} T^{41/24+\varepsilon}.$$

For the sum S_4, we have

$$\begin{aligned}S_4 \ll{}& \frac{\alpha^{1/4}}{8\pi^3\sqrt{2}}\left|\sum_{(i_0,i_1)\in\{0,1\}^2}\ \sum_{\substack{n\le M_0, m\le M_0\\ M_0<l\le M\\ \alpha_1=0}} \frac{\lambda_f(n)\lambda_f(m)\lambda_f(l)}{(nml)^{3/4}}\right|\left|\int_T^{2T} t^{3/4}\cos\left(\frac{\pi}{4}\right)dt\right|\\ &+\frac{\alpha^{1/4}}{8\pi^3\sqrt{2}}\sum_{(i_0,i_1)\in\{0,1\}^2}\ \sum_{\substack{n\le M_0\\ m\le M_0\\ M_0<l\le M\\ \alpha_1\neq 0}}\frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}}\left|\int_T^{2T} t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|.\end{aligned}$$

Lemma 3 (a) implies that

$$\begin{aligned}&\sum_{(i_0,i_1)\in\{0,1\}^2}\ \sum_{\substack{n\le M_0, m\le M_0\\ M_0<l\le M\\ \alpha_1\neq 0}} \frac{|\lambda_f(n)||\lambda_f(m)||\lambda_f(l)|}{(nml)^{3/4}}\left|\int_T^{2T} t^{3/4}\cos\left(4\pi\alpha_1\sqrt{t}-\frac{\pi}{4}\right)dt\right|\\ &\qquad\ll_{\alpha,\varepsilon} T^{3/2+\varepsilon}+T^{7/4-1/24+\varepsilon}\\ &\qquad\ll_{\alpha,\varepsilon} T^{41/24+\varepsilon}.\end{aligned}$$

Thus, by Formula (23) and the last bound, we get

$$S_4 \ll_{\alpha,\varepsilon} T^{41/24+\varepsilon}.$$

By the same arguments applied on the sums S_5 and S_6, we deduce that

$$S_5 \ll_{\alpha,\varepsilon} T^{41/24+\varepsilon} \qquad \text{and} \qquad S_6 \ll_{\alpha,\varepsilon} T^{41/24+\varepsilon}.$$

By summing up S_1 to S_6 above, we obtain

$$\int_T^{2T} (B_M(t))^2 B_M(\alpha t)dt = \frac{\alpha^{1/4}}{28\pi^3} c_1(\alpha) \left((2T)^{7/4} - T^{7/4}\right) + O_{\alpha,\varepsilon}\left(T^{41/24+\varepsilon}\right).$$

Therefore, we get the following formula

$$\int_T^{2T} A_f^2(t) A_f(\alpha t)dt = \frac{\alpha^{1/4}}{28\pi^3} c_1(\alpha) \left((2T)^{7/4} - T^{7/4}\right) + O_{\alpha,\varepsilon}\left(T^{41/24+\varepsilon}\right).$$

Put $T = X/2, X/4, ...$ After summing up, we find

$$\int_2^X A_f^2(t) A_f(\alpha t)dt = \frac{\alpha^{1/4}}{28\pi^3} c_1(\alpha) X^{7/4} + O_{\alpha,\varepsilon}\left(X^{41/24+\varepsilon}\right).$$

Let

$$C_f(\alpha) := \frac{\alpha^{1/4}}{28\pi^3} c_1(\alpha).$$

We finally get

$$\int_2^X A_f^2(t) A_f(\alpha t)dt = C_f(\alpha) X^{7/4} + O_{\alpha,\varepsilon}\left(X^{41/24+\varepsilon}\right).$$

Remark 1. If $\alpha = 1$, then we find the cubic moment for $A_f(x)$:

$$\int_2^X A_f^3(t)dt = O_{\varepsilon}\left(X^{41/24+\varepsilon}\right).$$

We just have to prove that $C_f(1) = 0$ or $c_1(1) = 0$. A simple way to get it is the use of the assertion 3 of Theorem 2 with $\alpha = 1 \in \mathbb{Q}_+^*$. Generally, if $\alpha \in \mathbb{Q}_+^*$ or $\alpha \in \sqrt{N}\mathbb{Q}_+^*$, we have the same upper bound which means

$$\int_2^X A_f^2(t) A_f(\alpha t)dt = O_{\alpha,\varepsilon}\left(X^{41/24+\varepsilon}\right)$$

and this is also a consequence of the assertion 3 of Theorem 2.

4. Sign of the Constant $C_f(\alpha)$

In this section, we are interested in the discussion of the sign of the constant $C_f(\alpha)$. Since $C_f(\alpha) = \frac{\alpha^{1/4}}{28\pi^3} c_1(\alpha)$, it suffices then to discuss the sign of $c_1(\alpha)$. For that, we shall prove Theorem 2.

Proof. 1. If α satisfies (4), then by using the previous steps in the proof of Lemma 2, for all positive integers n, m and l such that $\sqrt{n} \pm \sqrt{m} = \sqrt{\alpha l}$, we get

$$\begin{cases} n = d_1 tr^2 l'' \\ m = d_1 \frac{N}{t} s^2 l'' \\ l = a_2' b_2' d_2 l'' \end{cases} \quad \text{or} \quad \begin{cases} n = d_1 \frac{N}{t} s^2 l'' \\ m = d_1 tr^2 l'' \\ l = a_2' b_2' d_2 l''. \end{cases}$$

In this case, we have

$$\alpha = \frac{d_1}{d_2 a_2' b_2'} \left[\left(tr^2 + \frac{N}{t} s^2 \right) \pm 2rs\sqrt{N} \right].$$

Suppose that $d_1 = d_2 a_2' b_2' = 1$. Then, α becomes

$$\alpha = \left(tr^2 + \frac{N}{t} s^2 \right) \pm 2rs\sqrt{N}.$$

Therefore, we obtain

$$\begin{aligned} c_1(\alpha) \qquad &= \sum_{l''=1}^{+\infty} \frac{\lambda_f(tr^2 l'')\lambda_f\left(\frac{N}{t} s^2 l''\right)\lambda_f(l'')}{(Ns^2r^2l''^3)^{3/4}} \\ &= \frac{1}{(Ns^2r^2)^{3/4}} \sum_{l''=1}^{+\infty} \frac{\lambda_f(tr^2 l'')\lambda_f\left(\frac{N}{t} s^2 l''\right)\lambda_f(l'')}{l''^{9/4}}. \end{aligned}$$

According to (1), we have

$$\lambda_f(tr^2 l'')\lambda_f\left(\frac{N}{t} s^2 l''\right) = \sum_{d \mid \gcd\left(tr^2 l'', \frac{N}{t} s^2 l''\right)} \lambda_f\left(\frac{Nr^2 s^2 l''^2}{d^2}\right). \tag{24}$$

But $\gcd(r,s) = \gcd\left(t, \frac{N}{t}\right) = 1$, since N is a square-free integer. It follows that $\gcd\left(tr^2, \frac{N}{t}s^2\right) = 1$ and this implies that $\gcd\left(tr^2 l'', \frac{N}{t} s^2 l''\right) = l''$. Thus, Formula (24) leads to

$$\lambda_f(tr^2 l'')\lambda_f\left(\frac{N}{t} s^2 l''\right) = \sum_{d \mid l''} \lambda_f\left(Nr^2 s^2 \frac{l''^2}{d^2}\right). \tag{25}$$

By applying Formula (1), and since $\gcd\left(Nr^2 s^2 l'', l''\right) = l''$, one can deduce that

$$\sum_{d \mid l''} \lambda_f\left(Nr^2 s^2 \frac{l''^2}{d^2}\right) = \lambda_f\left(Nr^2 s^2 l''\right)\lambda_f\left(l''\right).$$

Thus, Formula (1) joined on to (25) results in

$$\lambda_f(tr^2 l'')\lambda_f\left(\frac{N}{t} s^2 l''\right)\lambda_f(l'') = \lambda_f\left(Nr^2 s^2 l''\right)\lambda_f\left(l''\right)^2. \tag{26}$$

By using Formula (26) and by the expression of $c_1(\alpha)$ in our case, we have

$$\begin{aligned}
c_1(\alpha) &= \frac{1}{(Ns^2r^2)^{3/4}} \sum_{l''=1}^{+\infty} \left(\frac{\lambda_f\left(Nr^2s^2l''\right)\lambda_f\left(l''\right)^2}{l''^{9/4}} \right) \\
&= \frac{1}{(Ns^2r^2)^{3/4}} \left[\sum_{\substack{l''=1 \\ Nr^2s^2l'' \in \mathcal{A}_+}}^{+\infty} \left(\frac{\lambda_f\left(Nr^2s^2l''\right)\lambda_f\left(l''\right)^2}{l''^{9/4}} \right) \right. \\
&\qquad \left. + \sum_{\substack{l''=1 \\ Nr^2s^2l'' \in \mathcal{A}_-}}^{+\infty} \left(\frac{\lambda_f\left(Nr^2s^2l''\right)\lambda_f\left(l''\right)^2}{l''^{9/4}} \right) \right] \\
&= \frac{1}{(Ns^2r^2)^{3/4}}(S_+ - S_-).
\end{aligned}$$

It follows that, if $S_+ > S_-$, then $c_1(\alpha) > 0$, and therefore, $C_f(\alpha) > 0$. Besides, if $S_+ < S_-$, then $c_1(\alpha) < 0$, and therefore, $C_f(\alpha) < 0$.

2. If α satisfies (5), then we have

$$\begin{cases} n = \frac{1}{2}(d_1 tr^2 l'') \\ m = \frac{1}{2}\left(d_1 \frac{N}{t} s^2 l''\right) \\ l = a_2' b_2' d_2 l'' \end{cases} \quad \text{or} \quad \begin{cases} n = \frac{1}{2}\left(d_1 \frac{N}{t} s^2 l''\right) \\ m = \frac{1}{2}(d_1 tr^2 l'') \\ l = a_2' b_2' d_2 l''. \end{cases}$$

Note that n and m are well-defined if l'' and d_1 are not both odd. Hence, we have

$$\alpha = \frac{d_1}{d_2 a_2' b_2'} \left[\frac{1}{2}\left(tr^2 + \frac{N}{t}s^2 \right) \pm rs\sqrt{N} \right].$$

Suppose that $d_1 = 2$ and $d_2 a_2' b_2' = 1$. So, the expression of α becomes

$$\alpha = \left(tr^2 + \frac{N}{t}s^2 \right) \pm 2rs\sqrt{N}$$

and thus,

$$c_1(\alpha) = \frac{1}{(Ns^2r^2)^{3/4}} \sum_{l''=1}^{+\infty} \frac{\lambda_f(tr^2l'')\lambda_f\left(\frac{N}{t}s^2l''\right)\lambda_f(l'')}{l''^{9/4}}.$$

By following the same procedure as in the proof of 1., we get:

$$c_1(\alpha) = \frac{1}{(Ns^2r^2)^{3/4}}(S_+ - S_-).$$

We conclude also that, if $S_+ > S_-$, then $C_f(\alpha) > 0$, and if not, then $C_f(\alpha) \leq 0$.

3. If α is a positive rational number and the equation

$$\sqrt{n} + (-1)^{i_0}\sqrt{m} + (-1)^{i_1}\sqrt{\alpha l} = 0, \qquad (i_0, i_1) \in \{0,1\}^2$$

is solvable in n, m and $l \in \mathbb{N}^*$, then by the result of Lemma 2 (a), we have

$$\alpha = \frac{d_1}{d_2 a_2' b_2'}\left[\left(tr^2 + \frac{N}{t}s^2\right) \pm 2rs\sqrt{N}\right]$$

or

$$\alpha = \frac{d_1}{d_2 a_2' b_2'}\left[\frac{1}{2}\left(tr^2 + \frac{N}{t}s^2\right) \pm rs\sqrt{N}\right].$$

Since $\alpha = \frac{a_1}{a_2} + \frac{b_1}{b_2}\sqrt{N}$, then $\alpha = \frac{a_1}{a_2}$, with $\gcd(a_1, a_2) = 1$ and $b_1 = 0$. It follows that $\gcd(b_1, b_2) = \gcd(0, b_2) = b_2 = 1$. Therefore, we obtain $d_1 = a_1 > 0$ and $a_2' = b_2' = d_2 = 1$. Thus, we get $a_2 = a_2' d_2 = 1$. Hence, we find that $\alpha = a_1$. So, we have

$$\begin{cases} \left(tr^2 + \frac{N}{t}s^2\right) = 1 \\ rs = 0 \end{cases} \text{ or } \begin{cases} \frac{1}{2}\left(tr^2 + \frac{N}{t}s^2\right) = 1 \\ rs = 0. \end{cases}$$

Now, if α satisfies (4), then $tr^2 + \frac{N}{t}s^2 = 1$. This implies that $tr^2 = 0$ or $\frac{N}{t}s^2 = 0$, and $r = 0$ or $s = 0$. But, we have

$$\begin{cases} n = a_1 tr^2 l'' \\ m = a_1 \frac{N}{t}s^2 l'' \end{cases} \text{ or } \begin{cases} n = a_1 \frac{N}{t}s^2 l'' \\ m = a_1 tr^2 l''. \end{cases}$$

Therefore, $n = 0$ or $m = 0$. This contradicts the fact that $n, m \in \mathbb{N}^*$. If α satisfies (5), then $\frac{1}{2}\left(tr^2 + \frac{N}{t}s^2\right) = 1$ and therefore, $tr^2 = \frac{N}{t}s^2 = 1$. This leads to

$$n = m = \frac{1}{2}a_1 l''.$$

Set $l'' = 2q$ to be even. Thus, we have $n = m = a_1 q$ and $l = l''$. The equation

$$\sqrt{n} + (-1)^{i_0}\sqrt{m} + (-1)^{i_1}\sqrt{\alpha l} = 0, \qquad (i_0, i_1) \in \{0,1\}^2$$

leads to

$$\sqrt{a_1 q} + (-1)^{i_0}\sqrt{a_1 q} + (-1)^{i_1}\sqrt{2a_1 q} = 0, \qquad (i_0, i_1) \in \{0,1\}^2.$$

This is equivalent to

$$\sqrt{a_1 q}\left[1 + (-1)^{i_0} + (-1)^{i_1}\sqrt{2}\right] = 0, \qquad (i_0, i_1) \in \{0,1\}^2.$$

It is clear that for all $(i_0, i_1) \in \{0,1\}^2$, we have

$$1 + (-1)^{i_0} + (-1)^{i_1}\sqrt{2} \neq 0.$$

Hence, the equation

$$\sqrt{n} + (-1)^{i_0}\sqrt{m} + (-1)^{i_1}\sqrt{\alpha l} = 0, \qquad (i_0, i_1) \in \{0,1\}^2$$

is not solvable in the case of $\alpha = a_1 = \frac{a_1}{a_2}$, which is a positive rational number, and therefore, $c_1(\alpha) = 0 = C_f(\alpha)$. Note that, in this case, we can also consider the fact that $rs = 0$ which implies that $r = 0$ or $s = 0$, and since $tr^2 = \frac{N}{t}s^2 = 1$, we then get the contradiction.

Suppose now that $\alpha \in \sqrt{N}\mathbb{Q}_+^*$ and that the equation

$$\sqrt{n} + (-1)^{i_0}\sqrt{m} + (-1)^{i_1}\sqrt{\alpha l} = 0, \qquad (i_0, i_1) \in \{0,1\}^2$$

is solvable in n, m and $l \in \mathbb{N}^*$. Lemma 2 (a) yields

$$\alpha = \frac{d_1}{d_2 a_2' b_2'}\left[\left(tr^2 + \frac{N}{t}s^2\right) \pm 2rs\sqrt{N}\right]$$

or

$$\alpha = \frac{d_1}{d_2 a_2' b_2'}\left[\frac{1}{2}\left(tr^2 + \frac{N}{t}s^2\right) \pm rs\sqrt{N}\right].$$

Since $\alpha = \frac{a_1}{a_2} + \frac{b_1}{b_2}\sqrt{N}$, then $\alpha = \frac{b_1}{b_2}\sqrt{N}$. Thus, $a_1 = 0$, $\gcd(a_1, a_2) = a_2 = 1$, $d_1 = b_1 > 0$ and $d_2 = a_2' = b_2' = 1$. Since $d_2 = \gcd(a_2, b_2) = b_2 = 1$, then $\alpha = b_1\sqrt{N}$. Hence, we obtain

$$\begin{cases} b_1\left(tr^2 + \frac{N}{t}s^2\right) = 0 \\ 2b_1 rs = b_1 \end{cases} \quad \text{or} \quad \begin{cases} \frac{b_1}{2}\left(tr^2 + \frac{N}{t}s^2\right) = 0 \\ b_1 rs = b_1. \end{cases}$$

In both cases, we have $tr^2 + \frac{N}{t}s^2 = 0$, and this is equivalent to $tr^2 = \frac{N}{t}s^2 = 0$.
If α satisfies (4), then

$$\begin{cases} n = b_1 tr^2 l'' \\ m = b_1 \frac{N}{t}s^2 l'' \end{cases} \quad \text{or} \quad \begin{cases} n = b_1 \frac{N}{t}s^2 l'' \\ m = b_1 tr^2 l'' \end{cases}$$

and if α satisfies (5), then

$$\begin{cases} n = \frac{1}{2}b_1 tr^2 l'' \\ m = \frac{1}{2}b_1 \frac{N}{t}s^2 l'' \end{cases} \quad \text{or} \quad \begin{cases} n = \frac{1}{2}b_1 \frac{N}{t}s^2 l'' \\ m = \frac{1}{2}b_1 tr^2 l''. \end{cases}$$

Therefore, in both cases, $n = m = 0$. Moreover, the equation

$$\sqrt{n} + (-1)^{i_0}\sqrt{m} + (-1)^{i_1}\sqrt{\alpha l} = 0, \qquad (i_0, i_1) \in \{0,1\}^2$$

is solvable, which implies that $n = m = l = 0$. This contradicts the fact that n, m and $l \in \mathbb{N}^*$. Hence, we get $c_1(\alpha) = 0 = C_f(\alpha)$.

Note that we can show the contradiction when α satisfies (4) by using the equality $2b_1 rs = b_1$, which is equivalent to $2rs = 1$ and this can not happen for positive integers r and s. $\square$

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